University of Texas
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson
CE 363

CE 363 ADVANCED STRUCTURAL ANALYSIS

Fall 2005

Unique No.: 14900

Course Purpose:

CE 363 focuses on computing the response of statically indeterminate structural systems. This course will extend many of the concepts treated in CE 329, and it will introduce matrix methods of analysis that form the basis of all modern structural analysis software.

Course Objectives:

By the end of the course, you should be able to do the following:

- Compute structural deflections accounting for applied loads, temperature effects, initial fabrication errors, support settlements, and flexible supports.
- Determine internal member forces and resulting stress distributions.
- Analyze statically indeterminate structures using both stiffness and flexibility approaches.
- Use and/or develop structural analysis software to analyze complicated structural systems.
- Interpret the output from computer-based analyses for the purpose of structural design.

Text (*optional*):

West, H. H. and Geschwindner, L. F. (2002) *Fundamentals of Structural Analysis, 2nd Ed.*. John Wiley & Sons, Inc., New York, NY.

Additional References:

Kassimali, A. (1999). *Matrix Analysis of Structures*. Brooks/Cole Publishing Company, Pacific Grove, CA.

Hibbeler, R. C. (2006). *Structural Analysis, 6th Ed.* Pearson Prentice Hall, Upper Saddle River, NJ.

McGuire, W., Gallagher, R. H., and Ziemian, R. D. (2000). *Matrix Structural Analysis: 2nd Ed.* John Wiley & Sons, Inc., New York, NY.

Sennet, R. E. (1994). *Matrix Analysis of Structures*. Waveland Press, Inc. Prospect Heights, IL.

Office Hours:

Tu/Th	1:30 - 3:00 P.M.
W	10:00 - 12:00 noon

Office: ECJ 4.722
Phone: 475-6175
email: ewilliamson@mail.utexas.edu

Class Hours:

Tu/Th 9:30 - 11:00 ECJ 5.410

Prerequisite:

CE 329

Conduct of Course:

Attendance: The course consists primarily of lectures and in-class problems. Attendance is essential and will follow the policies set forth in the Undergraduate and Graduate Catalogs.

Homework: Homework problems will be assigned regularly. *Late work (any that come in after the beginning of the period on the due date) will receive a maximum grade of 50%.* Late work will not be accepted after the solution has been made available, nor will late work be accepted from any student more than two times over the course of the semester.

Homework Format: The homework problems are probably the most important vehicle for learning the material presented in this course. There are two goals in doing the homework problems: *(a)* to learn the concept or method used in solving the given problem, and *(b)* to communicate your approach and results to someone else (the instructor or grader in this case). To encourage the achievement of these goals, I will insist that *all* homework assignments for the semester be done on engineering paper and/or printed out neatly from the computer. The evaluation of each homework will depend on both presentation aesthetics and technical correctness.

Tests: There will be two tests during the semester. Students will be given two hours to complete the midterm exams. Exact times and dates will be announced by the instructor at least two weeks prior to the exam. A final examination covering the entire course will be given during the regularly scheduled exam period (*Thursday, December 15, 9:00 - 12:00 noon*).

Missed Tests: If you miss a test without either a certified medical excuse or prior instructor approval, you may take a makeup test at a designated time near the end of the semester. Only one makeup test will be given. It will be fair but challenging! Tests missed with certified medical excuses or prior instructor approval will be dealt with individually. If you miss the final exam without a valid excuse, a zero will be averaged into your grade.

Grading: Grades will be determined according to the following format: Midterm exam with higher grade (30%), midterm exam with lower grade (25%), homework (15%), and final examination (30%). A grade of 90 or above will receive an A, 80 or above at least B, 70 or above at least C, and 60 or above at least D. *Exception:* In order to receive a passing grade, your exam average must be 60 or above.

Notice: I do not curve grades in this course. It is theoretically possible for everyone in the class to get an A (or an F). Your performance depends only on how you do, not on how everyone else in the class does. Therefore, it is in your best interest to help your classmates in every *legal* way possible.

Gray areas between guaranteed letter grades: There will be a "gray area" of several points below the specified numerical cutoff for letter grades. Thus, two people getting the same numerical grade (say an 89) might receive different grades for the course. If you are in one of these gray areas, whether or not you receive the higher or the lower grade depends upon your improvement over the semester and your participation in class and group work. If your test performance has shown improvement and you actively participate in class discussions, your grade will go up.

Academic Integrity: As engineers you will be responsible for upholding the canons of ethics for the profession. A test of your ability to do so is to uphold the University's Academic Honesty Policy. While I do not anticipate problems of this nature, any instances of academic dishonesty will be dealt with immediately and severely in accordance with published procedures. Students who violate University rules on scholastic dishonesty are subject to disciplinary penalties, including the possibility of failure in the course and/or dismissal from the University. Because such dishonesty harms the individual, all students, and the integrity of the University, policies on scholastic dishonesty will be strictly enforced. For further information, visit the Student Judicial Services web site http://deanofstudents.utexas.edu/sjs/.

Consulting with the instructor: You are strongly encouraged to discuss academic or personal questions with the instructor during office hours or by email.

Important Dates:

September 6	Last day to drop course without approval of Chairman and Dean
September 16	Last day to drop course for a possible refund
September 28	Last day to drop course without possible academic penalty
October 26	Last day to drop course with Dean's approval
November 24-26	Thanksgiving Holidays

Course Evaluation:

The students will evaluate the course and the instructor on forms provided by the Measurement and Evaluation Center.

Additional Information:

Web-based, password-protected class sites will be associated with all academic courses taught at the University. Syllabi, handouts, assignments and other resources are types of information that may be available within these sites. Site activities could include exchanging email, engaging in class discussions and chats, and exchanging files. In addition, electronic class rosters will be a component of the sites. Students who do not want their names included in these electronic class rosters must restrict their directory information in the Office of the Registrar, Main Building, Room 1. For information on restricting directory information, see the Undergraduate Catalog or go to: http://www.utexas.edu/student/registrar/catalogs/gi00-01/app/appc09.html.

The University of Texas at Austin provides upon request appropriate academic adjustments for qualified students with disabilities. For more information, contact the Office of the Dean of Students at 471-6259, 471-4241 TDD, or the College of Engineering, Director of Students with Disabilities at 471-4382.

Brief of History of Structural Mechanics and Analysis

1700-1800:	Fundamental concepts of elasticity (stress, strain, constitutive laws) are established. Work and Energy concepts for rigid and deformable bodies are developed.
1800-1900:	Systematic methods of structural analysis begin to emerge, and there is further development of energy/work concepts. • Reciprocal Theorems • Unit Load / Unit Displacement Methods • Castigliano's Theorems • Rayleigh-Ritz Methods (1900-1910)
1900-1950:	Slope-deflection equations are formulated (early stiffness-based approach), the moment-distribution method is developed, and approximate analysis techniques are established for regular frame structures.
1950-1955:	The first real, usable computers appear. The solution of many simultaneous equations becomes practical and leads to opportunities for new analysis methods.
1955-1965:	Early work on a formalized stiffness method of analysis begins. There are many debates over stiffness-based versus flexibility-based methods of analysis. Early general purpose computer programs start to emerge.
1965-present:	This period is recognized as the modern era of structural analysis. Stiffness-based analysis approaches are dominant. The finite element method comes into widespread use. Computer software becomes widely available on personal computers.

Classical methods of analysis (e.g., moment distribution) are now used for checking computer-generated output and for quick manual computations for simple frames.

Letters to the Editor

Sanity Checks Needed for Engineering Software Calculations

Re: Is it Ethical to Use an Engineering Software Program to Solve a Problem if You Cannot Complete the Calculations Manually? (Clay Forister PE, Summer 2005)

I found the article by Clay Forister PE in the Summer issue of the *Texas Civil Engineer* on the use of engineering software programs very interesting. It reflects many of my concerns on the current use of such programs and should be required reading for all engineers.

In my engineering experience, there have been instances where major errors in computer analyses were identified and fortunately corrected before transmitting the results to the client. However, the initial attitude was, the computer generated the results so they must be correct. Sadly, programming errors can have fatal consequences.

This was the case in the recent collapse of a rail tunnel in Singapore where four lives were lost. One of the causes of the collapse "was the use of an inappropriate soil simulation model, which overestimated the soil strength at the accident site and underestimated the forces on the retaining walls within the excavation."(*Engineering News Record*, May 23, 2005, page 15).

The use of computer programs is here to stay. Although in most cases one cannot perform check computations by hand, there should be a "sanity" check on the results. A critical issue mentioned by Mr. Forister was whether the user has a complete understanding of the assumptions embodied in the program, and whether they are appropriate to the problem being analyzed. This will be a function of the person's knowledge of the theory, boundary conditions, formulae, computational methods, and other factors used in the program. In today's legalistic environment, it is incumbent for engineering firms to have quality assurance and/or quality control procedures in place for checking the appropriateness of the program and results from computer analyses. I wonder if the problem is fully recognized in today's university environment where the trend in engineering education is to lessen the credit hours to obtain a degree and reduce the number of technical courses.

A concluding thought: what would be the position of an engineering licensing board on the engineer or group of engineers who performed the analyses in the Singapore tunnel collapse?

David Burgoine PE MASCE

Texas Civil Engineer *welcomes letters to the editor. Letters should be e-mailed to tce@silentpartners.com or mailed to: Managing Editor,* Texas Civil Engineer, *c/o Silent Partners, 8727 Shoal Creek Blvd., Austin, TX 78757-6815. Submission of letters implies permission to publish unless otherwise stated. We reserve the right to edit letters for style, length, and clarity.*

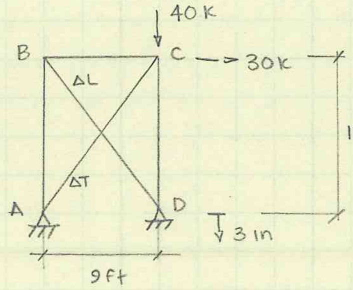
TEXAS CIVIL ENGINEER

CE 363: STRUCT. ANALYSIS | CATHERINE HOWELL 26 October 05 | $\boxed{92}$ | 1/5

MIDTERM

nice job!

1.



ΔT = 100° F

ΔL = 0.25 in

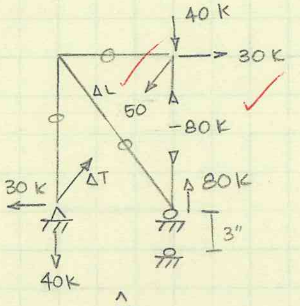
E = 15000 ksi

A = 12 in²

α = 6.5×10⁻⁶/°F

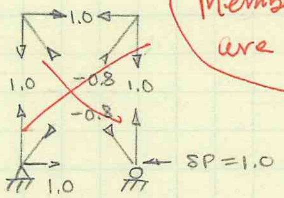
$EA = (15000\ \text{ksi})(12\ \text{in}^2) = 180000\ \text{K}$

Primary Structure:



$\hat{N}$

Secondary Structure:



n

Member forces are not correct

	L	$\hat{N}$	n	ΔL, ΔT	$\frac{nNL}{EA}$	$\frac{n\alpha\Delta TL}{n\Delta L}$	$\frac{n_i^2L}{EA}$	Nact
AB	144 in	0	~~1.0~~ $+\frac{4}{3}$	–	0	–	0.0008 in	~~217.7 K~~
BC	108	0	✓ 1.0	–	0	–	0.0006	~~217.7~~
CD	144	-80 K	~~1.0~~ $+\frac{4}{3}$	–	-0.064 in	–	0.0008	~~137.7~~
BD	180	0	~~-0.8~~ $-\frac{5}{3}$	0.25 in	0	-0.20 in	0.00064	~~-174.2~~
AC	180	50 K	~~-0.8~~ $-5/3$	100°F	-0.40	-0.0936	0.00064	~~124.2~~

OK PROCEDURE

$\Sigma = -0.464''$ $\quad -0.2936''$

$\Delta_{D_0} = 0.7576$ in right

$\Sigma = 0.00348\ \text{in/K} = f_{DD}$

$\Delta_{D0} + f_{DD} D_x = 0$

$0.7576\ \text{in} + D_x(-0.00348) = 0$

$D_x = 217.7\ \text{K}$ ✗

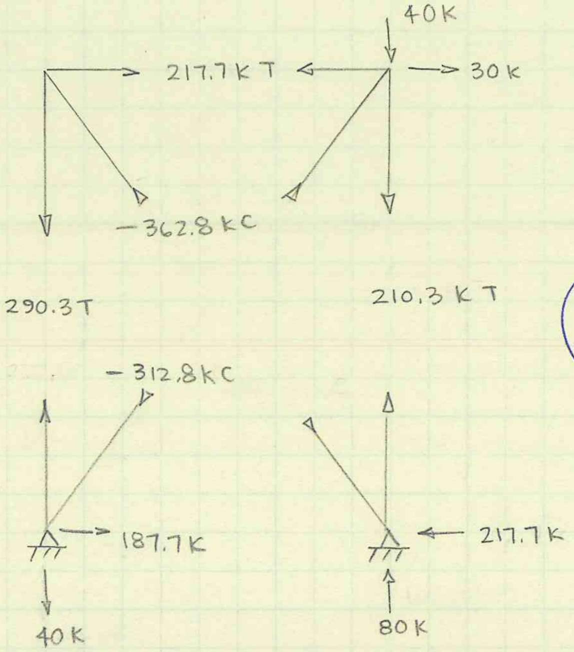
PROCEDURE OK - ERROR DUE TO VIRTUAL FORCE CALCULATION

$N = \hat{N} + D_x n$ ✓

MIDTERM

1. (cont'd)

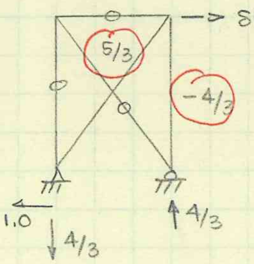
1.a)



(+) indicates tension
(also marked with T)

32/35

seems like a very, very large number ✓
... but, following through with it...



CORRECT FOR THIS CASE -

✓ ✓ ✓ ✓

$$\Delta = \sum \frac{nNL}{EA} + \sum n\alpha \Delta T L + \cancel{\sum n \Delta L} + RBM$$

$n_{AC} = 5/3$

$n_{CD} = -4/3$

$$\Delta = \frac{1}{180000k}\left[(5/3)(-312.8k)(180) + (-4/3)(210.3)(144)\right] +$$

$$(5/3)(6.5\times10^{-6}/°F)(100°F)(180) = -\xcancel{0.5506}\text{ in}$$
(left)

ΔRBM :

✓

$\theta = \frac{3in}{9ft} = 0.0278$

✓ ✓

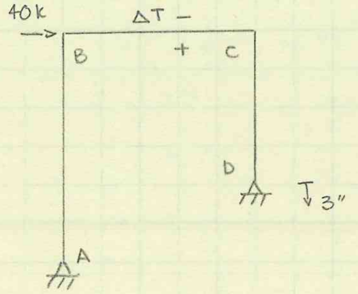
$\Delta_H = \theta(12ft) = 4.0$ in right

1.b) $\boxed{\Delta_{CH} = 3.45 \text{ in right} \times}$

But OK for your MEMBER FORCES

MIDTERM

2.



$I = 288 \text{ in}^4$

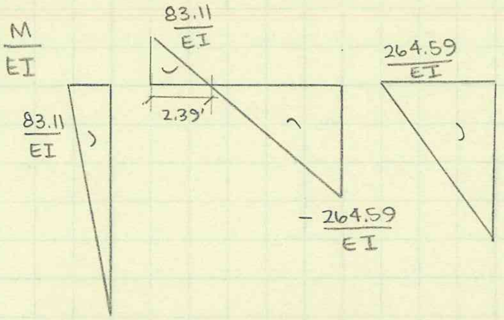
$E = 29000 \text{ ksi}$

$h = 12 \text{ in}$

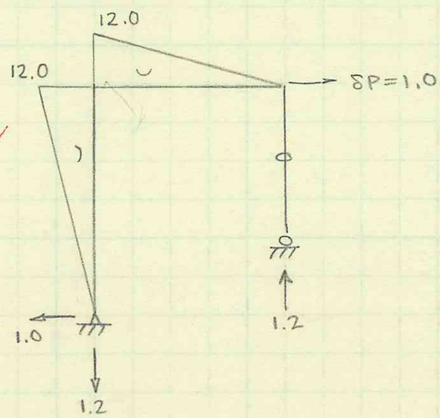
$\alpha = 6.5\times10^{-6}/°F$

$\Delta T = 100°F$

27/30

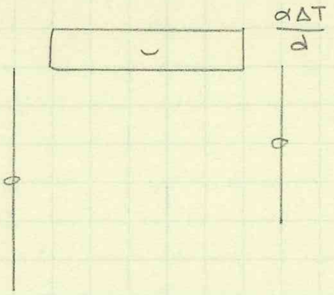


Virtual: ✓



$$\Delta = \int m \frac{M}{EI}dx + \int \frac{\alpha\Delta T}{d} m\, dx + RBM$$

Thermal:



$$\Delta_{C\,therm} = \frac{\alpha\Delta T}{d} L \cdot \frac{1}{2}(12.0\text{ft})$$

$$= \frac{(6.5\times10^{-6}/°F)(100°F)}{2(12\text{ in})}(10\text{ft})(12\text{ft})$$

$$= 0.468 \text{ in} \rightarrow$$

$$\Delta_{C\,load} = \frac{1}{2}\left(\frac{83.11}{EI}\right)(12)(2/3)(12.0) +$$

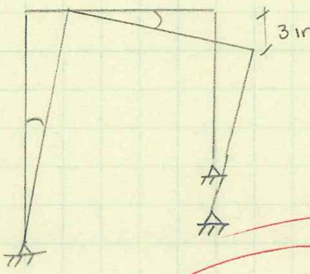
$$\frac{1}{2}\left(\frac{83.11}{EI}\right)(2.39)(12.0 \cancel{-0.7967}) + \quad 11.044$$

$$\frac{1}{2}\left(\frac{-264.59}{EI}\right)(7.61)(\cancel{4.5367}) \quad 3.04$$

$$= \frac{534.598}{EI} = 0.\cancel{1}06 \text{ in} \rightarrow$$

.418"

Rigid Body Motion



$$\theta = \frac{3\text{ in}}{10\text{ ft}} = 0.025$$

$$\Delta_{C\,RBM} = 0.025\,(12\text{ ft}) = 3.6 \text{ in} \rightarrow$$

$$\Delta_{C\,tot} = 0.468\text{in} + 0.\cancel{106}\text{in} + 3.6\text{ in}$$

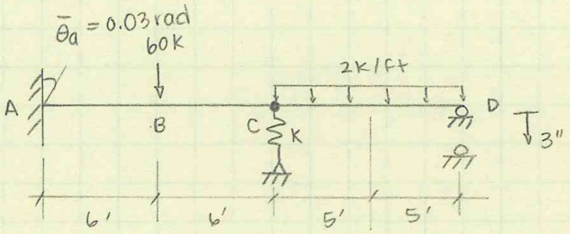
$$\boxed{\Delta_{CH} = 4.\cancel{18}\text{ in} \rightarrow}$$

4.48" →

PROCEDURE CORRECT

MIDTERM

3.



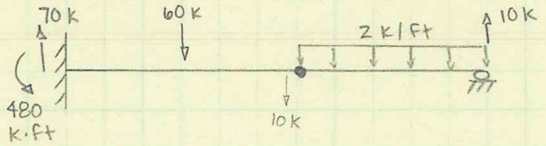
E = 29000 ksi

I = 288 in⁴

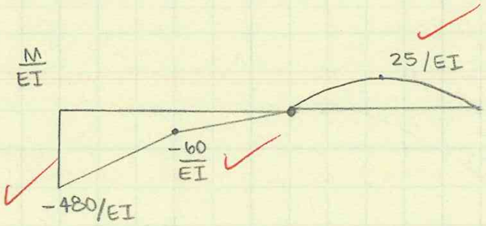
k = 14.5 k/in

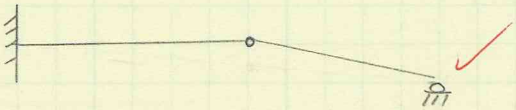
Redundant: Cy

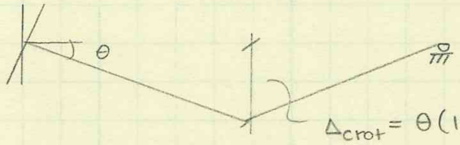
Primary:



loads



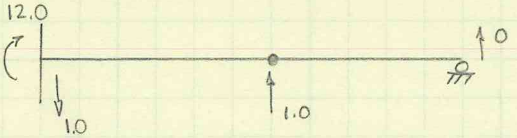

✓

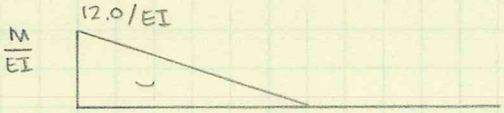


$\Delta_{crot} = \theta(12\text{ft}) = 4.32\text{in}$ ✓

33/35

Secondary:





also serves as virtual diagram to calculate Δ_{co}

compatibility:

$$\Delta_{\substack{co\\load}} = 1/2\left(\frac{-60}{EI}\right)(6.0\text{ft})(2/3)(6.0) + 1/2\left(\frac{-420}{EI}\right)(6\text{ft})(12-2)$$

$$+\left(\frac{-60}{EI}\right)(6)(9.0) = \frac{-16560}{EI} = -3.426\text{in}$$ ✓

negative sign indicates that virtual load indicates deflection the wrong way (Δ_{co} goes down) ✓

$$\Delta_{co} = 3.426\text{in} + 0 + 4.32\text{in} = 7.746\text{ in down}$$ ✓

$$f_{cc} = 1/2(12.0/EI)(2/3)(12.0)(12) = \frac{576}{EI} = 0.119\text{ in/k up}$$

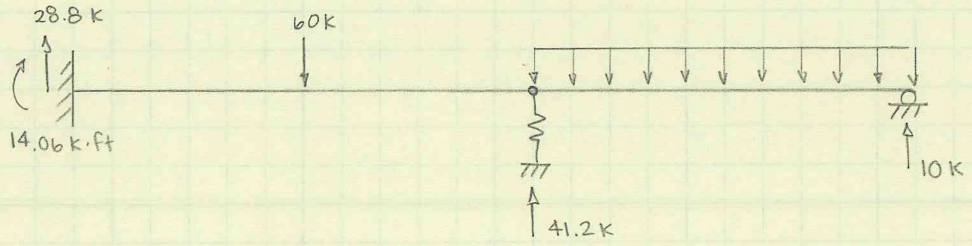
$$\Delta_{co} - f_{cc}C_y = C_y/K \quad , \quad \Delta_{co} + C_y\left(-f_{cc} - \frac{1}{K}\right) = 0$$

$$C_y = \frac{7.746\text{ in}}{0.119\text{in/k} + 1/14.5\text{ in/k}} = 41.172\text{ k up}$$ ✓

$$\underline{\underline{C_y = 41.2\text{ k up}}}$$ ✓

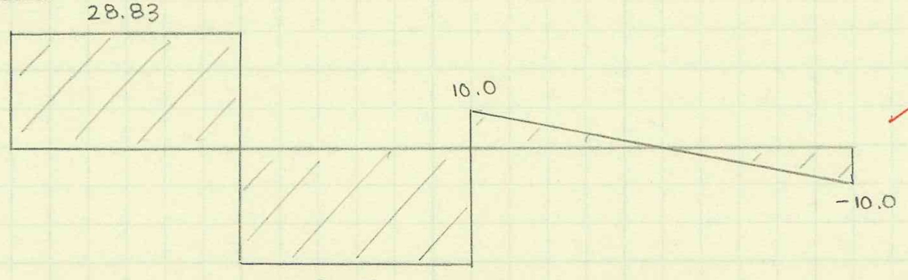
MIDTERM

3. (cont'd)

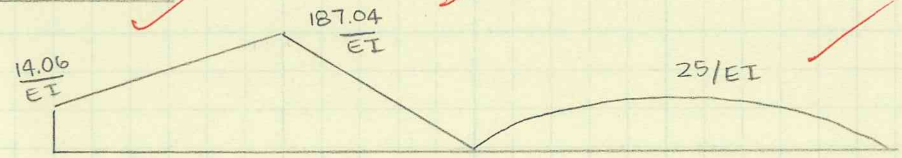


3. b)

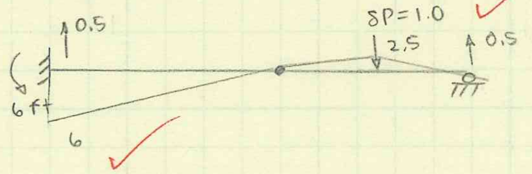
Shear (k)



Moment (k·ft)



Displacement at E:



$$\Delta = \int m \frac{M}{EI}\,dx + \int \frac{\alpha \Delta T}{d}\, m\,dx$$

$$\Delta_{E_{0}\,\text{axial}} = \frac{14.06}{EI}(6)(-4.5) + \frac{1}{2}\left(\frac{181.04-14.06}{EI}\right)(6)(-4) + \frac{1}{2}\left(\frac{187.04}{EI}\right)(6)(2) +$$

$$\frac{2}{3}\left(\frac{25}{EI}\right)(10)\cancel{(2.5)} = \frac{-916.47}{EI} \text{ up } = -0.01897\text{ in}$$

m NOT CONTINUOUS MUST DIVIDE INTO 2 SECTIONS

$$\Delta_{E_0\,\text{RBM}} = \tfrac{1}{2}(4.32\text{ in}) + \tfrac{1}{2}(3\text{ in}) = 3.66\text{ in down}$$

3.c) $\boxed{\Delta_{E\,tot} = 3.\cancel{64}\text{ in down}}$

CATHERINE HOVELL

University of Texas
Dept. of Civil, Arch. & Env. Eng.

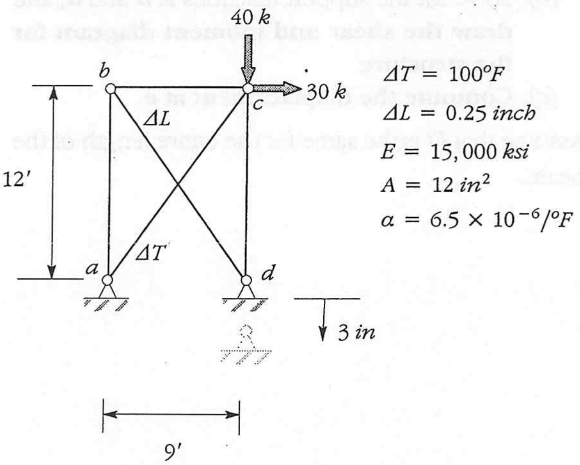
Instructor: E. B. Williamson

CE 363 Advanced Structural Analysis - Exam 1

Instructions: There are three (3) questions. Attempt to answer all of them. **Turn in the exam sheet with your exam.**

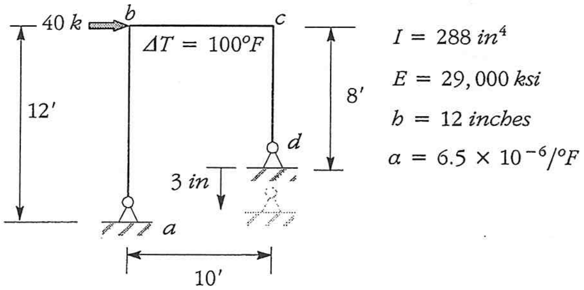
1. (35%) The truss shown in the figure is statically indeterminate to the first degree. In addition to the applied loads, member *ac* is subjected to a uniform temperature change, member *bd* has an initial misfit (it is longer than originally specified), and the support at *d* settles downward by 3 inches. All members have the same *EA*.

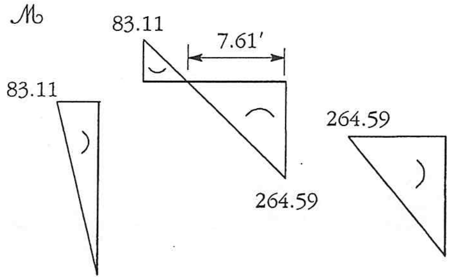
   - *(a)* **Select the horizontal support reaction at *d* as the redundant static quantity, and solve for all member forces**. Clearly indicate whether each member is in tension (T) or compression (C).
   - *(b).* **Determine the horizontal displacement at c ($\Delta_{c(H)}$)**



2. (30%) The structure shown in the sketch is statically indeterminate to the first degree. It is pin-supported at *a* and *d*. In addition to the applied 40 *k* load, the support at *d* settles downward by 3 inches and the beam is subjected a temperature change that is constant over its length. The temperature variation through the depth is linear, and the bottom is hotter than the top of the beam. The moment diagram due to these effects has been computed and provided for you in the figure.

Assume that *EI* is the same for all members, and **compute the horizontal displacement at *c*** ($\Delta_{c(H)}$).





CE363: STRUCT. ANALYSIS

92

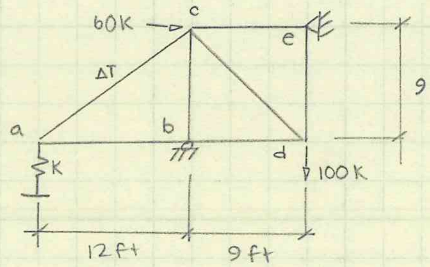
CATHERINE HOVELL
30 November 05

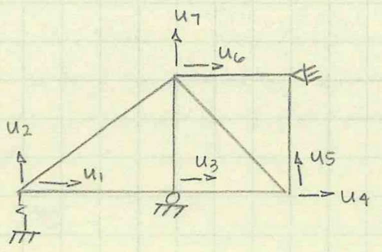
1/9

TEST #2

good job!

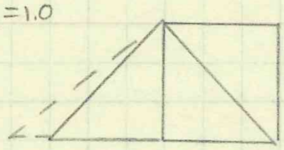
1.



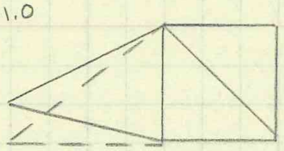


E=15000 ksi
A=24 in²
ΔT=100° F
$\alpha=6.5\times10^{-6}/°F$
k=1000 k/in

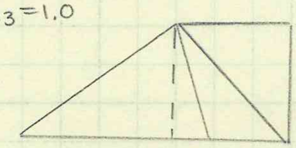
$u_1=1.0$



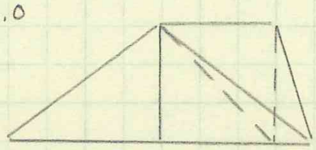
$u_2=1.0$



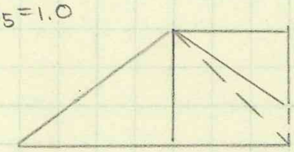
$u_3=1.0$



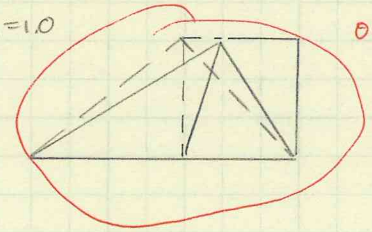
$u_4=1.0$



$u_5=1.0$

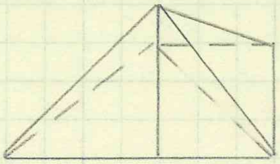


$u_6=1.0$

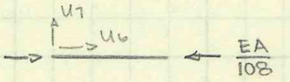


only deformed shape that is needed

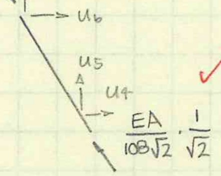
$u_7=1.0$



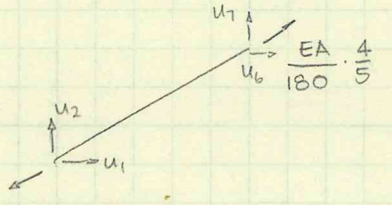
goal: K_{16} through K_{76}



✓



✓



✓

$K_{16}=\frac{-EA}{180}\cdot\frac{4}{5}\cdot\frac{4}{5}$ ✓

$K_{26}=\frac{-EA}{180}\cdot\frac{4}{5}\cdot\frac{3}{5}$ ✓

$K_{36}=0$ ✓

$K_{46}=\frac{-EA}{216}\frac{1}{\sqrt{2}}$ ✓

$K_{56}=\frac{EA}{216}\frac{1}{\sqrt{2}}$ ✓

$K_{66}=\frac{EA}{108}+\frac{EA}{216}\frac{1}{\sqrt{2}}+\frac{EA}{180}\cdot\frac{4}{5}\cdot\frac{4}{5}$ ✓ ✓ ✓

$K_{76}=\frac{-EA}{216}\cdot\frac{1}{\sqrt{2}}+\frac{EA}{180}\cdot\frac{4}{5}\cdot\frac{3}{5}$

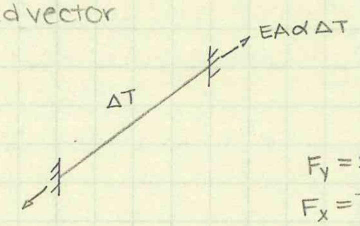
✓ ✓

EXAM II

1. (cont'd)

a) $\underset{\sim}{K}(\text{column}\,6) = \begin{bmatrix} -1280 \checkmark \\ -960 \checkmark \\ 0 \checkmark \\ -1178.51 \checkmark \\ 1178.51 \checkmark \\ 5791.84 \checkmark \\ -218.51 \checkmark \end{bmatrix}$ k/in

Load vector ✓

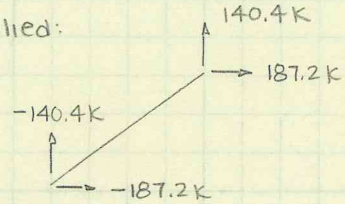


ΔT = 100°F

$F_y = \pm EA\alpha\Delta T \cdot \frac{3}{5} = 140.4K$ ✓

$F_x = \pm EA\alpha\Delta T \cdot 4/5 = 187.2K$ ✓

applied:



✓

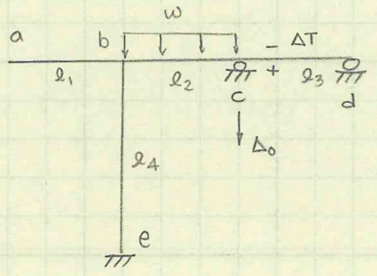
b) $\underset{\sim}{F} = \begin{bmatrix} -187.2 \\ -140.4 \checkmark \\ 0 \checkmark \\ 0 \checkmark \\ -100 \checkmark \\ 247.2 \checkmark \\ 140.4 \checkmark \end{bmatrix}$ K

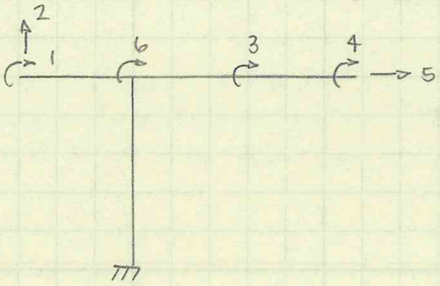
25/25

good job!

EXAM II

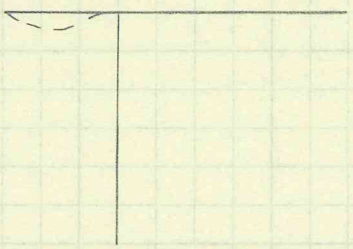
2.





b, h
EI constant
α, ΔT

$u_1 = 1.0$



$k_{11} = \frac{4EI}{l_1}$ ✓

$k_{21} = \frac{-6EI}{l_1^2}$ ✓

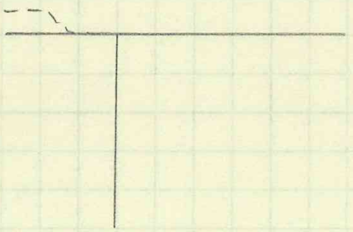
$k_{31} = 0$ ✓

$k_{41} = 0$ ✓

$k_{51} = 0$ ✓

$k_{61} = \frac{2EI}{l_1}$ ✓

$u_2 = 1.0$



$k_{12} = \frac{-6EI}{l_1^2}$

$k_{22} = \frac{12EI}{l_1^3}$ ✓

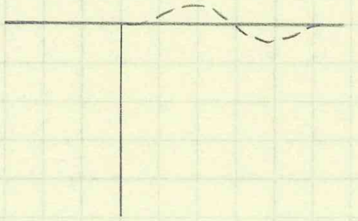
$k_{32} = 0$ ✓

$k_{42} = 0$ ✓

$k_{52} = 0$ ✓

$k_{62} = \frac{-6EI}{l_1^2}$ ✓

$u_3 = 1.0$



$k_{13} = 0$ ✓

$k_{23} = 0$ ✓

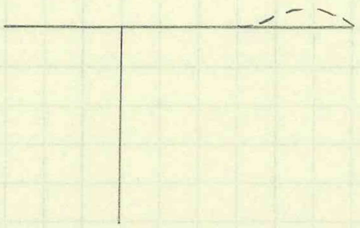
$k_{33} = \frac{4EI}{l_2} + \frac{4EI}{l_3}$ ✓

$k_{43} = \frac{2EI}{l_3}$ ✓

$k_{53} = 0$

$k_{63} = \frac{2EI}{l_2}$ ✓

$u_4 = 1.0$



$k_{14} = 0$

$k_{24} = 0$ ✓

$k_{34} = \frac{2EI}{l_3}$

$k_{44} = \frac{4EI}{l_3}$ ✓

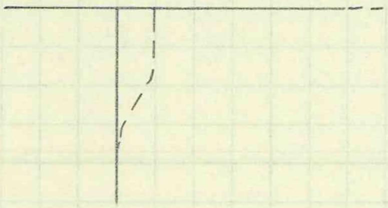
$k_{54} = 0$ ✓

$k_{64} = 0$ ✓

EXAM II

2. (cont'd)

$u_5 = 1.0$

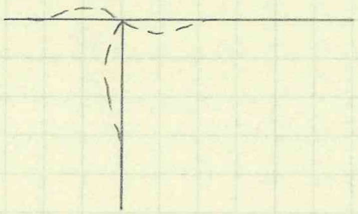


$K_{15} = 0$ $\qquad$ $K_{45} = 0$

$K_{25} = 0$ $\qquad$ $K_{55} = \frac{12EI}{\ell_4^3}$

$K_{35} = 0$ $\qquad$ $K_{65} = \frac{-6EI}{\ell_4^2}$

$u_6 = 1.0$



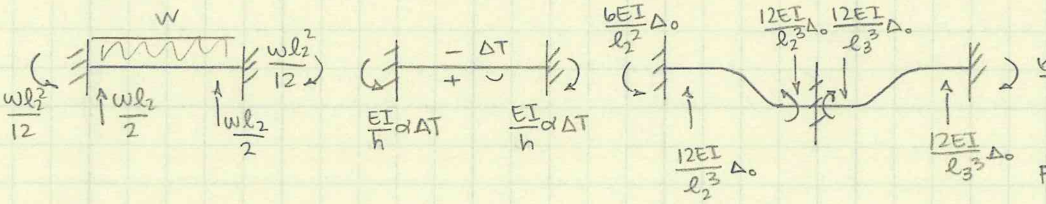
$K_{16} = \frac{2EI}{\ell_1}$ $\qquad$ $K_{46} = 0$

$K_{26} = \frac{-6EI}{\ell_1^2}$ $\qquad$ $K_{56} = \frac{-6EI}{\ell_4^2}$

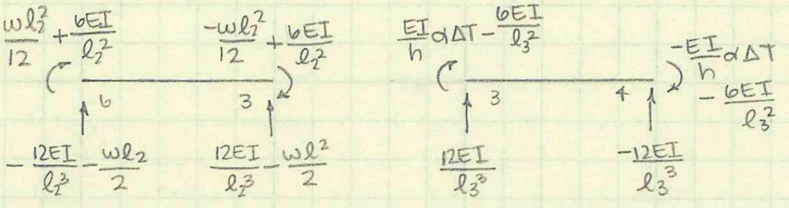
$K_{36} = \frac{2EI}{\ell_2}$ $\qquad$ $K_{66} = 4EI(1/\ell_1 + 1/\ell_2 + 1/\ell_4)$

a) $$\underset{\sim}{K} = \begin{bmatrix} 4EI/\ell_1 & -6EI/\ell_1^2 & 0 & 0 & 0 & 2EI/\ell_1 \\ -6EI/\ell_1^2 & 12EI/\ell_1^3 & 0 & 0 & 0 & -6EI/\ell_1^2 \\ 0 & 0 & 4EI(1/\ell_2+1/\ell_3) & 2EI/\ell_3 & 0 & 2EI/\ell_2 \\ 0 & 0 & 2EI/\ell_3 & 4EI/\ell_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12EI/\ell_4^3 & -6EI/\ell_4^2 \\ 2EI/\ell_1 & -6EI/\ell_1^2 & 2EI/\ell_2 & 0 & -6EI/\ell_4^2 & 4EI(1/\ell_1+1/\ell_2+1/\ell_4) \end{bmatrix}$$

Load vector



Fixed·ends



Applied

b) $$\underset{\sim}{F} = \begin{bmatrix} 0 \\ 0 \\ \frac{EI}{h}\alpha\Delta T - \frac{6EI}{\ell_3^2}\Delta_o - \frac{w\ell_2^2}{12} + \frac{6EI}{\ell_2^2}\Delta_o \\ -\frac{EI}{h}\alpha\Delta T - 6EI/\ell_3^2\Delta_o \\ 0 \\ \frac{w\ell^2}{12} + \frac{6EI}{\ell_2^2}\Delta_o \end{bmatrix}$$

great job!

25/25

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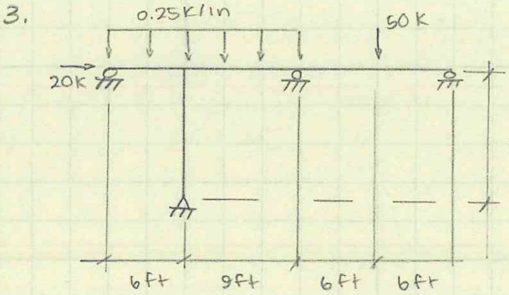
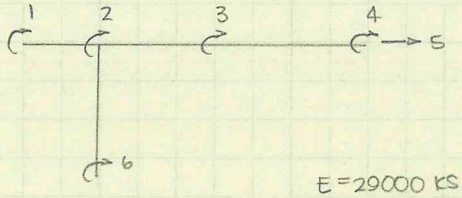
EXAM II

3.



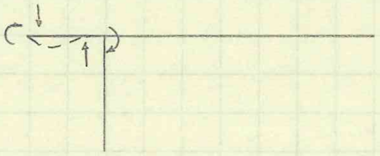


E = 29000 ksi

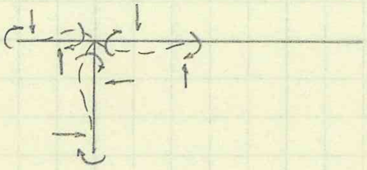
I = 500 in⁴

$$u = \begin{bmatrix} -0.0008736 \\ 0.00202 \\ 0.0006743 \\ -0.00257 \\ 1.03632 \\ 0.01195 \end{bmatrix}$$

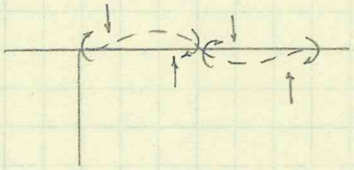
$u_1 = 1.0$

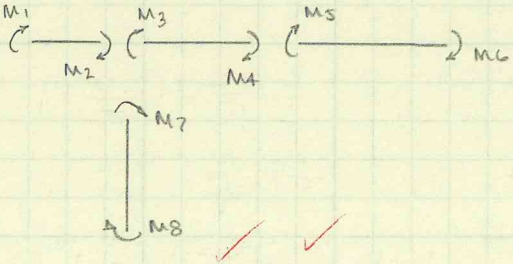


$u_2 = 1.0$

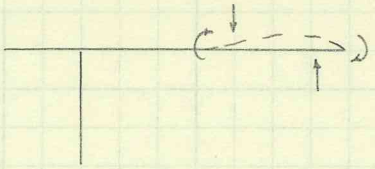


$u_3 = 1.0$





$u_4 = 1.0$

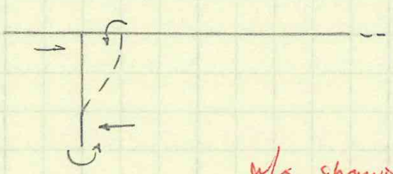


$$M_1 = \frac{4EI}{72}u_1 \checkmark + \frac{2EI}{72}u_2 \checkmark$$

$$M_2 = \frac{2EI}{72}u_1 \checkmark + \frac{4EI}{72}u_2 \checkmark \quad \text{etc.}$$

↑ for column, need $\frac{6EI}{l^2}\Delta$ term

$u_5 = 1.0$



$M_1 = 109.9$ k·in

$M_2 = 1275.4$ k·in

$M_3 = -327.97$ ~~k·in~~

$M_4 = -11$~~99.1~~ k·in

$M_5 = -245.98$ k·in

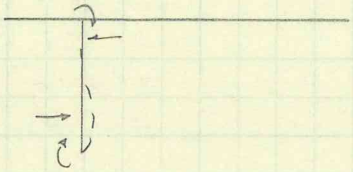
$M_6 = -899.3$ k·in

$M_7 = -2396.9$ k·in

$M_8 = 2.9$ k·in

w/o showing individual calcs, hard to tell where error is w, l, ui, etc.

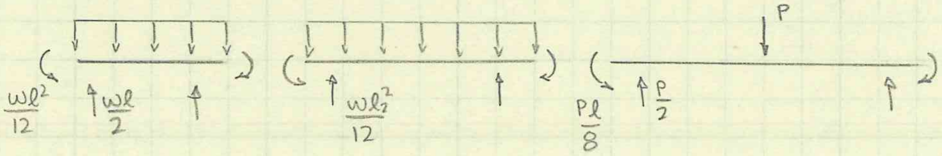
$u_6 = 1.0$



EXAM II

3. (cont'd)

consider fixed-end forces

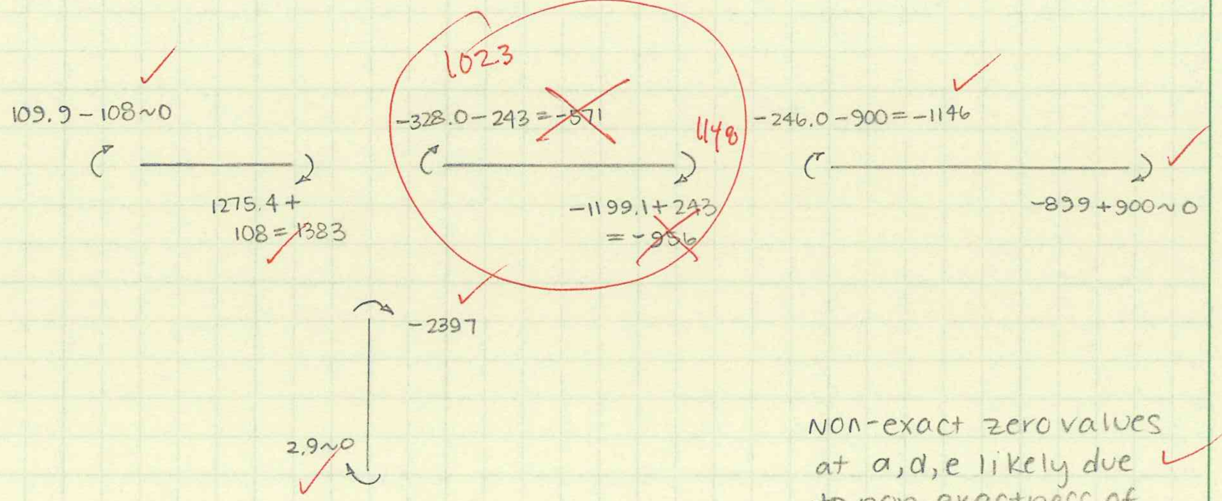


$$\frac{wl_1^2}{12} = \frac{(0.25K/in)(72in)^2}{12} = 108\ k\cdot in \qquad \frac{wl_2^2}{12} = 243 k\cdot in$$

$$\frac{wl_1}{2} = \frac{(0.25K/in)(72in)}{2} = 9K \qquad \frac{wl_2}{2} = 13.5K$$

$$\frac{P}{2} = 25K, \quad \frac{Pl}{8} = 900\ k\cdot in$$

must be subtracted from other values



Non-exact zero values at a, d, e likely due to non-exactness of u values. ✓

use same method to get shears

before fixed ends:

$F_1 = -19.2K$

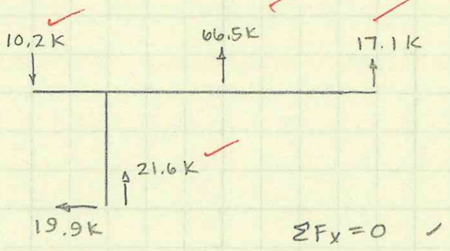
$F_3 = 28.0K$

$F_4 = -7.95K$

$F_{6y} = -0.86K$

$F_{6x} = -19.94K$

including fixed ends:



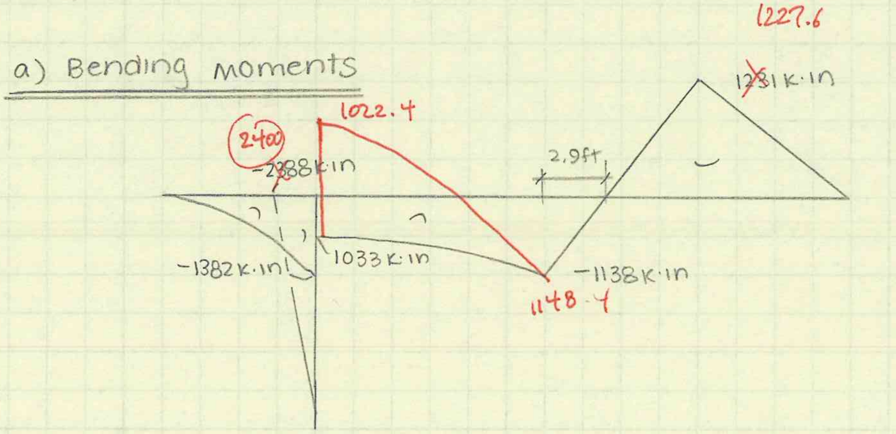
$\Sigma F_x = 0$ ✓

$\Sigma F_y = 0$ ✓

EXAM II

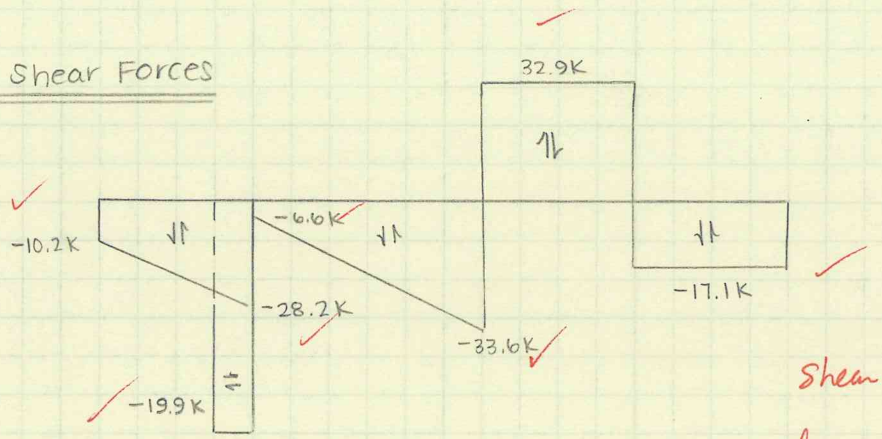
3. (cont'd)

a) Bending moments



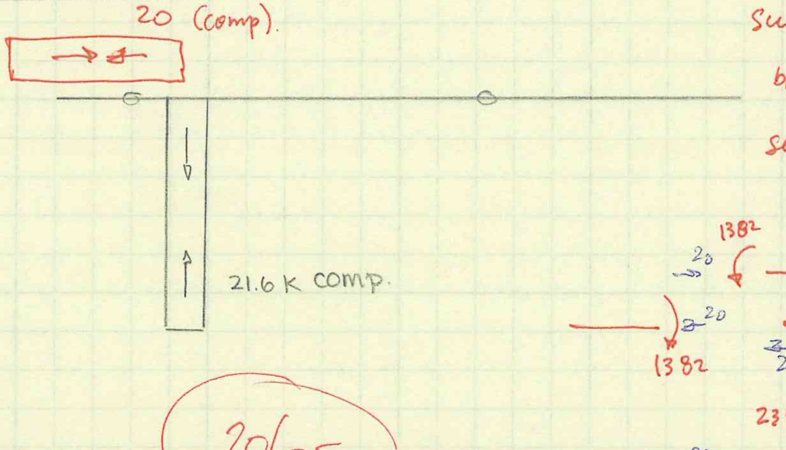
Moment diagram drawn from shear diagram, as values calculated did not have moment equilib. at joints – thus, likely an error in calcs.

b) Shear Forces

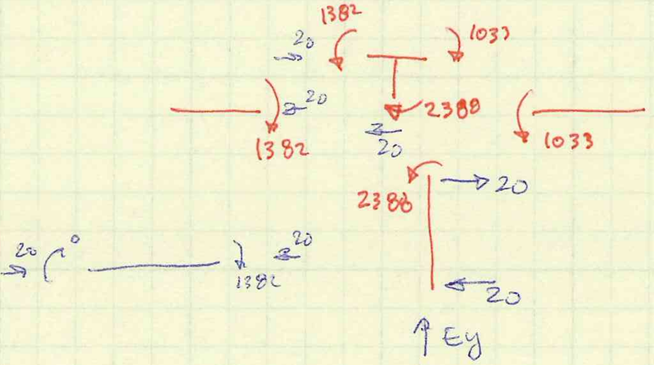


Shear diagram correct – from the work shown it is difficult to find your error – However, you should suspect an error w/ M b/c joint B does not satisfy moment equilibrium.

c) Axial Forces

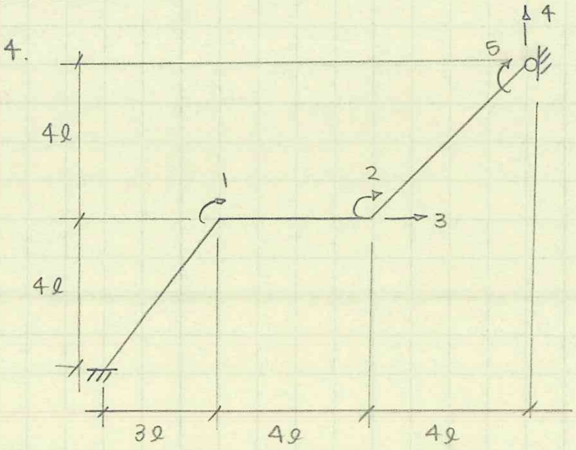


20/25



EXAM II

4.



$$K_{33} = 12EI\left[\frac{5/4}{(5\ell)^3}\cdot\frac{5}{4} + \frac{\sqrt{2}}{(4\ell\sqrt{2})^3}\sqrt{2}\right] + \frac{12EI}{(4\ell)^3}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$$

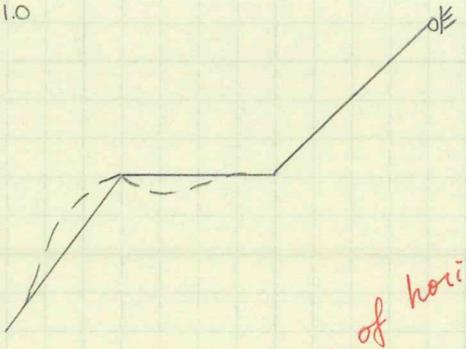
← bending of horizontal member

$$K_{34} = \frac{-12EI}{(4\ell)^3}\cdot\frac{1}{4} + \cancel{\frac{12EI}{(4\ell\sqrt{2})^3}}\sqrt{2}$$

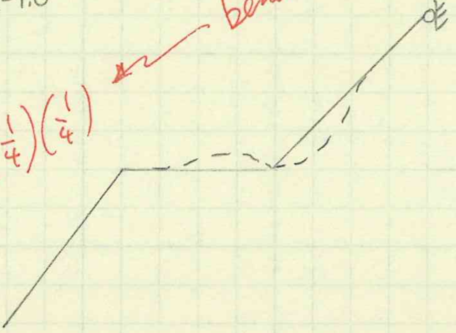
$$K_{35} = \frac{6EI}{(4\ell\sqrt{2})^2}\sqrt{2}$$

$$K_{44} = \frac{12EI}{(4\ell)^3}$$ ✓

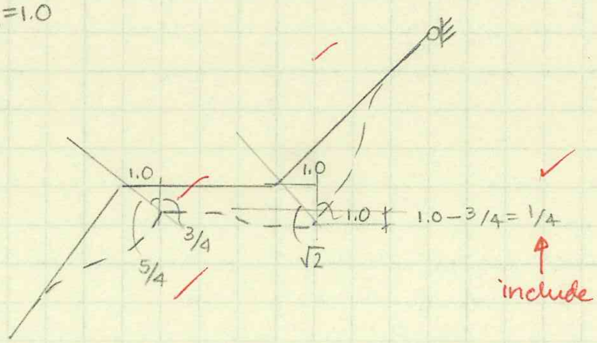
$u_1 = 1.0$



$u_2 = 1.0$



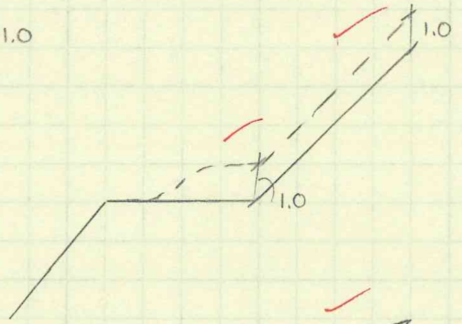
$u_3 = 1.0$



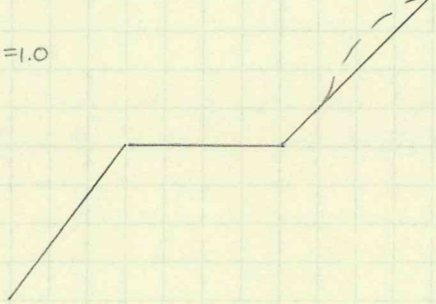
1.0 − 3/4 = 1/4 ✓

↑ include in K

$u_4 = 1.0$



$u_5 = 1.0$



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University of Texas
Department of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson

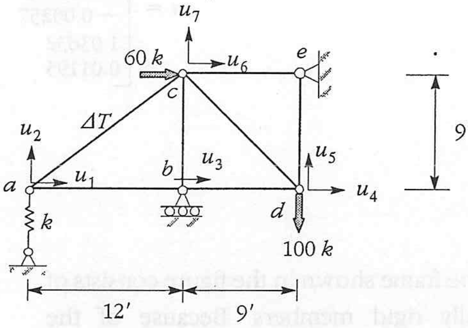
CE 363 Advanced Structural Analysis

Exam II

Instructions: There are four (4) questions. Attempt to answer all of them. **Turn in the exam sheet with your exam.**

1. (25%) The truss shown has seven kinematic degrees of freedom as indicated on the sketch. In addition to the applied loads, member *ac* is subjected to a temperature increase of 100°F. Assume that all members have the same properties (with values given in the figure) in responding to the following:

- *(a).* Develop the ***sixth*** column of ***K***.
- *(b).* Develop the load vector ***F***.

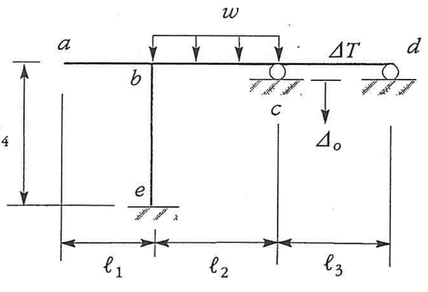


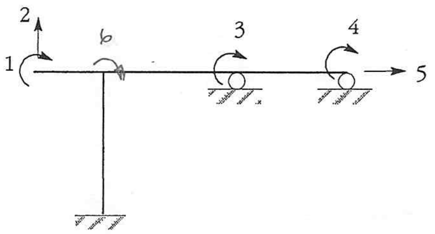
$\Delta T = 100°F$ $E = 15,000\ ksi$ $A = 24\ in^2$

$\alpha = 6.5 \times 10^{-6}/°F$ $k = 1000\ kips/in$

2. (25%) The frame shown in the sketch is fixed at *e* and supported by rollers at *c* and *d*. In addition to the uniform load acting on span *bc*, span *cd* is subjected to a thermal gradient, and the support at *c* settles downward. The thermal gradient is constant over the span, and the bottom of the beam is hotter than the top. All members have cross-sectional dimensions of width *b* and depth *h*. Assume that *EI* is the same for all members and that the coefficient of thermal expansion is equal to α.

- *(a)* Develop the stiffness matrix ***K*** using the degrees of freedom shown on the sketch (i.e., do ***not*** introduce an additional degree of freedom to account for the support displacement.)
- *(b).* Develop the load vector ***F***.



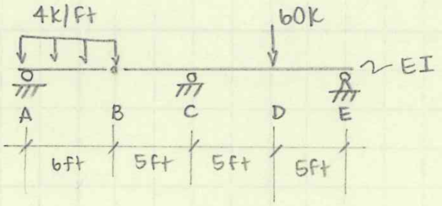


CE 363: ADV. STRUCT. ANALYSIS – HW #1

CATHERINE HOVELL
08 SEPT 2005

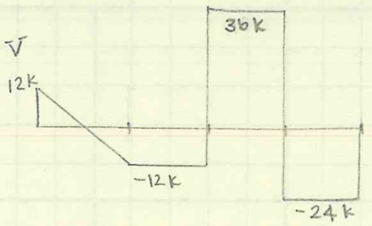
88

1.

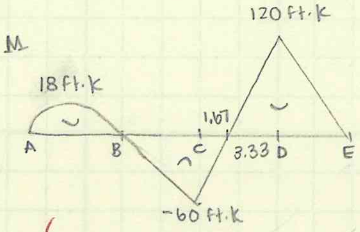


FIND: $\Delta_B, \Delta_D, \theta_A$

KNOWN: $\Delta_A = \Delta_C = \Delta_E = 0$



$t_{D/E} = (120\,ft{\cdot}k)(5ft)(1/2)(1/3)(5ft) = \dfrac{500}{EI}$

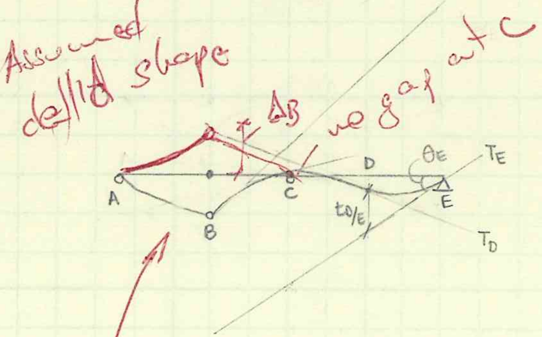


OR $\dfrac{M}{EI}$

$$t_{C/E} = \frac{(120\,ft{\cdot}k)}{2EI}\left[(5ft)(6.67ft) + (3.33ft)(3.89ft)\right] + \frac{-60\,ft{\cdot}k}{EI}\left(\frac{1}{2}\right)(1.67ft)\,{}^1/_3(1.67ft)$$

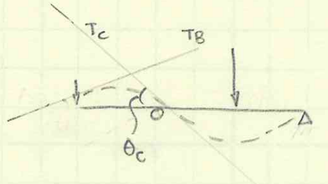
$$= \frac{2750}{EI} = \theta_E\,(10ft)$$

$$\theta_E = \frac{275}{EI}$$

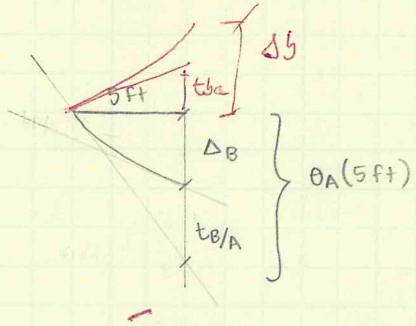


$\Delta_D = \theta_E(5ft) - t_{D/E} = \dfrac{875}{EI}$

$$\boxed{\Delta_D = \frac{875}{EI}\ \text{down}}$$



check.



$\Delta_B = \theta_C(5ft) - t_{B/C} = {}^1/_2\, t_{E/C} - t_{B/C}$

$\theta_C = \dfrac{t_{E/C}}{10\,ft}$

1750 how? / how?

$t_{E/C} \approx 1748/EI$, $t_{B/C} = 500/EI$

375

$$\boxed{\Delta_B \approx \frac{374}{EI}\ \text{up}}$$

$\theta_A = \dfrac{\Delta_B \overset{-}{+} t_{B/A}}{6ft}$

−10

$t_{B/A} = {}^4/_3(18ft{\cdot}k)\,{}^1/_2\,(6ft)^2 \times {}^1/_2$

$= 432/EI \times {}^1/_2$

$\theta_A = \dfrac{374 \overset{-}{+} 432/2}{6EI}$

26.5

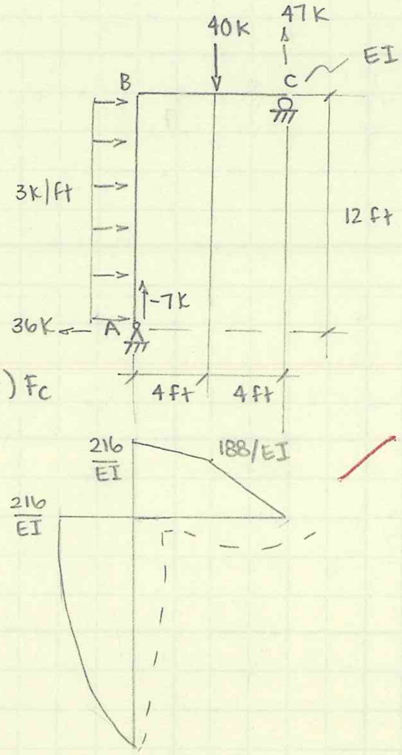
$$\boxed{\theta_A = \frac{134}{EI}}$$

C. HOVELL
08 SEPT 05

CE 363: HW #1

2. FIND: Δ_C
θ_A

KNOWN: $\Delta_A = 0$



REACTION FORCES

$$\Sigma M_A = 0$$

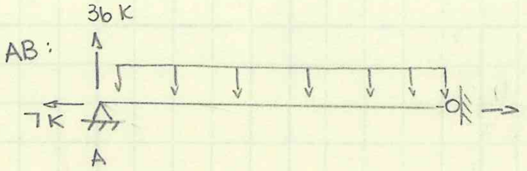
$$(3K/ft)(12ft)(6ft) + (40K)(4ft) = (8ft)F_C$$

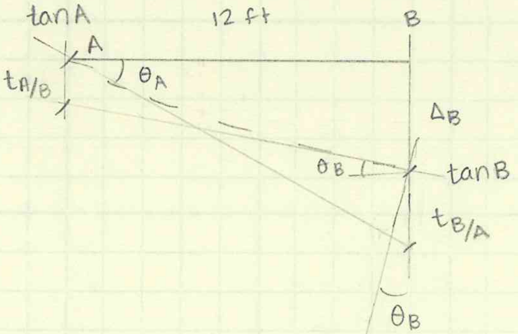
$$F_C = 47K$$

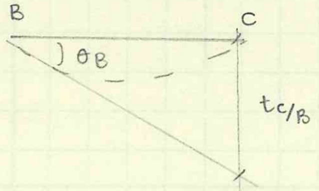
$$F_{Ay} + F_C = 40K, \quad F_{Ay} = -7K$$

$$F_{Ax} = 36 \text{ kip}$$

AB:







$$\theta_B(8ft) = t_{C/B}$$

$$t_{C/B} = \frac{188}{EI}(4)(1/2)\frac{2}{3}(4) + \frac{188}{EI}(4)(6) + \frac{28}{EI}(4)(1/2)\left[4 + 2/3(4)\right]$$

$$t_{C/B} = \frac{5888}{EI}, \quad \theta_B = \frac{736}{EI}$$

$$\theta_A = \theta_B + \theta_{A/B}$$

$$\theta_{A/B} = 2/3(12ft)\frac{216}{EI} = \frac{1728}{EI}$$

$$\theta_A = \frac{992}{EI}$$

-2

$$\theta_A(12ft) = \Delta_B + t_{B/A}, \quad t_{B/A} = 2/3\left(\frac{216}{EI}\right)\frac{3}{8}(12ft)^2 = \frac{7776}{EI}$$

$$(EI)\Delta_B = (12ft)(\frac{992}{C}) - \frac{7776}{EI} = \frac{4128}{ES}$$

Δ_B to right = Δ_c to right (no beam stretching)

$$\boxed{\theta_A = \frac{992}{EI}, \quad \Delta_C = \frac{4128}{EI} \text{ right}}$$

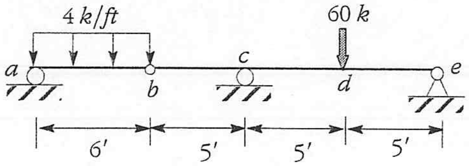
University of Texas
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson
CE 363 Advanced Structural Analysis

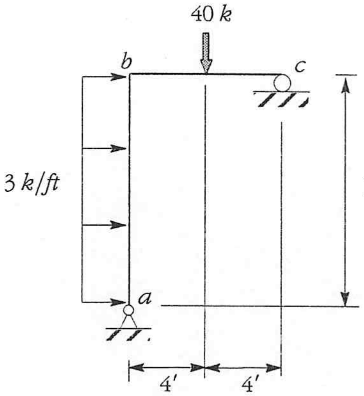
Homework 1

Due: Septmeber 8

1. The structure shown in the sketch is supported by a pin at *e* and rollers at *a* and *c*. The structure is hinged at *b*. Assuming that *EI* is the same for the entire length of the beam, use the moment-area equations to compute the deflection at *b* (Δ_b), the deflection at *d* (Δ_d), and the rotation at *a* (θ_a).



2. The frame shown to the right is supported by a pin at *a* and roller at *c*. The beam and column are rigidly connected at *b*. Assume *EI* is the same for both members. Using the moment-area theorems, compute the deflection at *c* (Δ_c) and the rotation at *a* (θ_a).



CE 363: STRUCT. ANALYSIS

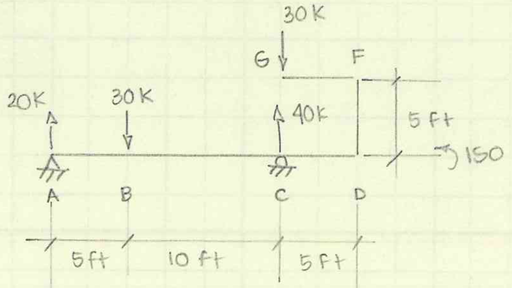
CATHERINE HOVELL
15 Sept 05

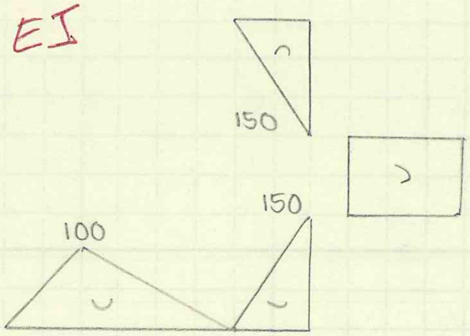
100

HOMEWORK #2

use $\frac{M}{EI}$

1.





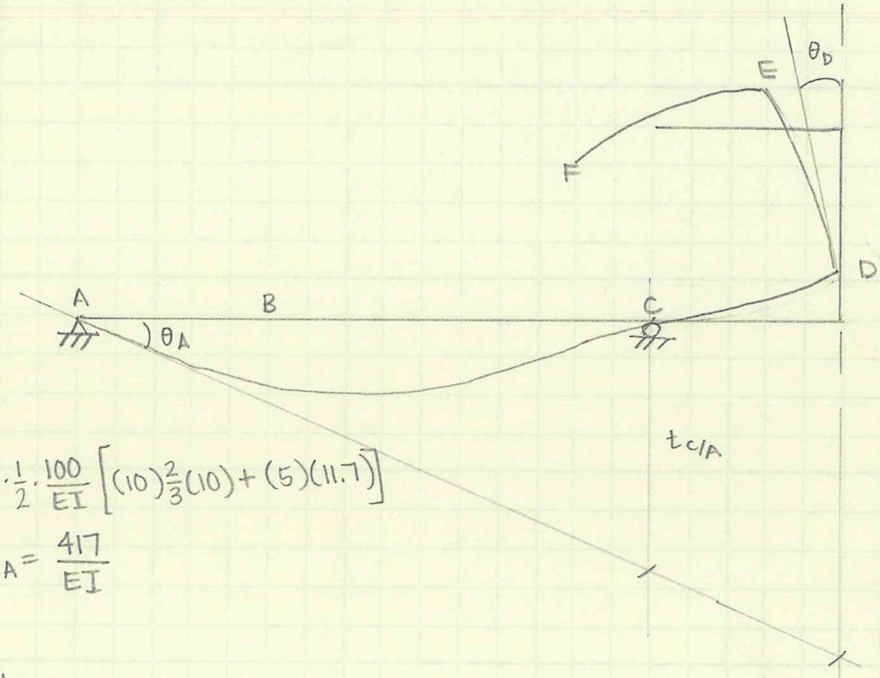
E = 29000 ksi

I = 1000 in^4

FIND: Δ_F (H & V)

Δ_D

θ_F



$$\theta_A = \frac{1}{15} t_{C/A} = \frac{1}{15} \cdot \frac{1}{2} \cdot \frac{100}{EI} \left[(10)\tfrac{2}{3}(10) + (5)(11.7) \right]$$

$$\theta_A = \frac{417}{EI}$$

$$\Delta_D + 20\theta_A = t_{D/A}$$

$$\Delta_D = \frac{1}{2} \cdot \frac{100}{EI} \left[(10)(11.7) + (5)(16.7) \right] + \frac{1}{2} \cdot \frac{150}{EI} (5) \tfrac{1}{3}(5) - 20\,\frac{417}{EI}$$

$$\Delta_D \simeq \frac{2285}{EI} \text{ up } = \Delta_{EV}$$

how? from you numbers I got $\frac{2810}{EI}$

exact $\Delta_D = \frac{2291.6666}{EI}$

$$\Delta_{EH} = 5\theta_D + t_{E/D}$$

$$\theta_D = \theta_{D/A} - \theta_A \quad , \quad \theta_{D/A} = \frac{1}{2}\left[(100)(15) + (150)(5) \right] = 1125/EI$$

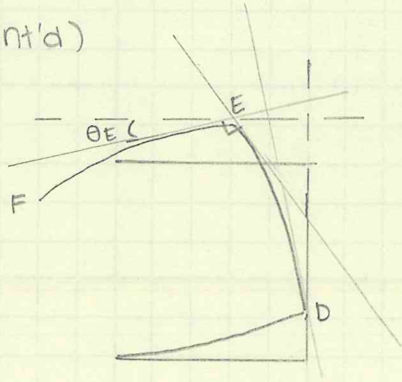
$$\theta_D \simeq \frac{708}{EI}$$

$$t_{E/D} = \frac{150}{EI}(5)\tfrac{1}{2}(5) = \frac{1875}{EI}$$

$$\Delta_{EH} = 5\left(\frac{708}{EI}\right) + \frac{1875}{EI} \simeq \frac{5415}{EI} \text{ left } = \Delta_{FH}$$

HOMEWORK #2

1. (cont'd)



$$\theta_E = \theta_D + \theta_{E/D} = \frac{708}{EI} + \frac{150}{EI}(5) = \frac{1458}{EI}$$

$$\Delta_{FV_E} = 5\theta_E + t_{F/E}$$

$$t_{F/E} = \frac{150}{EI}\cdot\frac{1}{2}(5)\frac{2}{3}(5) = \frac{1250}{EI}$$

$$\Delta_{FV_E} = 5\,\frac{1458}{EI} + \frac{1250}{EI} = \frac{8540}{EI} \text{ down}$$

↳ WRT final position of E

$$\Delta_{FV} = \Delta_{FV_E} - \Delta_D = \frac{6255}{EI} \text{ down}$$

Modify numbers to include EI values

$$\frac{\Delta(12\,in/ft)^3}{(29000\,ksi)(1000\,in^4)} = in, \quad \frac{\theta(12\,in/ft)^2}{(29000\,ksi)(1000\,in^4)} = rad$$

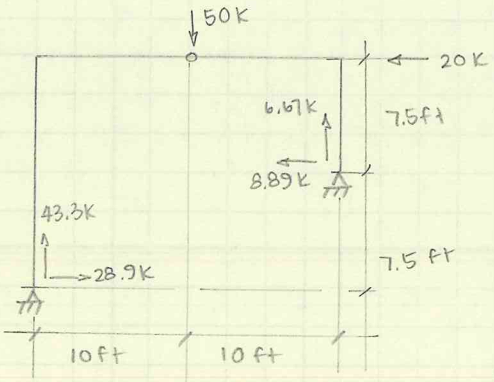
$$\theta_F = \theta_E + \theta_{F/E} = \frac{1458}{EI} + \frac{150}{EI}(5)\frac{1}{2} = \frac{1833}{EI} \circlearrowright$$

$\Delta_D = 0.14$ in up
$\Delta_{FH} = 0.32$ in left
$\Delta_{FV} = 0.37$ in down
$\theta_F = 0.009$ rad $\circlearrowright$

CE 363: STRUCT. ANALYSIS | 15 Sept 05 | 3/4

HOMEWORK #2

2.



E = 29000 ksi

I = 1000 in⁴

FIND: Δc (H & V)

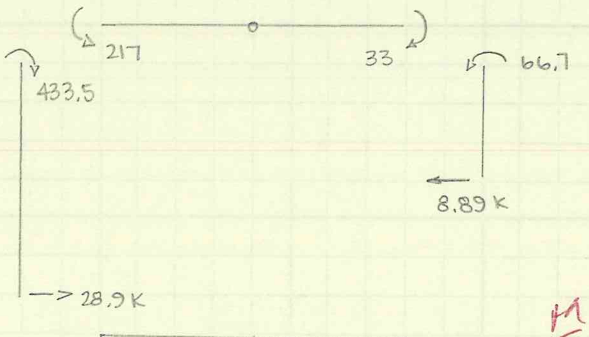
θc (L & R)

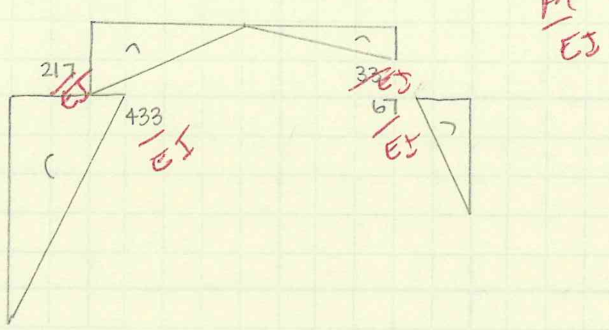
$A_x = 2/3 A_y$

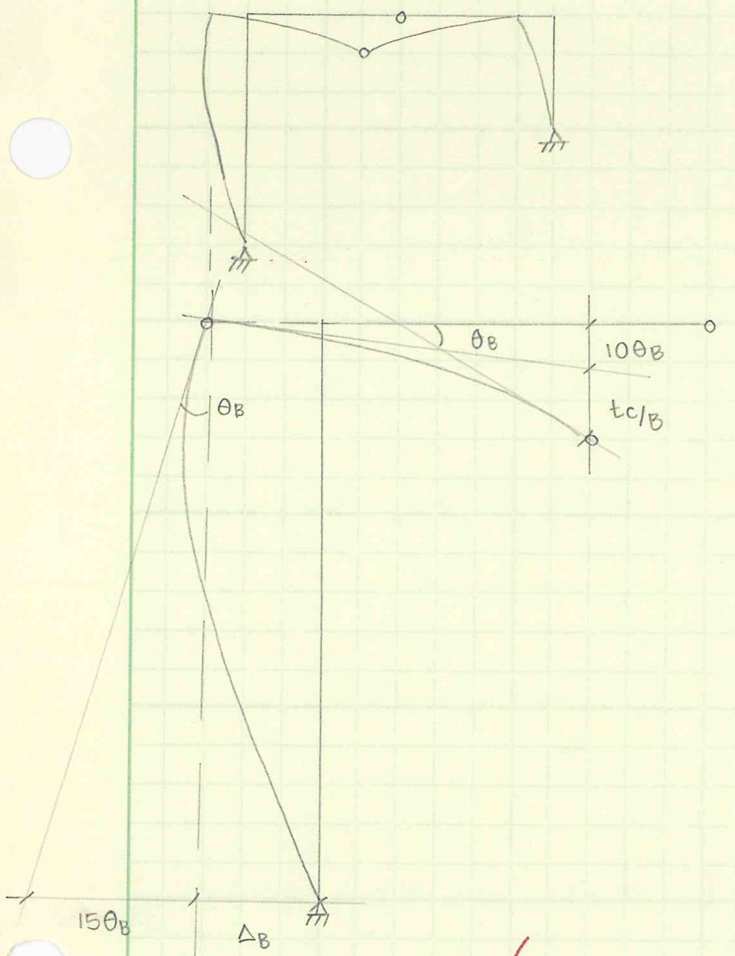
$E_x = 4/3 E_y$

$E_x - A_x = 20K$

$A_y + E_y = 50K$







$\Delta_{cv} = 10\theta_B + t_{c/B}$

$\Delta_{cv} = 2/3 (t_{A/B} - \Delta_B) + t_{c/B}$

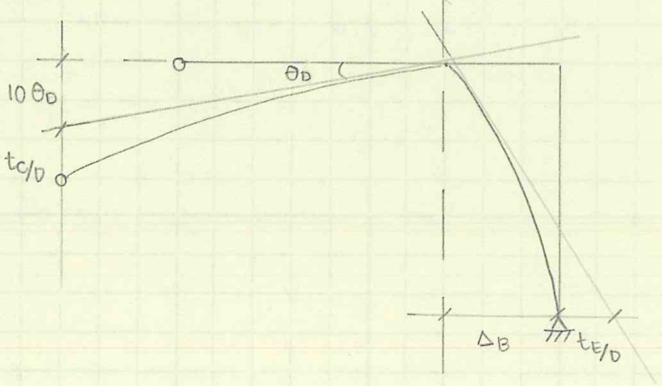
$\theta_{cL}(10) = \Delta_{cV} + t_{B/C}$

$\Delta_B + 15\theta_B = t_{A/B}$

$$\theta_B = \frac{1}{15}(t_{A/B} - \Delta_B)$$

HOMEWORK #2

2. (cont'd)



$\Delta_{CV} = 10\theta_D + t_{C/D}$

$\Delta_{CV} = \frac{4}{3}(t_{E/D} + \Delta_B) + t_{C/D}$

$7.5\theta_D = t_{E/D} + \Delta_B$

$\theta_D = \frac{1}{7.5}(t_{E/D} + \Delta_B)$

$4/3(t_{E/D} + \Delta_B) + t_{C/D} = \Delta_{CV} = 2/3(t_{A/B} - \Delta_B) + t_{C/B}$

$t_{A/B} \overset{\sim}{=} -32475/EI$
$t_{C/B} \overset{\sim}{=} -7233/EI$
$t_{C/D} \overset{\sim}{=} -1100/EI$
$t_{E/D} \overset{\sim}{=} 1256/EI$

$2\Delta_B = 2/3\, t_{A/B} + t_{C/B} - t_{C/D} - 4/3\, t_{E/D}$

$\Delta_B \overset{\sim}{=} \frac{13,054}{EI}$ or 0.78 in left $= \Delta_{CH}$

$\Delta_{CV} \overset{\sim}{=} \frac{20180}{EI}$ or 0.84 in down

?

$10\theta_{CL} = \Delta_{CV} + t_{B/C}$

$t_{B/C} = 3617/EI$

$\theta_{CL} \overset{\sim}{=} \frac{2380}{EI}$, or 0.012 rad ↻

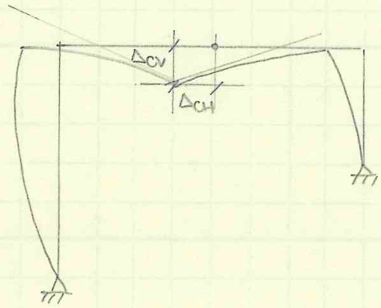
?

$10\theta_{CR} = \Delta_{CV} + t_{D/C}$

$t_{D/C} = 550/EI$

$\theta_{CR} \overset{\sim}{=} \frac{2073}{EI}$, or 0.010 rad ↺

$\Delta_{CH} = 0.78$ in left
$\Delta_{CV} = 1.20$ in down
$\theta_{CL} = 0.012$ rad ↻
$\theta_{CR} = 0.010$ rad ↺

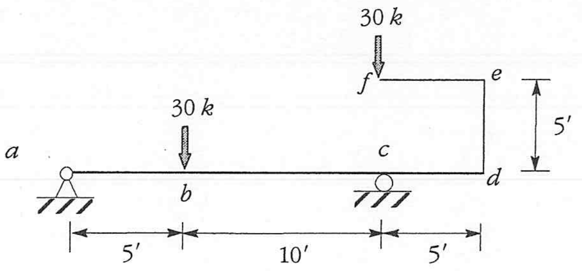


University of Texas
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson
CE 363 Advanced Structural Analysis

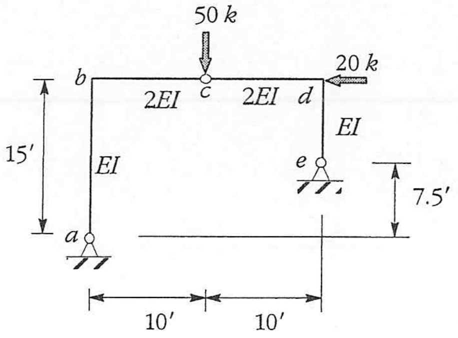
Homework 2 **Due: September 15**

1. The structure shown in the sketch is supported by a pin at *a* and a roller at *c*. Assuming that *EI* is the same for the entire structure, use the moment-area equations to compute the deflection at *f* (Δ_f) (both horizontal and vertical components), the deflection at *d* (Δ_d), and the rotation at *f* (θ_f).



$E = 29,000\ ksi \qquad I = 1,000\ in^4$

2. The frame shown to the right is pin-supported at both *a* and *e* and hinged at *c*. *EI* for the beam is twice that of the columns. For the loading shown, use the moment-area equations to compute both the vertical and horizontal components of the deflection at *c* ($\Delta_{c(V)}$ and $\Delta_{c(H)}$), the rotation to the left of *c* ($\theta_{c(L)}$), and the rotation the right of *c* ($\theta_{c(R)}$).

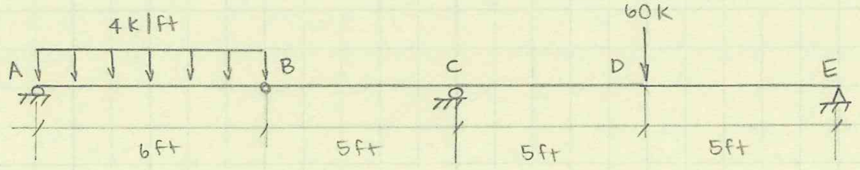


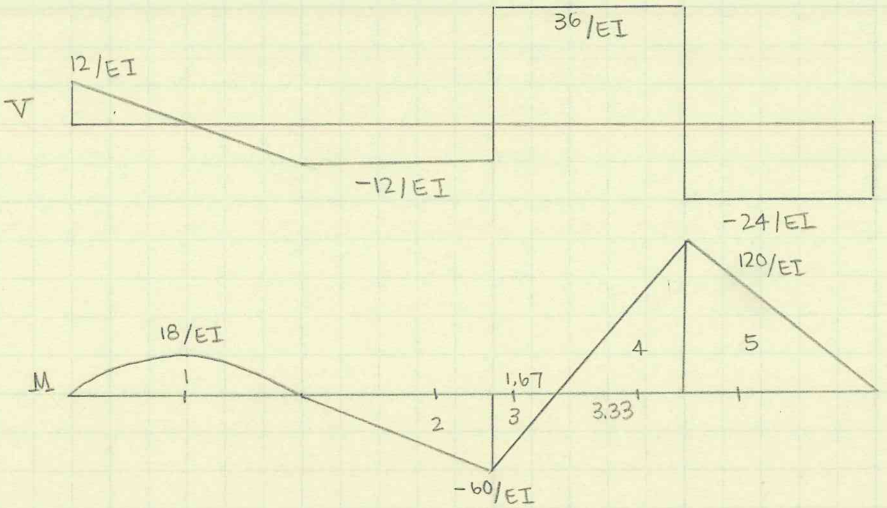
$E = 29,000\ ksi \qquad I = 1,000\ in^4$

100

HOMEWORK #3

1.





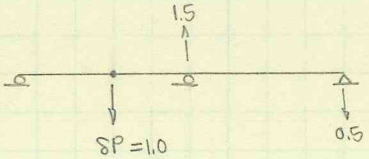
$A_1 = \frac{2}{3}\left(\frac{18}{EI}\right)(6) = \frac{72}{EI}$

$A_2 = \frac{1}{2}\left(\frac{-60}{EI}\right)(5) = \frac{-150}{EI}$

$A_3 = \frac{1}{2}\left(\frac{-60}{EI}\right)(1.67) = \frac{-50}{EI}$

$A_4 = \frac{1}{2}\left(\frac{120}{EI}\right)(3.33) = \frac{200}{EI}$

$A_5 = \frac{1}{2}\left(\frac{120}{EI}\right)(5) = \frac{300}{EI}$

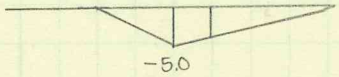


$\delta M(\bar{x}_2) = \frac{2}{3}(-5) = -3.33$

$\delta M(\bar{x}_3) = {}^{17}/_{18}(-5.0) = -4.72$

$\delta M(\bar{x}_4) = {}^{11}/_{18}(-5) = -3.06$

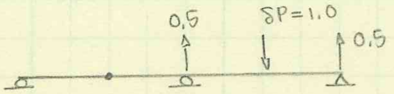
$\delta M(\bar{x}_5) = \frac{2}{3}(-2.5) = -1.67$

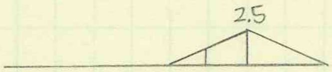


$\Delta_B(EI) = (-150)(-3.33) + (-50)(-4.72) + (200)(-3.06) + (300)(-1.67)$

$\Delta_B = \frac{-375}{EI}$, so δP is wrong

$\boxed{\Delta_B = \frac{375}{EI} \uparrow}$ ✓





$\delta M(\bar{x}_3) = {}^{1}/_{9}(2.5) = 0.278$

$\delta M(\bar{x}_4) = {}^{7}/_{9}(2.5) = 1.94$

$\delta M(\bar{x}_5) = {}^{2}/_{3}(2.5) = 1.67$

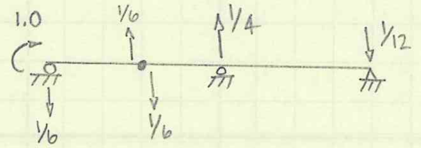
$\Delta_D(EI) = (-50)(0.278) + (200)(1.94) + (300)(1.67) = 875$

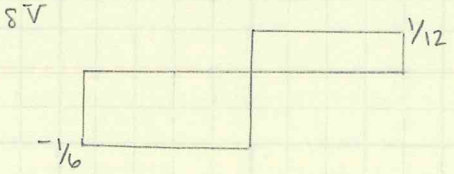
$\boxed{\Delta_D = \frac{875}{EI} \downarrow}$ ✓

HOMEWORK #3

1. (cont'd)

θ_A





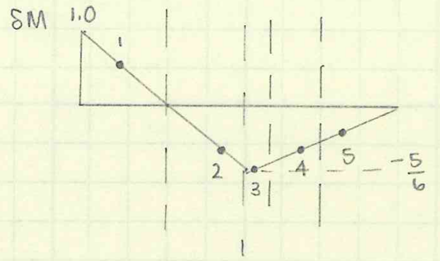
$A_1 = \frac{72}{EI}$

$A_2 = \frac{-150}{EI}$

$A_3 = \frac{-50}{EI}$

$A_4 = \frac{200}{EI}$

$A_5 = \frac{300}{EI}$



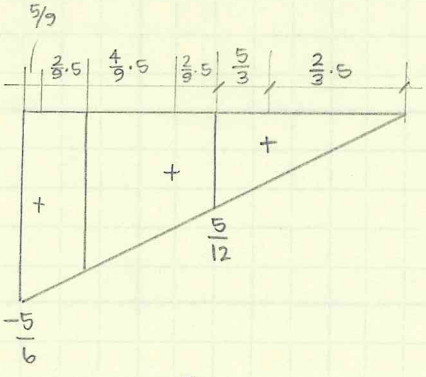
$\bar{x}_1 = \frac{1}{2}(6) = 3$, $\delta M(\bar{x}) = \frac{1}{2}(1.0) = 0.5$

$\bar{x}_2 = \frac{2}{3}(5) + 6 = 9.33$, $\delta M = \frac{2}{3}(-5/6) = -5/9$

$\bar{x}_3 = 5/9$ from c, $\delta M = \frac{5}{60}\cdot\frac{5}{9} - \frac{5}{6} = \frac{-85}{108}$

$\bar{x}_4 = 3\,8/9$ from c, $\delta M = \frac{5}{60}(3.89) - \frac{5}{6} = \frac{-55}{108}$

$\bar{x}_5 = 6\,2/3$ from c, $\delta M = \frac{5}{60}(6.67) - \frac{5}{6} = \frac{-5}{18}$



$\frac{5}{60}(x) - \frac{5}{6} = \delta M$

$\theta_A(EI) = (72)(0.5) + (-150)(-5/9) + (-50)(-85/108) + (200)(-55/108) + (300)(-5/18)$

$\theta_A = -\frac{26.5}{EI}$ (−) means ↺

$\boxed{\theta_A = \frac{26.5}{EI} ↺}$ ✓

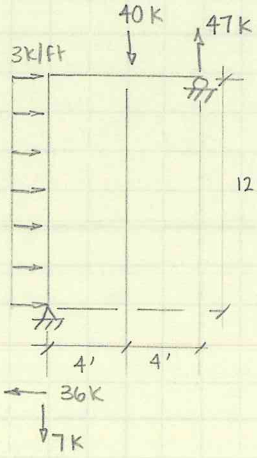
CE 363: STRUCT. ANALYSIS

C. HOVELL
27 Sept 05

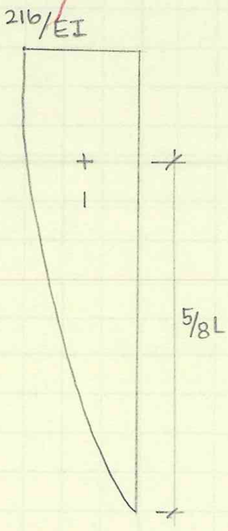
3/7

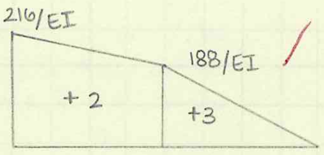
HOMEWORK #3

2.



FIND: Δ_c, θ_A



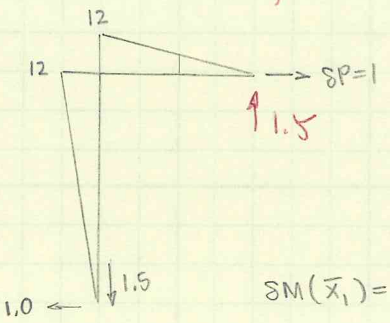


$$A_1 = 2/3(12)\left(\frac{216}{EI}\right) = \frac{1728}{EI}$$

$$A_2 = \frac{1}{2}\left(\frac{216}{EI} + \frac{188}{EI}\right)(4) = \frac{808}{EI}$$

$$A_3 = 1/2\left(\frac{188}{EI}\right)(4) = \frac{376}{EI}$$

Δ_c:



$$\bar{x}_1 = 5/8(12) = 7.5$$

$$\bar{x}_2 = \left[1/2(4)^2(188) + \frac{1}{3}(4)^2\frac{1}{2}(28)\right]/A_2$$

$$= 1.95$$

$$\bar{x}_3 = \overset{2}{\cancel{1}}/3(4) = 1.33 \times 2$$

$$\delta M(\bar{x}_1) = 5/8(12) = 7.5$$

$$\delta M(\bar{x}_2) = 12 - \frac{1.95}{4}(6) = 9.07$$

$$\delta M(\bar{x}_3) = 2/3(6) = 4$$

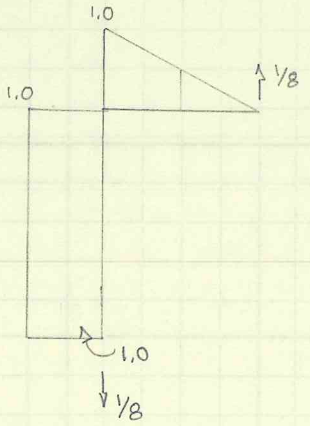
$$\Delta_c = \frac{1728}{EI}(7.5) + \frac{808}{EI}(9.07) + \frac{376}{EI}(4) = \sim\frac{21793}{EI} \rightarrow$$

$$\boxed{\Delta_c = \sim\frac{21793}{EI} \rightarrow}$$

HOMEWORK #3

2. (cont'd)

θ_A:



$\delta M(\bar{x}_1) = 1.0$

$\delta M(\bar{x}_3) = \frac{2}{3}(0.5) = 0.33$

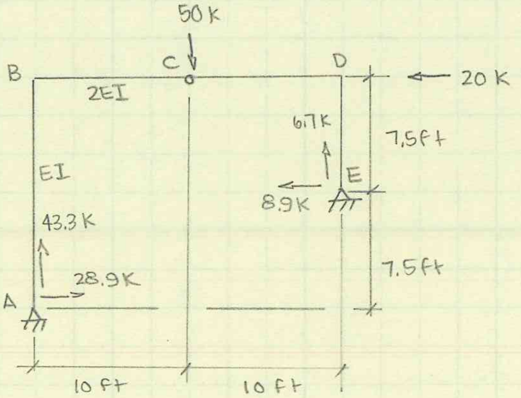
$\delta M(\bar{x}_2) = 1.0 - \frac{1.95}{8}(1.0) = 0.756$

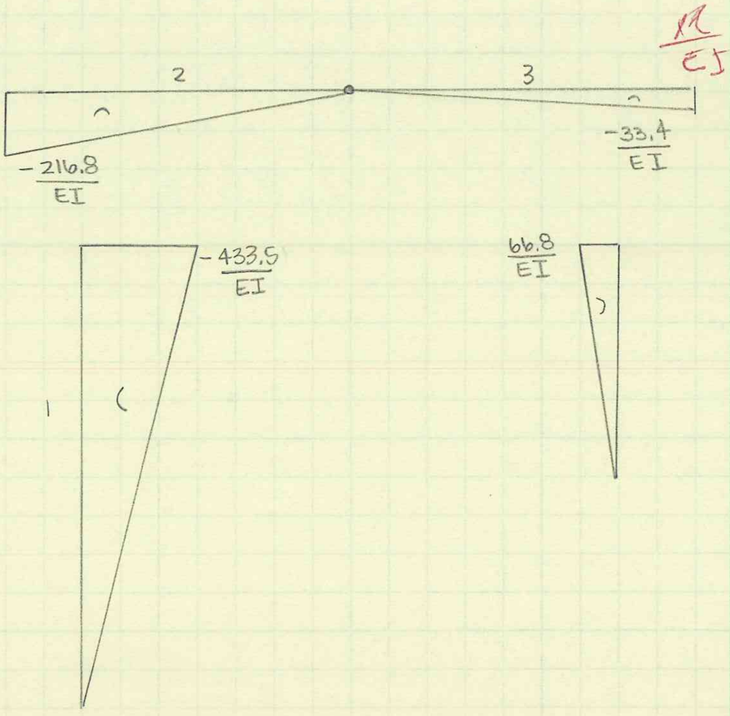
$$\theta_A = \frac{1728}{EI}(1.0) + \frac{808}{EI}(0.756) + \frac{376}{EI}(0.33) = \frac{2464}{EI} \circlearrowleft$$

$$\boxed{\theta_A = \frac{2464}{EI} \circlearrowleft}$$

HOMEWORK #3

3.





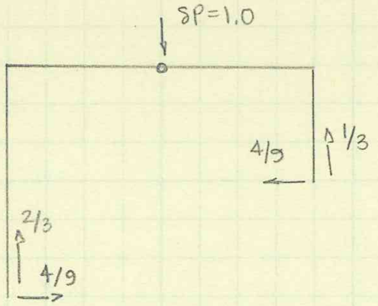
$A_1 = \frac{1}{2}\left(\frac{-433.5}{EI}\right)(15) = \frac{-3251}{EI}$

$A_2 = \frac{1}{2}\left(\frac{-216.8}{EI}\right)(10) = \frac{-1084}{EI}$

$A_3 = \frac{1}{2}\left(\frac{-33.4}{EI}\right)(10) = \frac{-167}{EI}$

$A_4 = \frac{1}{2}\left(\frac{66.8}{EI}\right)(7.5) = \frac{250.5}{EI}$

Δ_{cv}:

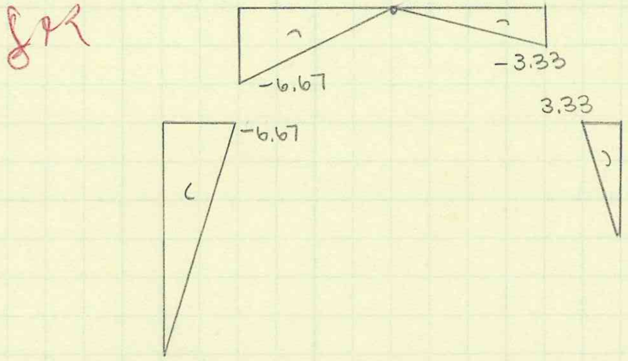


$A_x - E_x = 0$

$A_y + E_y = 1.0$

$15A_x - 10A_y = 0$

$7.5E_x - 10E_y = 0$



$\delta M(\bar{x}_1) = \frac{2}{3}(-6.67) = -4.44$

$\delta M(\bar{x}_2) = \frac{2}{3}(-6.67) = -4.44$

$\delta M(\bar{x}_3) = \frac{2}{3}(-3.33) = -2.22$

$\delta M(\bar{x}_4) = \frac{2}{3}(3.33) = 2.22$

$$\Delta_{cv}(EI) \cong (-3251)(-4.44) + (-1084)(-4.44) + (-167)(-2.22) + (250.5)(2.22)$$

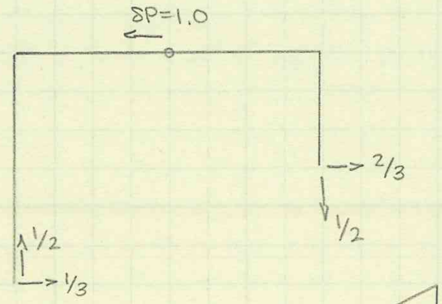
$$\cong \frac{20174}{EI}$$

$$\Delta \cong \frac{\Delta_{cv}(12)^3}{EI}$$

$$\boxed{\Delta_{cv} = 1.20 \text{ in} \downarrow}$$

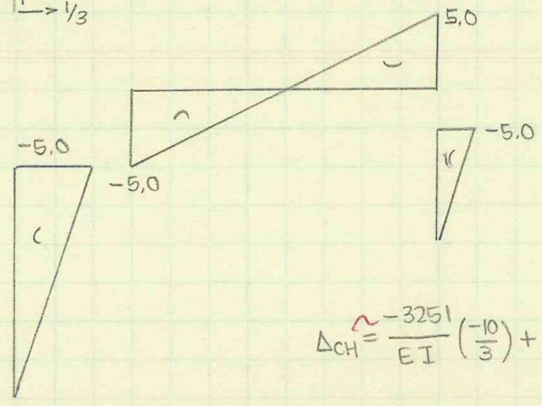
HOMEWORK #3

3. (cont'd)



$A_1 = \frac{-3251}{EI}$ $A_2 = \frac{-1084}{EI}$

$A_3 = \frac{-167}{EI}$ $A_4 = \frac{250.5}{EI}$

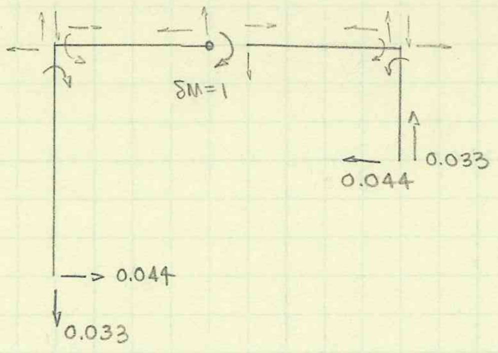


$$\Delta_{CH} \cong \frac{-3251}{EI}\left(\frac{-10}{3}\right) + \frac{-1084}{EI}\left(\frac{-10}{3}\right) + \frac{-167}{EI}\left(\frac{10}{3}\right) + \frac{250.5}{EI}\left(\frac{-10}{3}\right)$$

$$\Delta_{CH} \cong \frac{13058}{EI} \leftarrow$$

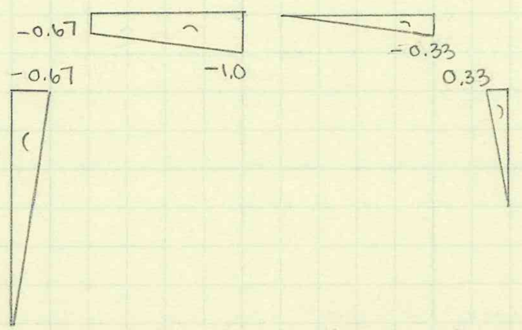
$$\Delta_{CH} = \frac{(13058)(12\,in/ft)^3}{(29000\,ksi)(1000\,in^4)} = 0.78\,in$$

$$\boxed{\Delta_{CH} = 0.78\,in \leftarrow}$$



$A_x(15) + A_y(10) = 1$

$A_x(7.5) + A_y(20) - 1 = 0$



$\delta M(\bar{x}_1) = \frac{2}{3}(-2/3) = -4/9$

$\delta M(\bar{x}_2) = -2/3 - \frac{1}{3}(1/3) = -7/9$

$\delta M(\bar{x}_3) = \frac{2}{3}(-1/3) = -2/9$

$\delta M(\bar{x}_4) = 2/3(1/3) = 2/9$

$$\theta_{CL}(EI) = (-3251)(-4/9) + (-1084)(-7/9) + (-167)(-2/9) + (250.5)(2/9)$$

$$\theta_{CL} = \frac{2381}{EI}$$

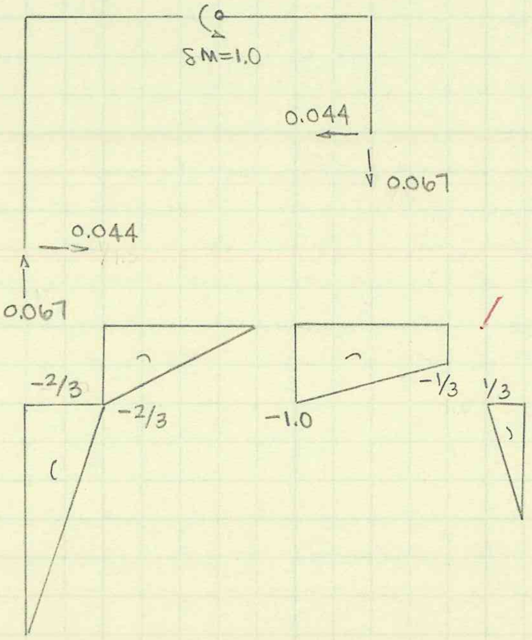
$$\boxed{\theta_{CL} = 0.012\,rad \circlearrowright}$$

HOMEWORK #3

3. (cont'd)

$E_x(7.5) + E_y(10) = 1$

$E_x(7.5) - E_y(20) + 1 = 0$



$\delta M(\bar{x}_1) = 2/3(-2/3) = -4/9$

$\delta M(\bar{x}_2) = 2/3(-2/3) = -4/9$

$\delta M(\bar{x}_3) = -1/3 - 1/3(2/3) = -5/9$

$\delta M(\bar{x}_3) = 2/3(1/3) = 2/9$

$(EI)\theta_{CR} = (-3251)(-4/9) + (-1084)(-4/9) + (-167)(-5/9) + (250.5)(2/9)$

$\theta_{CR} = \frac{2075}{EI}$

$\boxed{\theta_{CR} = 0.010 \text{ rad } \curvearrowright}$

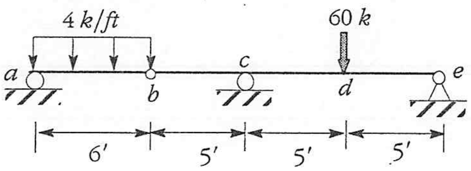
University of Texas
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson
CE 363 Advanced Structural Analysis

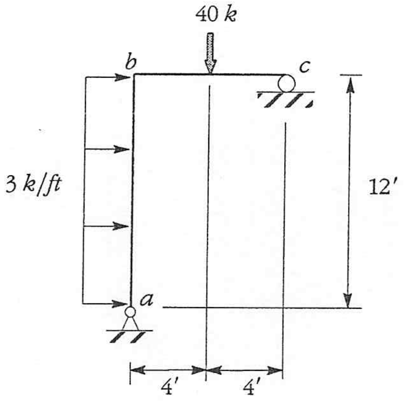
Homework 3

Due: September ~~26~~ 27

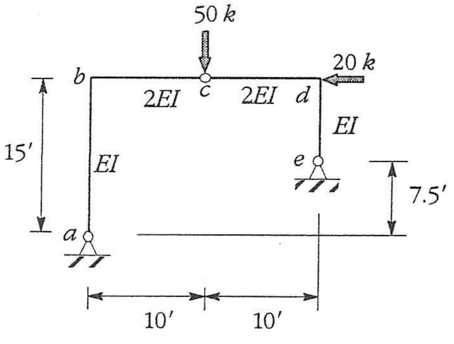
1. The structure shown in the sketch is supported by a pin at *e* and rollers at *a* and *c*. The structure is hinged at *b*. Assuming that *EI* is the same for the entire length of the beam, use the principle of complimentary virtual work to compute the deflection at *b* (Δ_b), the deflection at *d* (Δ_d), and the rotation at *a* (θ_a).



2. The frame shown to the right is supported by a pin at *a* and roller at *c*. The beam and column are rigidly connected at *b*. Assume *EI* is the same for both members. Using the principle of complimentary virtual work, compute the deflection at *c* (Δ_c) and the rotation at *a* (θ_a).



3. The frame shown to the right is pin-supported at both *a* and *e* and hinged at *c*. *EI* for the beam is twice that of the columns. For the loading shown, use the principle of complimentary virtual work to compute both the vertical and horizontal components of the deflection at *c* ($\Delta_{c(V)}$ and $\Delta_{c(H)}$), the rotation to the left of *c* ($\theta_{c(L)}$), and the rotation the right of *c* ($\theta_{c(R)}$).

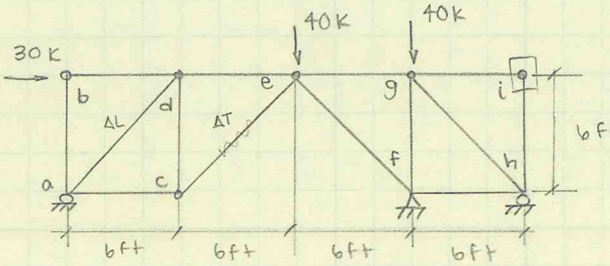


$E = 29,000\ ksi \qquad I = 1,000\ in^4$

100

HOMEWORK #4

1.



Δe_v, Δi_h

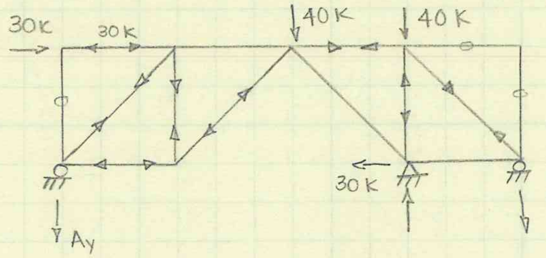
$A = 4\ in^2$

$E = 29000\ ksi$

ce: $\Delta T = 50°F$

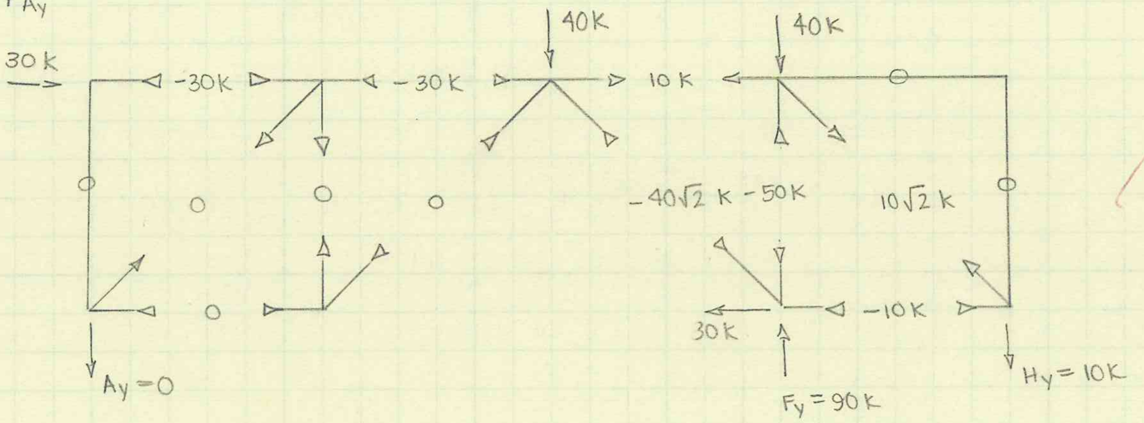
ad: $\Delta L = 0.75\ in$

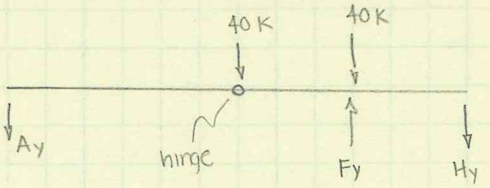
$\alpha = 6.5 \times 10^{-6}\ 1/°F$



$A_y + F_y + H_y = 80K$

$F_y(18ft) = H_y(24ft) + 1380K \cdot ft$





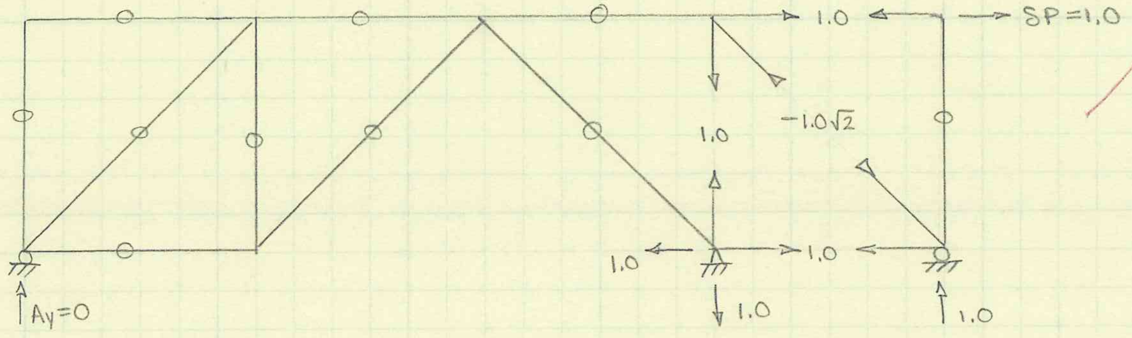
$A_y = 0$ for

$\Sigma M_E = 0$

$(40K)(6ft) = H_y(6ft) + 30K(6ft)$

$H_y = 10K$

$F_y = 90K$



HOMEWORK #4

1. (cont'd)

$$\Delta = \sum \frac{nNL}{EA} + \sum n\alpha\Delta TL + \sum n\Delta L$$

$E = 29000\,ksi$
$A = 4\,in^2$
$\alpha = 6.5\times10^{-6}/°F$
$\Delta T = 50°F$ (ce), $\Delta L = 0.75$ in (ad)

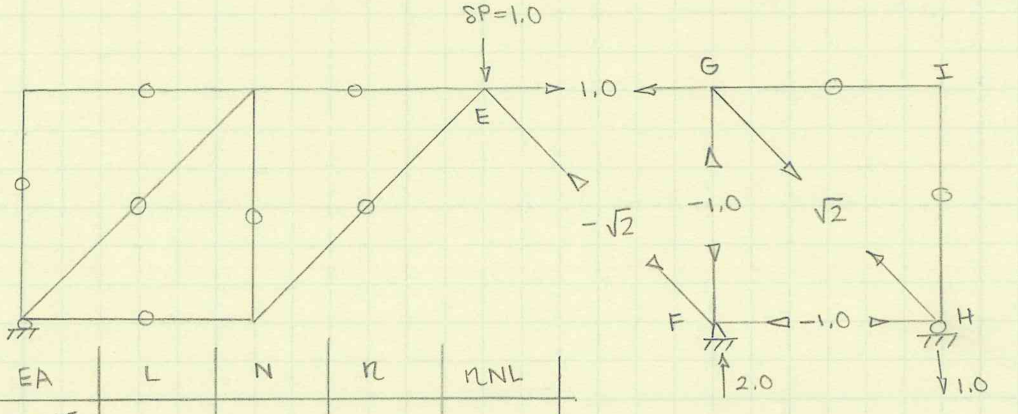
	EA	L	N	n	nNL
FG	1.16×10^5 kip	6 ft	−50K	1.0	−3600 k·in
GI	↓	6	0	1.0	0
GH		$6\sqrt{2}$	$10\sqrt{2}$	$-\sqrt{2}$	−2036
FH		6	−10	1.0	−720

$\Sigma = \frac{1}{EA}(-6356\,k\cdot in)$

all other members
have n=0, so Δ
components = 0 (including ce, ad, with ΔT, ΔL effects)

$$\Delta_H = \frac{6356\,k\cdot in}{1.16\times10^5\,kip} = 0.055\,in \text{ left}$$

$$\boxed{\Delta_{iH} = 0.055\,in \leftarrow}$$



	EA	L	N	n	nNL
EG	1.16×10^5 kip	6 ft	10 K	1.0	720 k·in
EF	↓	$6\sqrt{2}$	$-40\sqrt{2}$	$-\sqrt{2}$	8145.9
FG		6	−50	−1.0	3600
GH		$6\sqrt{2}$	$10\sqrt{2}$	$\sqrt{2}$	2036.5
FH		6	−10	−1.0	720

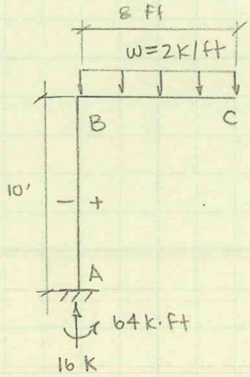
$\Sigma nNL = 15222.4\,k\cdot in$

$$\Delta_{EV} = \frac{15{,}222.4\,k\cdot in}{(29000\,ksi)(4\,in^2)}$$

$$\boxed{\Delta_{EV} = 0.131\,in \text{ down}}$$

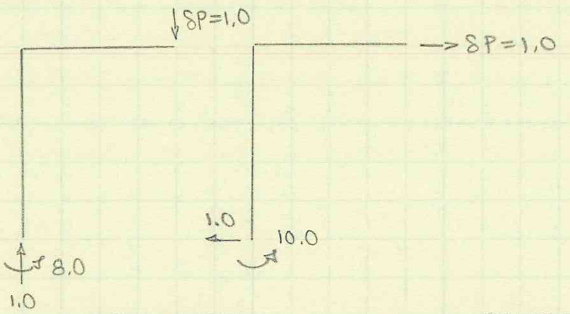
HOMEWORK #4

2.

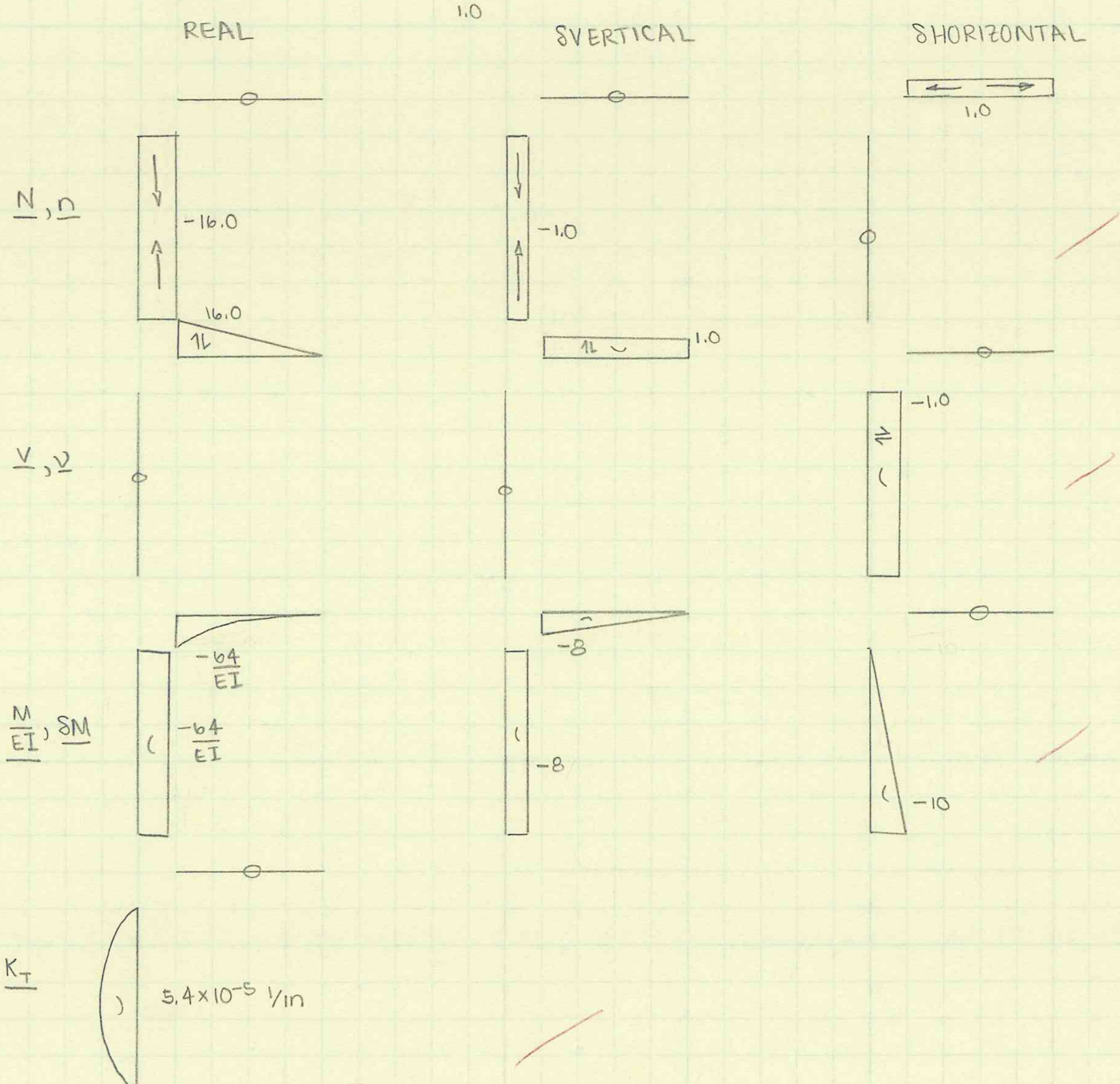


E = 29000 ksi
G = 12,000 ksi
A = 24 in^2
I = 288 in^4
k = 1.2

$\alpha = 6.5\times10^{-6}$ 1/°F
$\Delta T_o = 100°F$
$\Delta T = 4\Delta T_o\left[x/l-(x/l)^2\right]$
b = 2 in
h = 12 in



Solve for Δ_{CH}, Δ_{CV}
- axial – N
- shear – V
- flexural – M
- thermal – K



$K_T = 4\Delta T_o\left[x/l-(x/l)^2\right]\frac{\alpha}{h}$

HOMEWORK #4

2. (cont'd)

$$\Delta_C = \int_0^\ell \frac{mM}{EI}\,dx + \sum \frac{nNL}{EA} + \int_0^\ell \frac{vV}{GA}\,k\,dx + \int_0^\ell m K_T\,dx$$

$$\Delta_{CV_M} = \frac{(-64)(10)(-8.0)}{(29000)(288)}(12)^3 + \frac{(-64)(1/3)(8)(3/4)(-8.0)}{(29000)(288)}(12)^3 = 1.27 \text{ in}$$

$$\Delta_{CV_N} = \frac{(-1.0)(-16.0)(10)}{(29000)(24)}(12) = 0.0028 \text{ in}$$

$$\Delta_{CV_V} = \frac{1/2(16.0)(8)(1.0)}{(12000)(24)}(12)(1.2) = 0.0032 \text{ in}$$

$$\Delta_{CV_T} = \frac{2}{3}(10)\frac{6.5\times10^{-6}}{12}(4)(100)\left[1/4\right](-8)(12)^2 = -0.42 \text{ in}$$

$$\boxed{\Delta_{CV} = 0.88 \text{ in down}}$$

$$\Delta_{CH_M} = \frac{(-64)(10)(-10)}{(29000)(288)}\frac{1}{2}(12)^3 = 0.66 \text{ in}$$

$$\Delta_{CH_N} = 0$$

$$\Delta_{CH_V} = 0$$

$$\Delta_{CH_T} = \frac{2}{3}(10)\frac{6.5\times10^{-6}}{12}(4)(100)(1/4)(-10)(1/2)(12)^2 = -0.26 \text{ in}$$

$$\boxed{\Delta_{CH} = 0.40 \text{ in right}}$$

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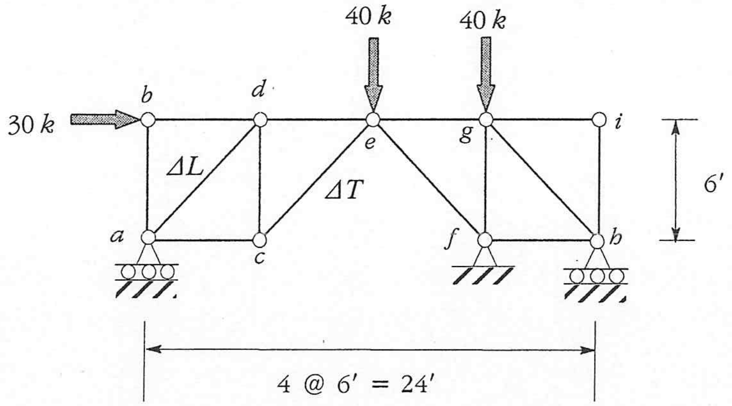
CE 363 Advanced Structural Analysis

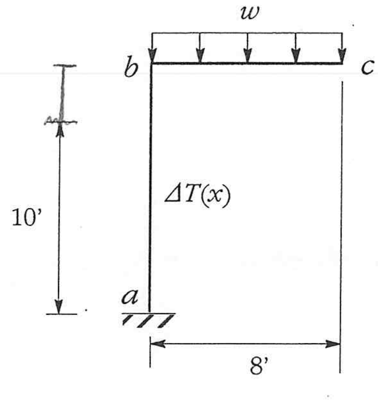
Homework 4

Due: October 13, 2005

1. For the truss shown below, compute the vertical deflection at e and the horizontal deflection at i. Assume that all members have a cross-sectional area of $A = 4$ in^2 and a modulus of elasticity $E = 29{,}000$ *ksi*. Note that, in addition to the applied loads, member *ce* has been heated by 50° F, and member *ad* has an initial misfit of $\Delta L = 0.75$ inches. Assume that the coefficient of thermal expansion for all members in the truss is $\alpha = 6.5 \times 10^{-6}/°\text{F}$.



2. The structure shown in the figure below consists of a cantilevered frame in which the beam and column have the same properties. In addition to the uniformly distributed load applied to the beam, the column is subjected to a temperature gradient that varies linearly through its depth and quadratically over its length. Referring to the figure below, the right side of the column is heated by 50° F above its mean temperature at the mid-height of the column, and the left side of the column is cooled by 50°F at the mid-height of the column (i.e., the column face on the inside of the frame is at a greater temperature than the column face on the outside of the frame). Thus, the total temperature variation at the mid-height of the column is $\Delta T_o = 100°\text{F}$, and the *mean* temperature is unaffected. Using Virtual Work Principles, compute both the horizontal and vertical components of displacement at point c. In performing these computations, be sure to account for deformations due to axial, shear, flexural, and thermal effects. Parameters needed to solve the problem are provided in the figure.



$E = 29{,}000\ ksi$

$G = 12{,}000\ ksi$

$A = 24\ in^2$

$I = 288\ in^4$

$k = 1.2$

$w = 2\ k/ft$

$\alpha = 6.5 \times 10^{-6}/°F$

$\Delta T_o = 100°$

$\Delta T(x) = 4\,\Delta T_o\left(\frac{x}{l} - \frac{x^2}{l^2}\right)$

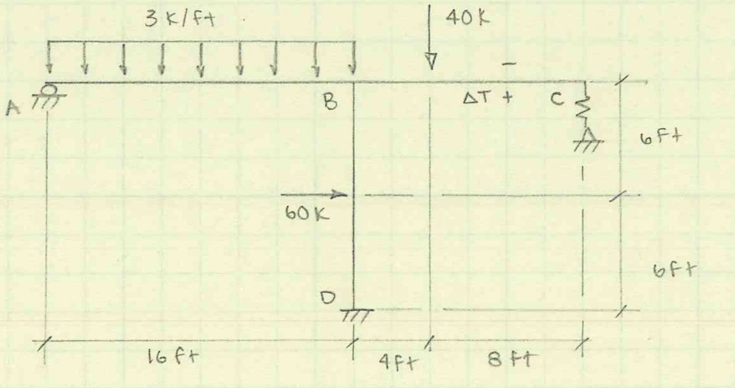
$b = 2\ in$

$h = 12\ in$

HOMEWORK #5

93

1.



E = 29000 ksi

$I = 288\ in^4$

$\alpha = 6.5 \times 10^{-6}/°F$

K = 3 k/in

$\Delta_A = 3$ in ↓

$\Delta_D = 5$ in ↓

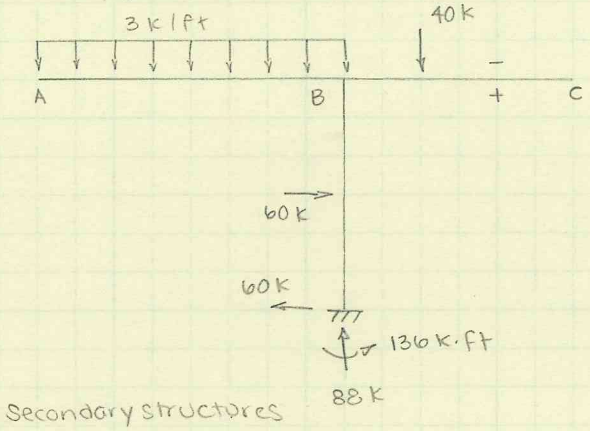
$\theta_D = 0.03$ rad ↷

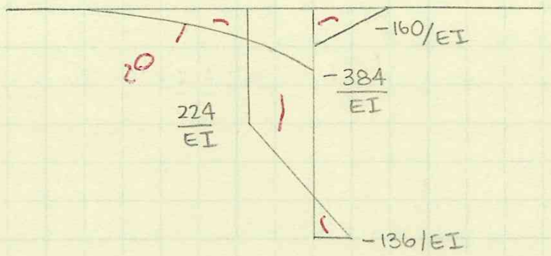
$\Delta T_o = 100°F$

d = 12 in

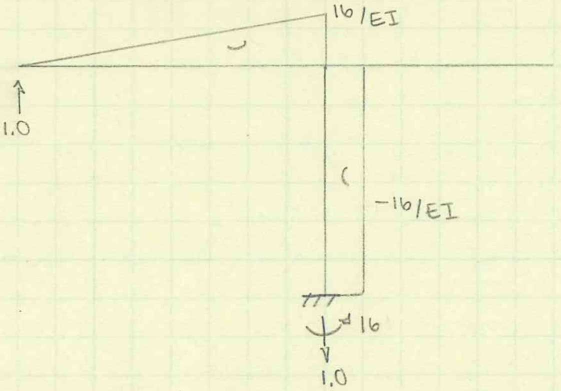
Rs, V, N, M, Δ_{CH}

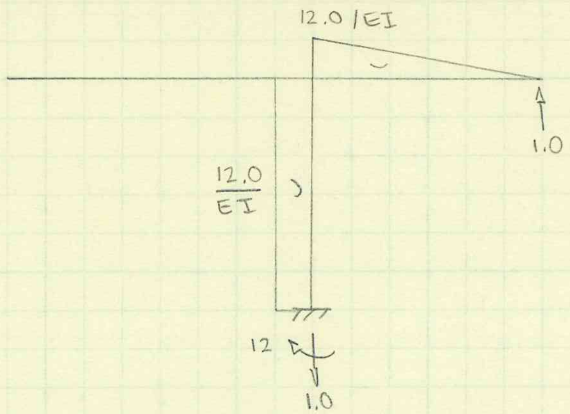
Primary structure





Secondary structures





$f_{AA} = 1/2\left(\frac{16}{EI}\right)(16)\,2/3\,(16) + \left(\frac{-16}{EI}\right)(12)(-16) = 4437.33/EI$ up

$f_{CA} = 0 + \left(\frac{-16}{EI}\right)(12)(12) = -2304/EI$ down

$f_{CC} = 1/2\left(\frac{12}{EI}\right)(12)\,2/3\,(12) + \left(\frac{12}{EI}\right)(12)(12) = 2304/EI$ up

$f_{AC} = 0 + \left(\frac{12}{EI}\right)(12)(-16) = -2304/EI$ down check: $f_{CA} = f_{AC}$ ✓

$\Delta_{AO} = 1/3\left(\frac{-384}{EI}\right)(16)(3/4)(16) + (-16)\left[\left(\frac{224}{EI}\right)(6) + \frac{1}{2}\left(\frac{224}{EI}\right)(3.73) + \frac{1}{2}\left(\frac{-136}{EI}\right)(2.27)\right]$

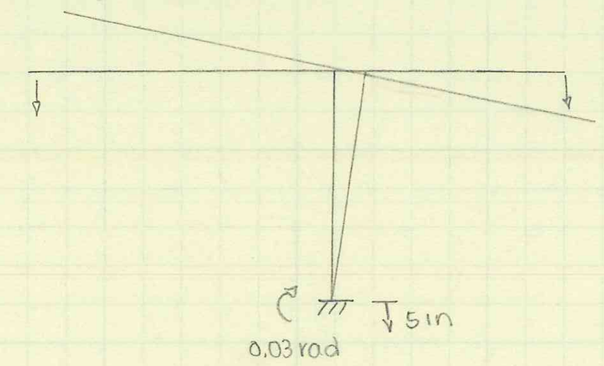
$= \frac{-50304}{EI}$

$\Delta_{CO} = 1/2\left(\frac{-160}{EI}\right)(4)(8 + 2/3 \cdot 4) + (12)\left[1608/EI\right] = \frac{15882.7}{EI}$

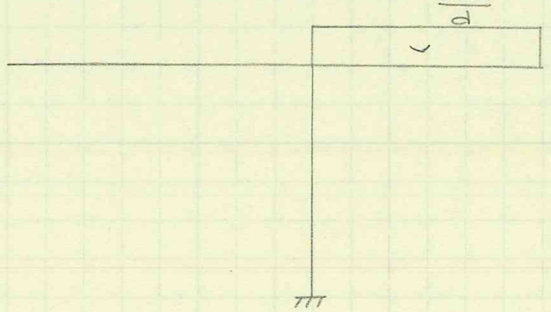
HOMEWORK #5

1. (cont'd)

settling/rotation at D



Temperature — $\Delta T_0 = 100°\ F$



$\Delta_{A\theta} = 0.03\,(16\,ft) = 5.76$ in up

$\Delta_{AS} = 5$ in down

$\Delta_{C\theta} = 0.03\,(12\,ft) = 4.32$ in down

$\Delta_{CS} = 5$ in down

$\Delta_{AT} = 0$

$$\Delta_{CT} = \frac{6.5\times10^{-6}/°F\,(100°F)}{(12\,in)}(12)(6)\ X12^2$$

$= 0.5616$ in up

compatibility

$$\Delta_{AO} = \frac{-50304\,(12)^3}{(29000\,KSI)(288\,in^4)} + (-5\,in) + (5.76\,in) + 0 = 9.65 \text{ in down}$$

$$\Delta_{CO} = \frac{15882.7\,(12)^3}{(29000\,KSI)(288\,in^4)} + (-5\,in) + (-4.32\,in) + (0.5616\,in) = 5.472 \text{ in down}$$

$\Delta_{AO} - F_A f_{AA} + F_C f_{CA} = 3$ in , $F_A(-f_{AA}) + F_C(f_{AC}) = -6.65$ in

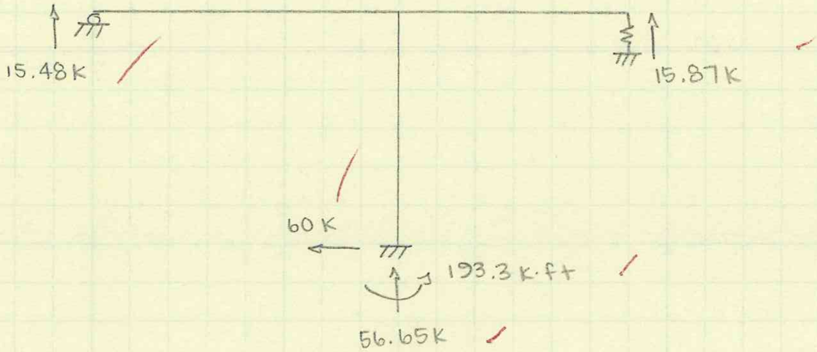
$\Delta_{CO} + F_A f_{AC} - F_C f_{CC} = F_C(1/K)$ $F_A(f_{AC}) + F_C(-f_{CC} - 1/K) = -5.472$ in

$$[f][F] = [\Delta]$$

$$[f] = \begin{bmatrix} -0.9181 & 0.4767 \\ 0.4767 & -0.8100 \end{bmatrix}$$

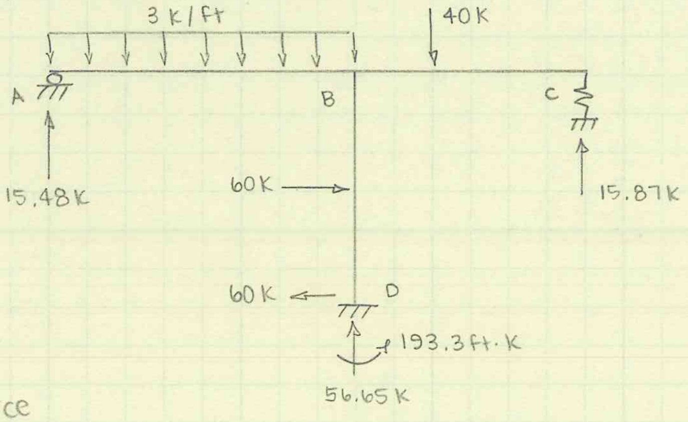
$F_A = 15.48$ K up

$F_C = 15.87$ K up $F_{DY} = 58.53$ K up

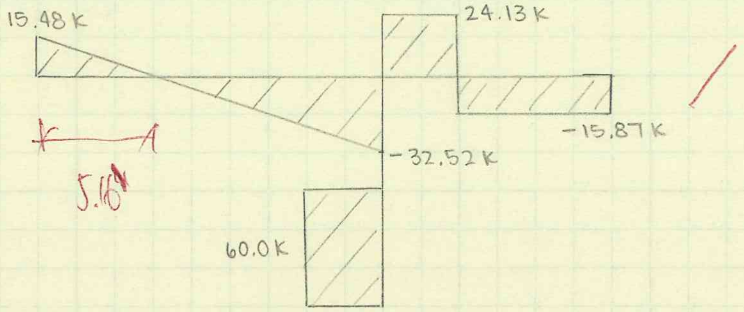


HOMEWORK #5

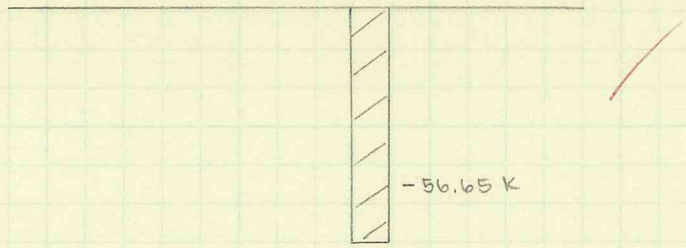
1. (cont'd) Structure, with Reactions



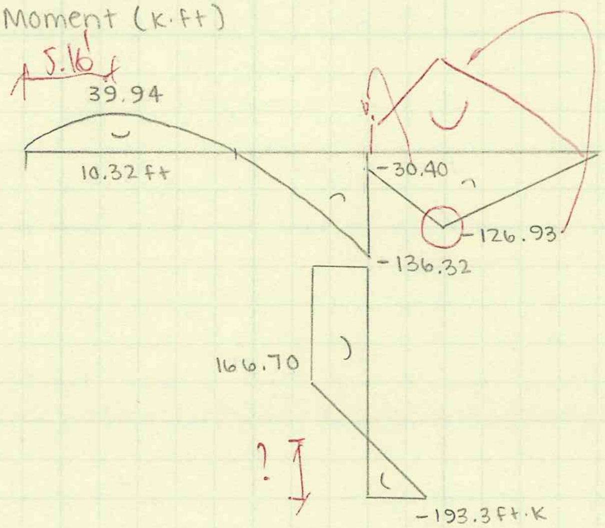
Shear Force



Axial Force



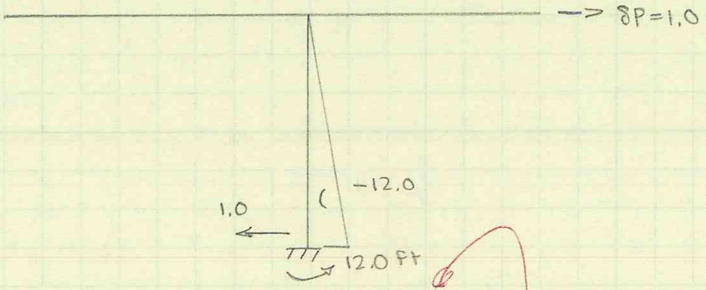
Bending Moment (k·ft)



HOMEWORK #5

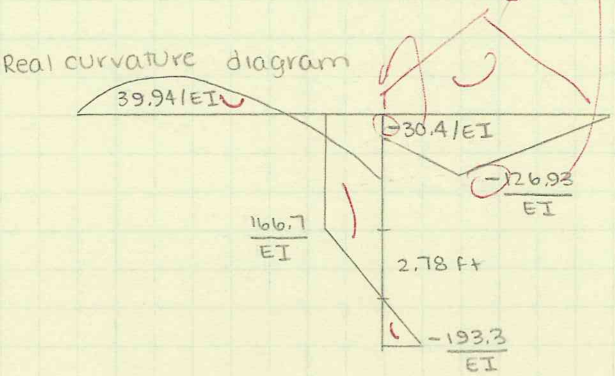
1. (cont'd)

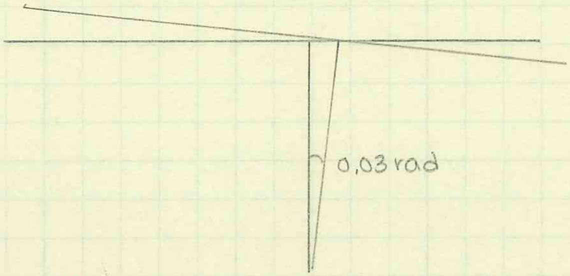
virtual load at C:



assume $F_A = F_{CY} = 0$

Real curvature diagram





$\Delta_{CH\theta} = 0.03\,(12\,ft) = 4.32\,in \rightarrow$

$$-\left(6+\frac{2.778}{3}\right)$$

$$\Delta_{C\,loads} = \left[\left(\frac{166.7}{EI}\right)\left[(6)(-3)+\frac{1}{2}(2.778)(-7.852)\right]+\frac{1}{2}\left(\frac{-193.3}{EI}\right)(3.221)(-10.926)\right](12)^3$$

$\Delta_{C\,load} = 0.948$ in right left

$\Delta_{CH} = \Delta_{C\,load} + \Delta_{CH\theta} + \Delta_{CH\,THERM}$

$\hookrightarrow = 0$

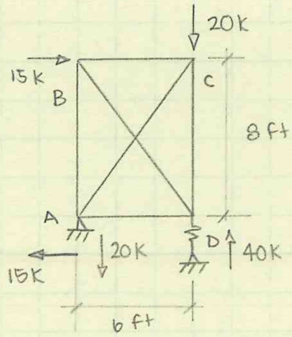
ok!

$= 0.948\,in + 4.32\,in$

$$\boxed{\Delta_{CH} = 5.27\,in \text{ right}}$$

HOMEWORK #5

2.

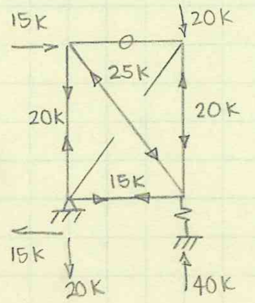


$A = 4\,in^2$
$E = 15{,}000$ ksi
$\Delta L_{AC} = 0.10$ in
$\Delta L_{BD} = 0.05$ in

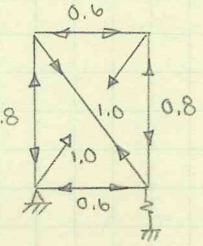
$\alpha = 6.5 \times 10^{-6} /°F$
$\Delta T_{BC} = 50°F$
$\Delta T_{AD} = 80°F$

Redundant: AC

Primary structure



Secondary Structure



?

compatibility:

$\Delta_{AC} + F_{AC} f_{AC/AC} = 0.10\text{ in}$

$$\Delta = \Sigma \frac{nLN}{EA} + \Sigma n\alpha \Delta T L + \Sigma n \Delta L$$

$$f = \Sigma \frac{n^2 L}{EA}$$

	EA	L	$\hat{N}$	n	ΔT	ΔL	$\Sigma \frac{nLN}{EA}$	ΔT, ΔL
AB	60,000 kip	8 ft	20 K	-0.8	–	–	-0.0256 in	–
BC		6	0	-0.6	50°F	–	0	-0.014 in
CD		8	-20	-0.8	–	–	0.0256	–
AD		6	15	-0.6	80	–	-0.0108	-0.022
BD		10	-25	1.0	–	0.05 in	-0.05	0.05
AC		10	0	1.0	–	0.1 in	0	0.10
							Σ = -0.0608	0.114

-2

$\Delta_0 \overset{\sim}{=} 0.0532$ in

$$f_{AC/AC} = \Sigma \frac{n^2 L}{EA} = \frac{(2)(-0.8)^2(8) + (2)(-0.6)^2(6) + (2)(1.0)^2(10)}{60{,}000\text{ kip}}(12) = 0.006912$$

$\Delta_0 + F_{AC} f_{AC/AC} = 0$

↑ virtual structure guessed separation

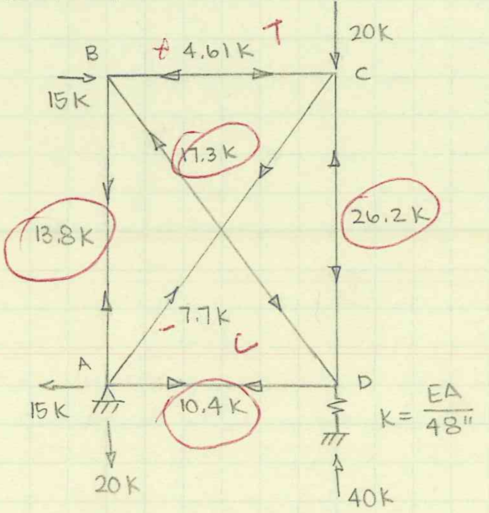
$$F_{AC} = \frac{-0.0532''}{0.006912} = -7.698\text{ K}$$ tension

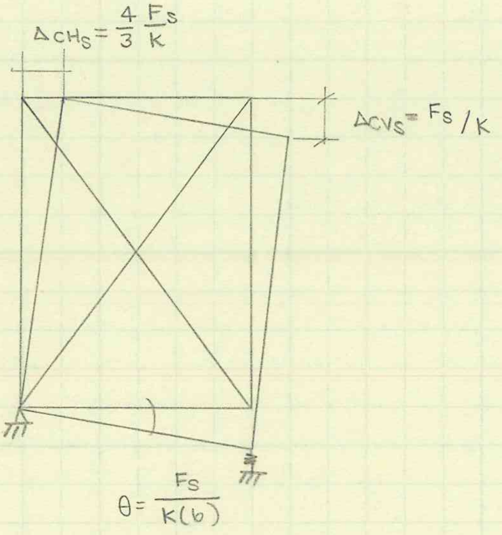
C

6/6

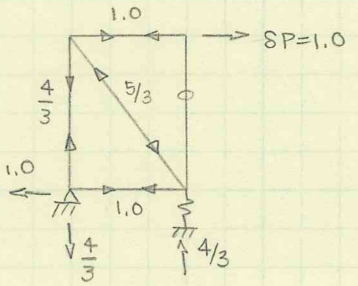
HOMEWORK #5

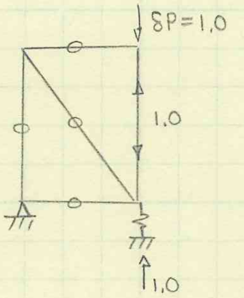
2. (cont'd)





Virtual structures - assume $F_{AC}=0$





	EA	L	N	n_H	n_V	$\frac{n_H L N}{EA}$	$\frac{n_V L N}{EA}$	ΔL, ΔT H	ΔL, ΔT V
AB	60,000 kip	8.0 ft	13.8K	4/3	0	0.0294 in	–	0	0
BC		6.0	-4.61	1.0	0	-0.0055	–	0.0234 in	0
CD		8.0	-26.2	0	-1.0	0	0.0419 in	0	0
AD		6.0	10.4	1.0	0	0.01248	–	0.0374	0
AC		10.0	7.70	0	0	0	–	0	0
BD		10.0	-17.3	-5/3	0	0.0577	–	-0.0833	0

$\Sigma = 0.0941$ in right, 0.0419 in down, 0.0225 in left

Rigid rotations

$$\Delta_{CV_S}=\frac{40\text{ K }(48\text{ in})}{60000\text{ KIP}}=0.032\text{ in down},\quad \Delta_{CH_S}=\frac{4}{3}(0.032\text{ in})=0.04267\text{ in right}$$

$$\boxed{\begin{array}{l}\Delta_{CV}=0.074\text{ in down}\\ \Delta_{CH}=0.114\text{ in right}\end{array}}$$

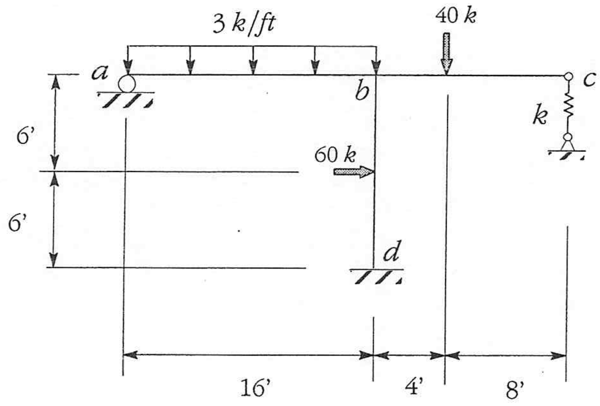
University of Texas
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson
CE 363 Advanced Structural Analysis

Homework 5

Due: October ~~25~~ 20, 2005

1. All members shown in the figure below have the same cross-sectional properties. In addition to the applied loads, the structure experiences displacements at the supports, and member *bc* experiences a temperature change that varies linearly through it depth. The bottom of the member becomes hotter by 50 degrees, and the top becomes cooler by 50 degrees. The member has a depth of 12 inches. Using the method of consistent deformations, determine the support reactions for the structure. Afterward, draw the shear force, axial force, and bending moment diagrams. Finally, compute the horizontal displacement at *c*. For this problem, ignore axial and shear deformations. Parameters needed to solve the problem are provided in the figure.



$E = 29,000\ ksi$

$I = 288\ in^4$

$\alpha = 6.5 \times 10^{-6}/°F$

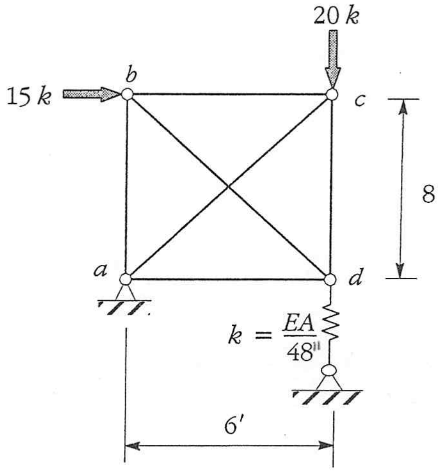
$k = 3\ k/in$

$\Delta_a = 3\ in \downarrow$

$\Delta_d = 5\ in \downarrow$

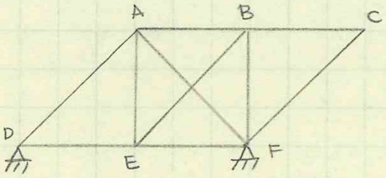
$\theta_d = 0.03\ rad \circlearrowleft$

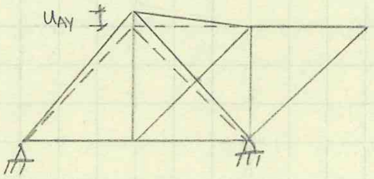
2. For the truss shown below, compute the horizontal and vertical deflection at *c*. Assume that all members have a cross-sectional area of $A = 4\ in^2$ and a modulus of elasticity $E = 15,000\ ksi$. In addition to the applied loads, member *bc* is heated by 50 degrees F and member *ad* is heated by 80 degrees F. Also, member *ac* has an initial misfit of 0.10 inch and member *bd* has an initial misfit of 0.05 inch. The coefficient of thermal expansion is equal to $6.5 \times 10^{-6}/°F$.



95

HOMEWORK #6

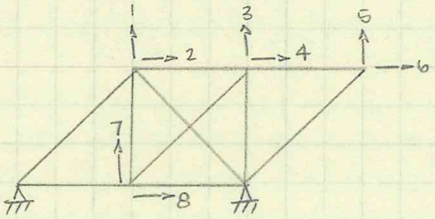




A, E, L, $L\sqrt{2}$

u_{AY}, u_{BX}, u_{CY}, u_{EY}
1 4 5 7

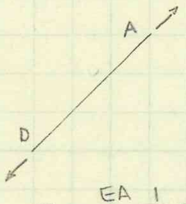
Degrees of indeterminacy



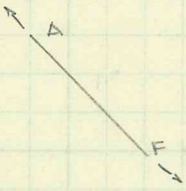
$\underset{\sim}{K} \cdot \underset{\sim}{U} = \underset{\sim}{F}$

Calculate K matrix

$u_1 = 1.0$, all others $= 0$



$F_{AD} = \frac{EA}{\sqrt{2}L}\frac{1}{\sqrt{2}} = \frac{EA}{2L}$



$F_{AF} = \frac{EA}{\sqrt{2}L}\frac{1}{\sqrt{2}} = \frac{EA}{2L}$

$F_{AE} = \frac{EA}{L}$

$k_{11} = \frac{EA}{\sqrt{2}L} + \frac{EA}{L}$

$k_{21} = 0$

$k_{31} = 0$

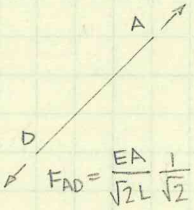
$k_{41} = 0$

$k_{51} = k_{61} = 0$

$k_{71} = -\frac{EA}{L}$

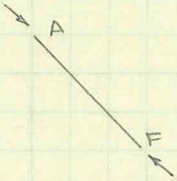
$k_{81} = 0$

$u_2 = 1.0$



$F_{AD} = \frac{EA}{\sqrt{2}L}\frac{1}{\sqrt{2}}$

$F_{AB} = -\frac{EA}{L}$



$F_{AF} = \frac{-EA}{2L}$

$k_{22} = \frac{EA}{\sqrt{2}L}\frac{1}{2} + \frac{EA}{L} + \frac{EA}{\sqrt{2}L} \cdot \frac{1}{2}$

$k_{32} = 0$

$k_{42} = -\frac{EA}{L}$

$k_{52} = 0$

$k_{62} = 0$

$k_{72} = 0$

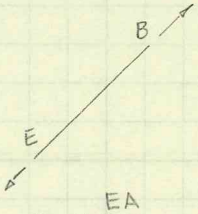
$k_{82} = 0$

2/5

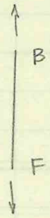
HOMEWORK #6

K matrix calcs

$u_3 = 1.0$



$F_{BE} = \frac{EA}{2L}$



$F_{BF} = \frac{EA}{L}$

$k_{33} = \frac{EA}{L} + \frac{EA}{\sqrt{2}L}\frac{1}{2}$

$k_{43} = \frac{EA}{\sqrt{2}L}\frac{1}{2}$

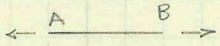
$k_{53} = 0$

$k_{63} = 0$

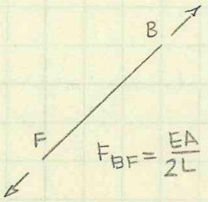
$k_{73} = \frac{-EA}{\sqrt{2}L}\frac{1}{2}$

$k_{83} = -\frac{EA}{\sqrt{2}L}\frac{1}{2}$

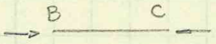
$u_4 = 1.0$



$F_{AB} = \frac{EA}{L}$



$F_{BF} = \frac{EA}{2L}$



$F_{BC} = \frac{-EA}{L}$

$k_{44} = \frac{EA}{L} + \frac{EA}{L} + \frac{EA}{\sqrt{2}L}\frac{1}{2}$

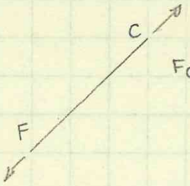
$k_{54} = 0$

$k_{64} = \frac{-EA}{L}$

$k_{74} = \frac{-EA}{\sqrt{2}L}\frac{1}{2}$

$k_{84} = \frac{-EA}{\sqrt{2}L}\frac{1}{2}$

$u_5 = 1.0$



$F_{CF} = \frac{EA}{2L}$

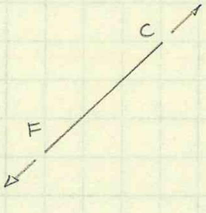
$k_{55} = \frac{EA}{\sqrt{2}L} \cdot \frac{1}{2}$

$k_{65} = \frac{EA}{\sqrt{2}L}\frac{1}{2}$

$k_{75} = k_{85} = 0$

$u_6 = 1.0$

$F_{BC} = \frac{EA}{L}$



$F_{CF} = \frac{EA}{2L}$

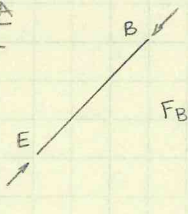
$k_{66} = \frac{EA}{L} + \frac{EA}{\sqrt{2}L}\frac{1}{2}$

$k_{76} = 0$

$k_{86} = 0$

$u_7 = 1.0$

$F_{AE} = \frac{-EA}{L}$



$F_{BE} = \frac{-EA}{2L}$

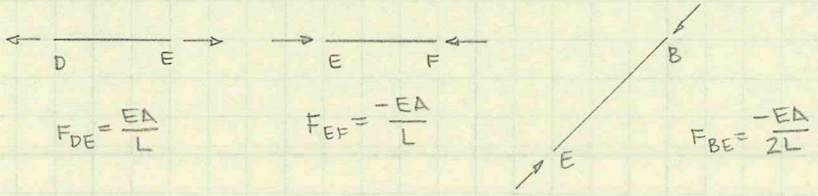
$k_{77} = \frac{EA}{L} + \frac{EA}{\sqrt{2}L} \cdot \frac{1}{2}$

$k_{87} = \frac{EA}{\sqrt{2}L} \cdot \frac{1}{2}$

HOMEWORK #6

K matrix calcs

$u_8 = 1.0$



$F_{DE} = \frac{EA}{L}$ $F_{EF} = \frac{-EA}{L}$ $F_{BE} = \frac{-EA}{2L}$

$K_{88} = \frac{EA}{L}(2) + \frac{EA}{\sqrt{2}L}\frac{1}{2}$

Stiffness matrix:

$$\begin{bmatrix} 1.707 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1.707 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 \\ 0 & -1 & 0.354 & 2.354 & 0 & -1 & -0.354 & -0.354 \\ 0 & 0 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0.354 & 1.354 & 0 & 0 \\ -1 & 0 & -0.354 & -0.354 & 0 & 0 & 1.354 & 0.354 \\ 0 & 0 & -0.354 & -0.354 & 0 & 0 & 0.354 & 2.354 \end{bmatrix} \frac{EA}{L}$$

Finding applied forces

$\underset{\sim}{K}\,\underset{\sim}{u} = \underset{\sim}{F}$

$\underset{\sim}{K}$ is known

$\underset{\sim}{F}$ is wanted

$\underset{\sim}{u}$ given by criteria: $u_{AY} = \begin{bmatrix} u_{AY} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, etc

$u_{AY} = u_1$

$u_{BX} = u_4$

$u_{CY} = u_5$

$u_{EY} = u_7$

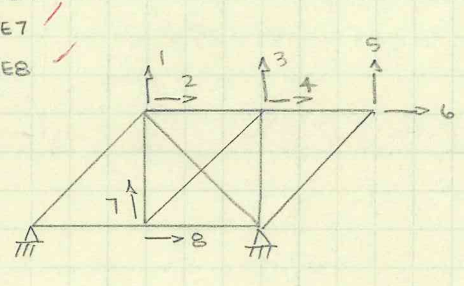
$$u_{AY}\begin{cases} 1.707\, u_{AY} = F_{A1} \\ -1\, u_{AY} = F_{A7} \end{cases} \qquad u_{CY}\begin{cases} u_{CY}/2\sqrt{2} = F_{C5} \\ u_{CY}/2\sqrt{2} = F_{C6} \end{cases}$$

$$u_{BX}\begin{cases} -1\, u_{BX} = F_{B2} \\ u_{BX}/2\sqrt{2} = F_{B3} \\ 2.354\, u_{BX} = F_{B4} \\ -1\, u_{BX} = F_{B6} \\ -u_{BX}/2\sqrt{2} = F_{B7} \\ -u_{BX}/2\sqrt{2} = F_{B8} \end{cases} \qquad u_{EY}\begin{cases} -1\, u_{EY} = F_{E1} \\ -u_{EY}/2\sqrt{2} = F_{E3} \\ -u_{EY}/2\sqrt{2} = F_{E4} \\ 1.354\, u_{EY} = F_{E7} \\ u_{EY}/2\sqrt{2} = F_{E8} \end{cases}$$

where forces are applied as shown below; (-) sign indicates down/left.

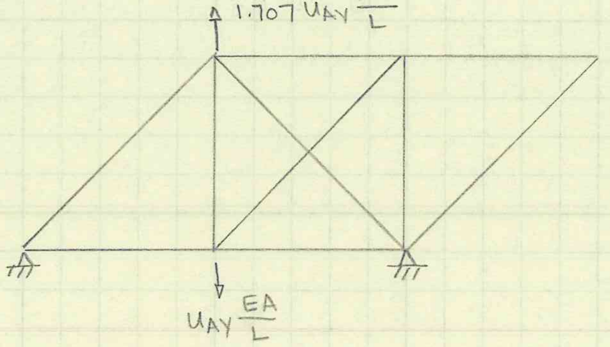
(all xEA/L)

shown on next page.

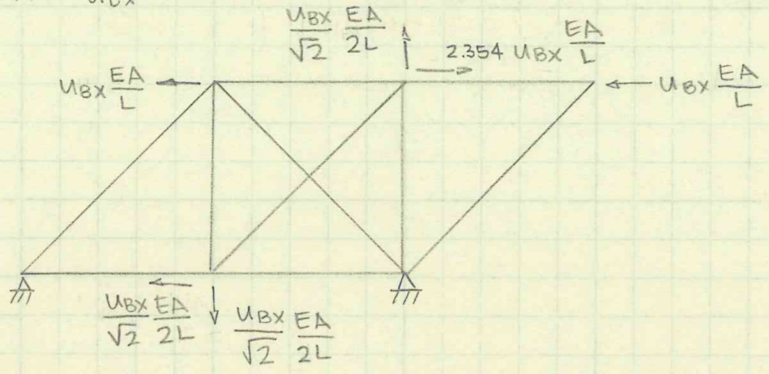


HOMEWORK #6

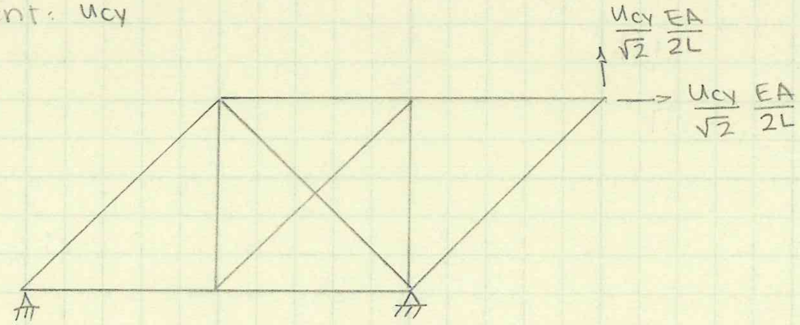
Displacement: u_{AY}



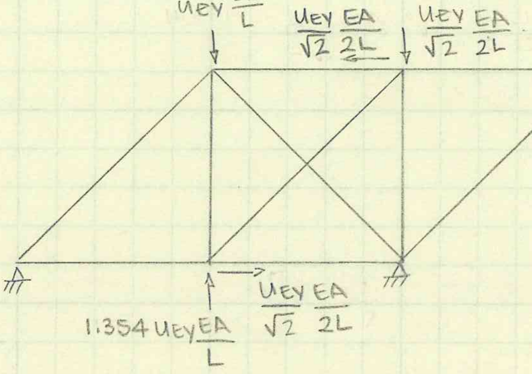
Displacement: u_{BX}



Displacement: u_{CY}



Displacement: u_{EY}



HOMEWORK #6

DISCUSSION:

Static indeterminacy (internal and external) do not affect these calculations, as we assign values to all but one unknown in every step. Removing a redundant adds a degree of freedom and with it, another equation. For instance, changing the support at F to a roller would not change the K values obtained above. It would add another row/column to the stiffness matrix, from loads induced due to a horizontal unit load at joint F.

?

-J

External -> reduces # d.o.f.

Internal -> may not affect # d.o.f.

(eliminate b.e for example)

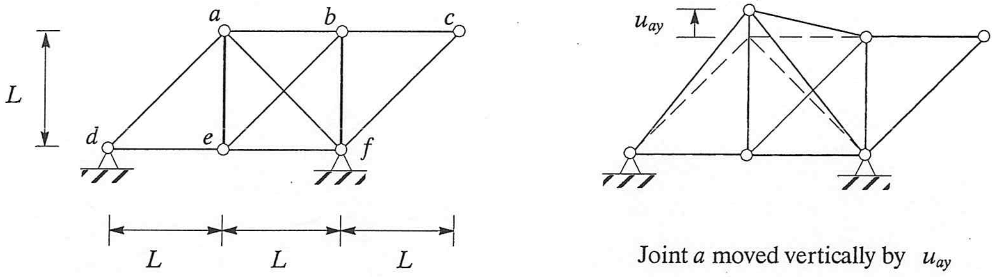
University of Texas
Department of Civil Engineering

Instructor: E. Williamson
CE 363 Advanced Structural Analysis

Homework 6

Due: November 1, 2005

The truss shown in Fig. 1 is statically indeterminate to the second degree. It has 8 kinematic degrees of freedom (show them on a sketch). All of the members have area A and modulus of elasticity E. Find the forces that must be applied at the joints to move joint a an amount u_{ay} with all other joints fixed against motion, in order for the structure to be in equilibrium. Repeat the exercise for the horizontal movement at joint b, that is, find the forces required (for equilibrium) to move joint b an amount u_{bx} with all other joints fixed against motion. Repeat the exercise for a vertical movement at joints c and e.



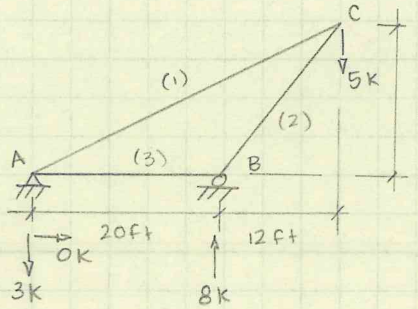
Joint a moved vertically by u_{ay}

Fig. 1. Description of structure and example of joint movement

For discussion: A problem in which the motion of the structure is completely known is called *kinematically determinate*. If we specify all of the joint movements in a truss, we know the motion completely, and thus we are solving a kinematically determinate structure. What role does static indeterminacy play in the solution of these kinematically determinate problems? Explain why.

95

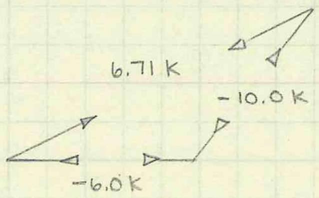
HOMEWORK #7



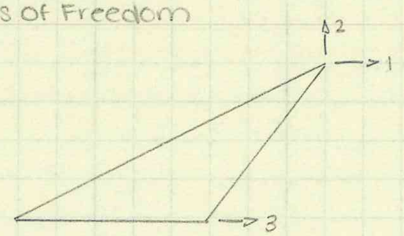
$A = 4.0\,in^2$
$E = 10{,}000\,ksi$

$L_{AB} = (20ft)(12in/ft) = 240\,in$
$L_{BC} = 240\,in$
$L_{AC} = 429.3\,in = 16\sqrt{5}\,ft$

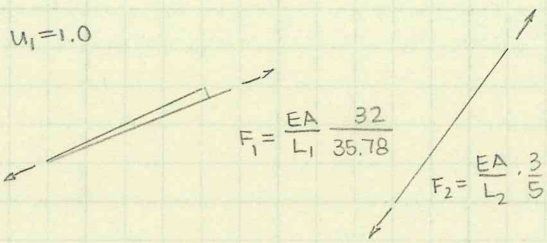
Expected member Forces (by statics)



Degrees of Freedom



$u_1 = 1.0$



$F_1 = \frac{EA}{L_1}\,\frac{32}{35.78}$

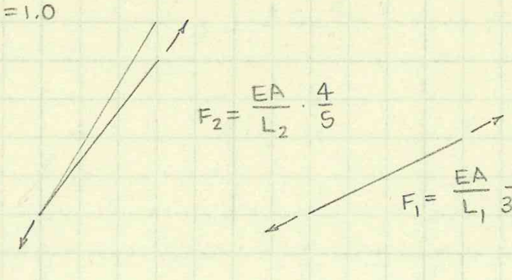
$F_2 = \frac{EA}{L_2}\cdot\frac{3}{5}$

$$k_{11} = \frac{EA}{(429.3)^3}(384in)^2 + \frac{EA}{240}\cdot\frac{3}{5}\cdot\frac{3}{5}$$

$$k_{21} = \frac{EA}{(429.3)^3}(384)(192) + \frac{EA}{240}\cdot\frac{3}{5}\cdot\frac{4}{5}$$

$$k_{31} = 0 - \frac{EA}{240}\cdot\frac{3}{5}\cdot\frac{3}{5}$$

$u_2 = 1.0$



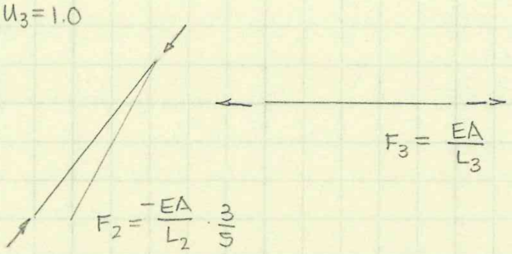
$F_2 = \frac{EA}{L_2}\cdot\frac{4}{5}$

$F_1 = \frac{EA}{L_1}\,\frac{16}{35.78}$

$$k_{22} = \frac{EA}{240}\cdot\frac{4}{5}\cdot\frac{4}{5} + \frac{EA}{(429.3)^3}(192)^2$$

$$k_{32} = \frac{-EA}{240}\cdot\frac{4}{5}\cdot\frac{3}{5}$$

$u_3 = 1.0$



$F_3 = \frac{EA}{L_3}$

$F_2 = \frac{-EA}{L_2}\cdot\frac{3}{5}$

$$k_{33} = \frac{EA}{240}\cdot\frac{3}{5}\cdot\frac{3}{5} + \frac{EA}{240}$$

$$\underset{\sim}{K}\,\underset{\sim}{u} = \underset{\sim}{F} \qquad \begin{bmatrix} 134.536 & 117.268 & -60.0 \\ 117.268 & 125.300 & -80.0 \\ -60.0 & -80.0 & 226.667 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -5 \\ 0 \end{Bmatrix}$$

$u_1 = 0.21$ in right
$u_2 = 0.26$ in down
$u_3 = 0.036$ in left

$u_1 = 0.210\,in,\ u_2 = -0.260\,in,\ u_3 = -0.036$

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HOMEWORK #7

Axial Forces

$u_1 = 0.210$ in

$u_2 = -0.260$ in

$u_3 = -0.036$ in

$$\Delta_1 = \frac{384}{192\sqrt{5}} u_1 + \frac{1}{\sqrt{5}} u_2 \qquad \Delta_1 = 0.072 \text{ in}$$

$$\Delta_2 = 4/5\,(u_2) + 3/5\,(u_1 - u_3) \qquad \Delta_2 = -0.06 \text{ in}$$

$$\Delta_3 = u_3 \qquad \Delta_3 = -0.036 \text{ in}$$

$$F_1 = \Delta_1 \frac{EA}{L_1} = (40000 \text{ K}) \frac{0.072 \text{ in}}{192\sqrt{5} \text{ in}} = 6.708 \text{ K}$$

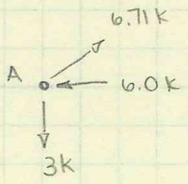
$$F_2 = (40000 \text{ K}) \frac{-0.06 \text{ in}}{240 \text{ in}} = -10.0 \text{ K}$$

$$F_3 = (40000 \text{ K}) \frac{-0.036 \text{ in}}{240 \text{ in}} = -6.0 \text{ K}$$

which match values expected from statics.

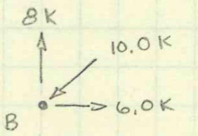
$$\boxed{\begin{array}{l} F_1 = 6.71 \text{ K T} \\ F_2 = 10.0 \text{ K C} \\ F_3 = 6.0 \text{ K C} \end{array}}$$

Check joint equilibrium



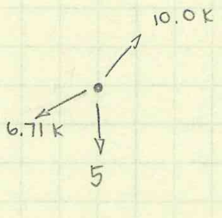
$$\Sigma F_x: \; 6.0\text{K} = \frac{6.71\text{K}}{\sqrt{5}} \cdot 2 \quad ✓$$

$$\Sigma F_y: \; 3.0\text{K} = \frac{6.71\text{K}}{\sqrt{5}} \quad ✓$$



$$\Sigma F_x: \; 6.0\text{K} = 10.0\text{K}\,(3/5) \quad ✓$$

$$\Sigma F_y: \; 8.0\text{K} = 10.0\text{K}\,(4/5) \quad ✓$$

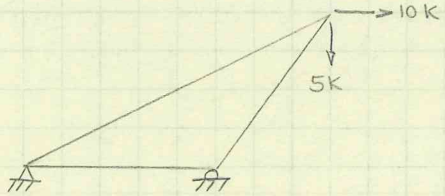


$$\Sigma F_x: \; 10.0\text{K}\,(3/5) = \frac{6.71\text{K}}{\sqrt{5}}\, 2 \quad ✓$$

$$\Sigma F_y: \; 10.0\text{K}\,(4/5) = \frac{6.71\text{K}}{\sqrt{5}} + 5 \quad ✓$$

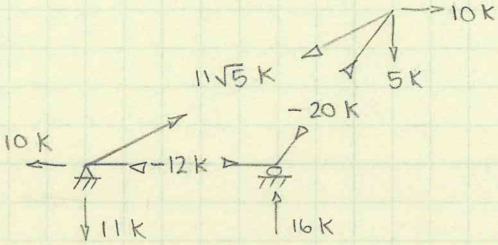
HOMEWORK #7

For DISCUSSION:



Changing the external forces only affects the right side of your equilibrium equations — the force vector. The stiffness terms stay the same.

with $\underset{\sim}{F} = \begin{bmatrix} 10 \\ -5 \\ 0 \end{bmatrix} k$, $\underset{\sim}{u} = \begin{bmatrix} 0.635 \\ -0.681 \\ -0.072 \end{bmatrix}$ in, $\underset{\sim}{\Delta} = \begin{bmatrix} 0.264 \\ -0.12 \\ -0.072 \end{bmatrix}$ in



and the member forces are $F_1 = 24.60$ K T
$F_2 = 20.0$ K C
$F_3 = 12.0$ K C

which are again confirmable with statics.

Using a force-based method for solving (eg, virtual work) would require new real and virtual moment diagrams, an additional virtual case for the horizontal load and (if indeterminate), primary and secondary structures.

for every new load case

Using this stiffness-based method, indeterminacy would not matter, whereas it would dictate many extra steps and choices (redundants, for example) in a force-based approach.

Force Method, determ. structure ?

University of Texas
Department of Civil Engineering

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CE 363 Advanced Structural Analysis

Homework 7 **Due: November 8, 2005**

The truss shown in Fig. 1 is subjected to a downward force of $5k$ at joint c. The structure has three kinematic degrees of freedom. All of the members have area A and modulus of elasticity E, with numerical values as indicated. Find the displacements at the three degrees of freedom caused by the loading. Find the values of the deformation in each member in terms of the nodal displacements, and then find the axial force in each member by multiplying the deformation by the stiffness of that member (EA/L). Set up three equations in three unknowns (the nodal displacements) by establishing joint equilibrium at each joint. The structure is statically determinate, so check the member forces using only statics. (Note: watch your units).

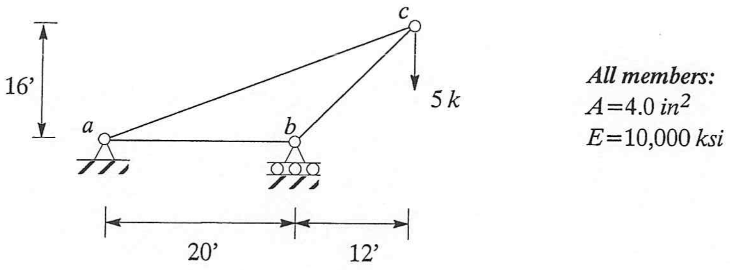


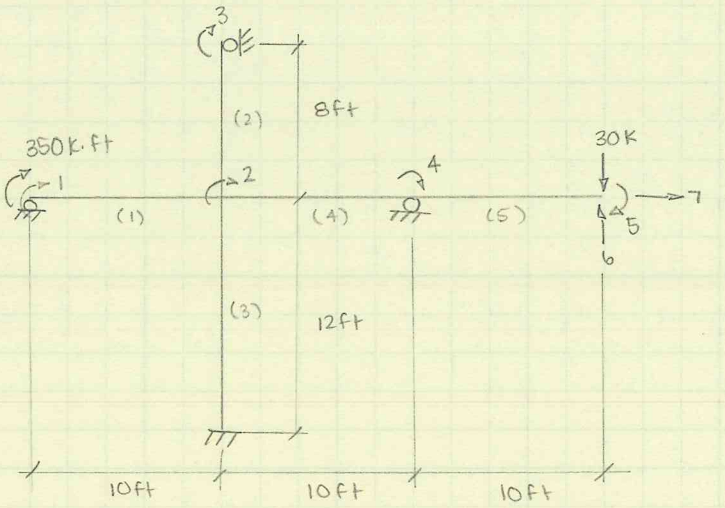
Fig. 1. Description of structure and loading

For discussion: Consider a new load case in which a 10 *k* force acts in the vertical direction at joint *c* in addition to horizontal force of 5 *k*. Compute the new member forces using the approach described above, and comment on which terms in your equilibrium equations change and if any remain the same. Compare and contrast the solution for the new load case to that which would be required using a force-based solution procedure. What parts would be similar, and what parts would be different had the structure been statically indeterminate?

99
+1
100

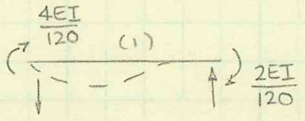
HOMEWORK #8

1.



E=29000 ksi
I=288 in⁴

$u_1 = 1.0$

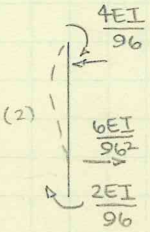


$K_{11} = \frac{EI}{30}$ ✓

$K_{21} = \frac{EI}{60}$ ✓

all others = 0

$u_3 = 1.0$



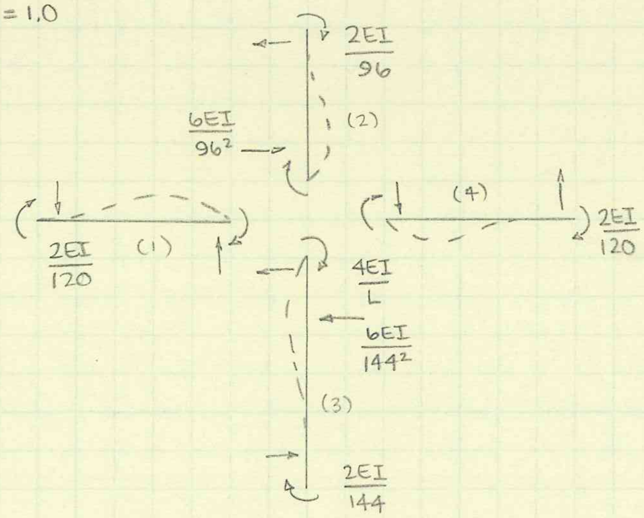
$K_{33} = \frac{EI}{24}$ ✓

$K_{73} = \frac{EI}{1536}$ ✓

all others = 0

$K_{23} \neq 0$

$u_2 = 1.0$



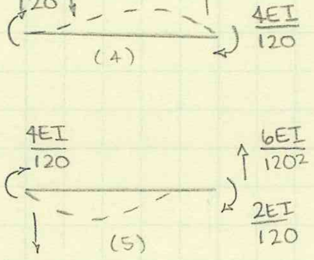
$K_{22} = \frac{EI}{3}\left(\frac{1}{8} + \frac{1}{10} + \frac{1}{10} + \frac{1}{12}\right)$ ✓

$K_{32} = \frac{EI}{48}$ ✓

$K_{42} = \frac{EI}{60}$ ✓ $\quad K_{52} = K_{62} = 0$

$K_{72} = 6EI\left(\frac{1}{96^2} - \frac{1}{144^2}\right)$ ✓

$u_4 = 1.0$



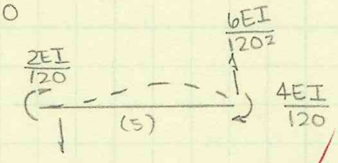
$K_{44} = \frac{EI}{15}$ ✓

$K_{54} = \frac{EI}{60}$ ✓

$K_{64} = \frac{EI}{2400}$ ✓

$K_{74} = 0$ ✓

$u_5 = 1.0$



$K_{55} = \frac{EI}{30}$ ✓

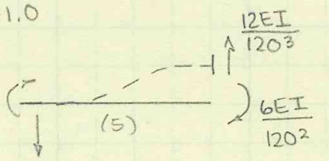
$K_{65} = \frac{EI}{2400}$ ✓

$K_{75} = 0$ ✓

HOMEWORK #8

1. (cont'd)

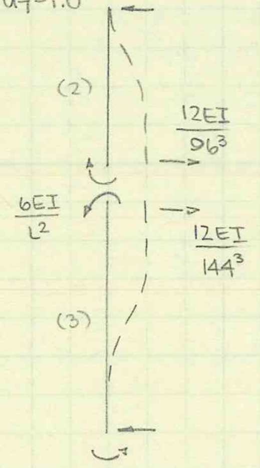
$u_6 = 1.0$



$k_{66} = \frac{12EI}{120^3}$

$k_{76} = 0$

$u_7 = 1.0$



$k_{77} = 12EI\left(\frac{1}{96^2} + \frac{1}{144^2}\right)$

$k_{27}, k_{37} \neq 0$ ✓

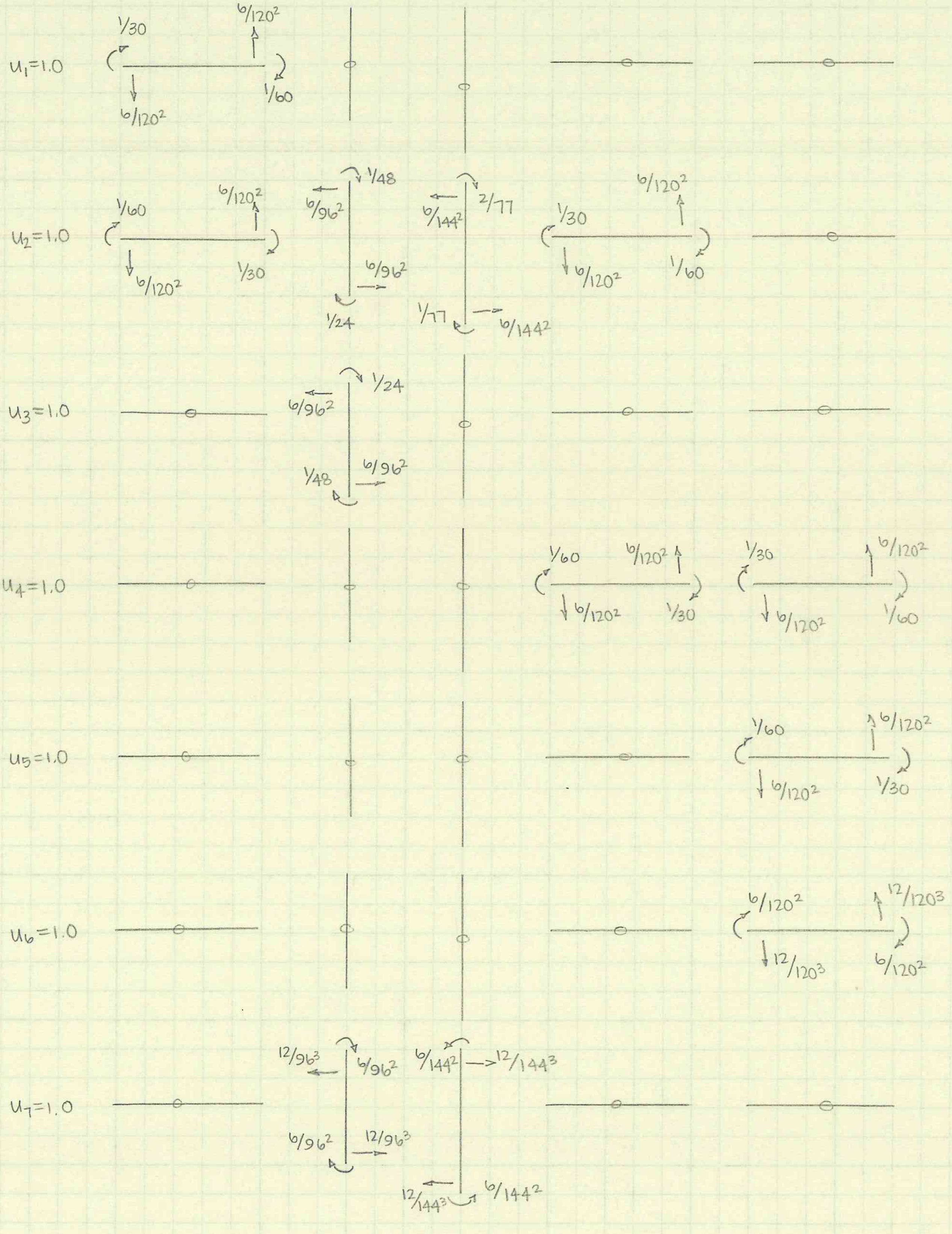
a) $$\underset{\sim}{K} = EI\begin{bmatrix} 1/30 & 1/60 & 0 & 0 & 0 & 0 & 0 \\ 1/60 & 49/360 & 1/48 & 1/60 & 0 & 0 & 3.62\times10^{-4} \\ 0 & 1/48 & 1/24 & 0 & 0 & 0 & 1/1536 \\ 0 & 1/60 & 0 & 1/15 & 1/60 & 1/2400 & 0 \\ 0 & 0 & 0 & 1/60 & 1/30 & 1/2400 & 0 \\ 0 & 0 & 0 & 1/2400 & 1/2400 & 6.94\times10^{-6} & 0 \\ 0 & 3.62\times10^{-4} & 1/1536 & 0 & 0 & 0 & 1.76\times10^{-5} \end{bmatrix}$$

b) $$\underset{\sim}{F} = \begin{bmatrix} 4200 \\ 0 \\ 0 \\ 0 \\ 0 \\ -30 \\ 0 \end{bmatrix} \begin{array}{l} k\cdot in \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ k \\ k \end{array}$$

c) $$\underset{\sim}{u} = \begin{bmatrix} 0.01723 \\ -0.00428 \\ 0.00182 \\ 0.01507 \\ 0.04093 \\ -3.8777 \\ 0.02094 \end{bmatrix} \begin{array}{l} rad \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ in \\ in \end{array}$$

HOMEWORK #8

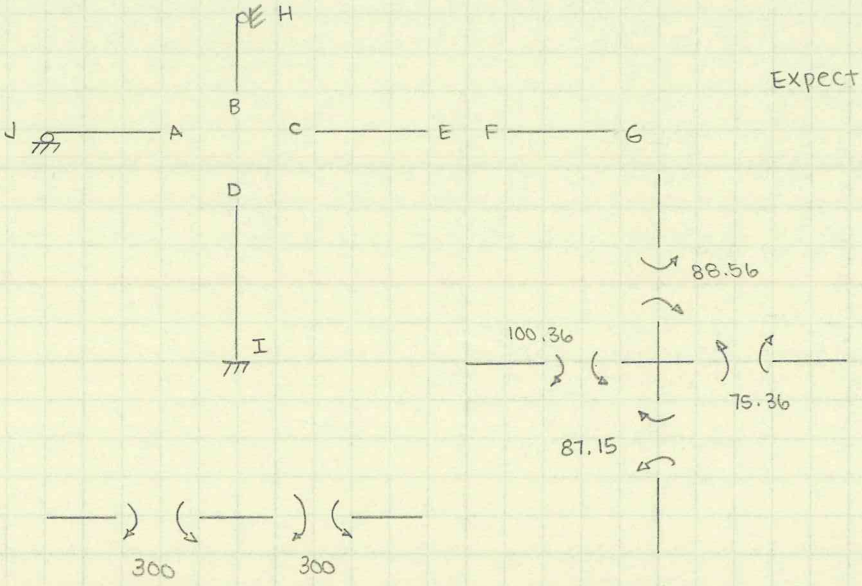
1. (cont'd)



HOMEWORK #8

1. (cont'd)

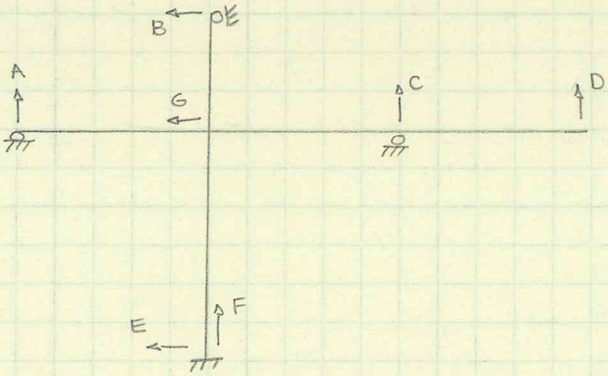
Solving for moments



Expect: J = 350 ft·k
H, G = 0
E = −F

M_I = 43.0 k·ft ↻

Solving for shears



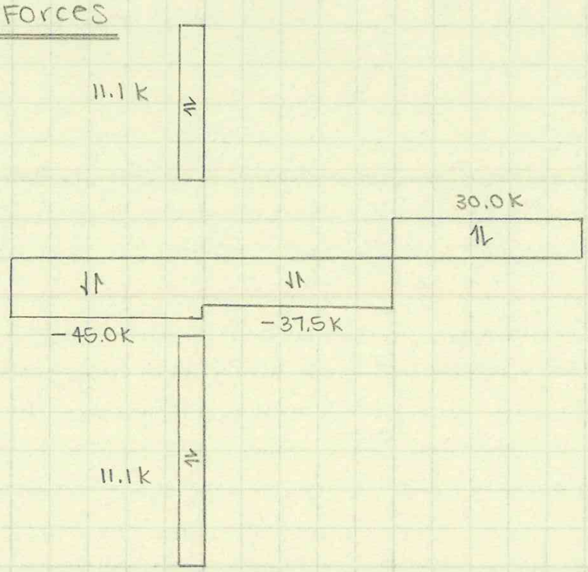
A = −45.04 k
B = −11.07 k
C = 67.54 k
D = −30 k
E = 11.07 k
F = 7.5 k
G = 4.74 k

A + C + D + F = 0 , D = −30 k (applied)
B + E + G = 0

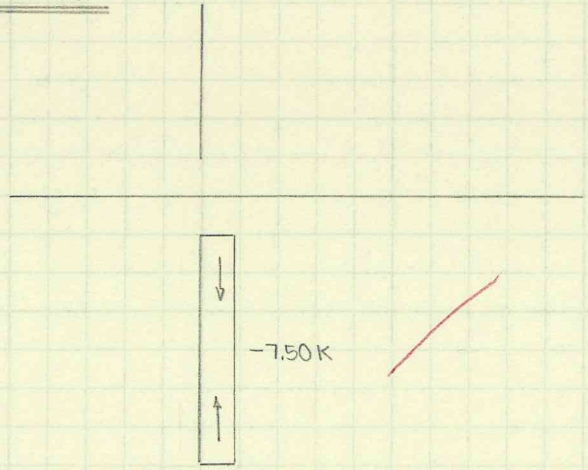
HOMEWORK #8

1. (cont'd)

d) Shear Forces

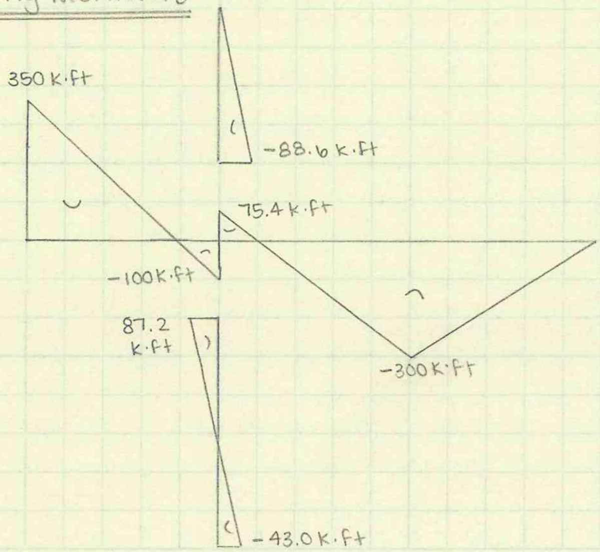


Axial Forces



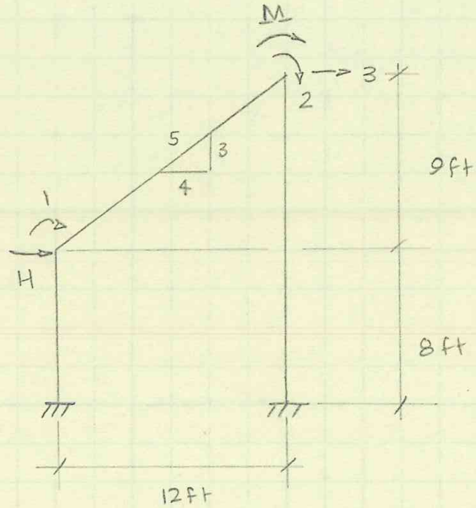
Beam cannot hold axial force (would just roll)

Bending Moments



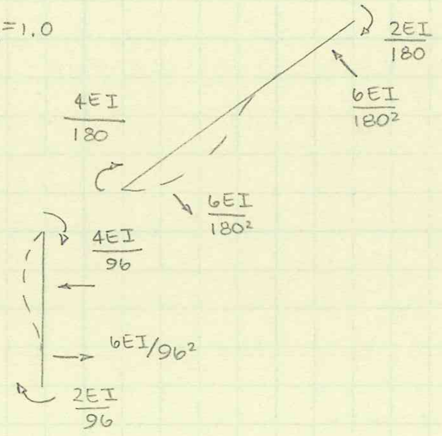
HOMEWORK #8

2.



Axially rigid

$u_1 = 1.0$

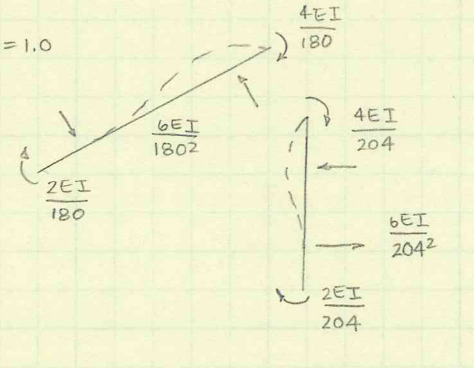


$$K_{11} = \frac{4EI}{96} + \frac{4EI}{180}$$ ✓

$$K_{21} = \frac{2EI}{180}$$ ✓

$$K_{31} = \frac{-6EI}{96^2}$$ ✓

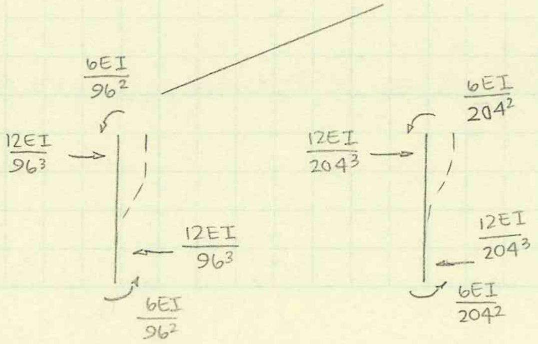
$u_2 = 1.0$



$$K_{22} = \frac{4EI}{180} + \frac{4EI}{204}$$ ✓

$$K_{32} = \frac{-6EI}{204^2} + \frac{6EI}{180^2}\left(-\frac{4}{5} + \frac{4}{5}\right)$$ ✓

$u_3 = 1.0$



$$K_{33} = \frac{12EI}{96^3} + \frac{12EI}{204^3}$$ ✓

HOMEWORK #8

2. (cont'd)

$$a)\ \underset{\sim}{K} = EI \begin{bmatrix} 1/24 + 1/45 & 1/90 & -1/1536 \\ 1/90 & 1/45 + 1/51 & -1.44\times10^{-4} \\ -1/1536 & -1.44\times10^{-4} & 1.50\times10^{-4} \end{bmatrix}$$ ✓

$$b)\ \underset{\sim}{F} = \begin{bmatrix} 0 \\ M \\ H \end{bmatrix}$$ ✓

how?

-1

University of Texas
Department of Civil Engineering

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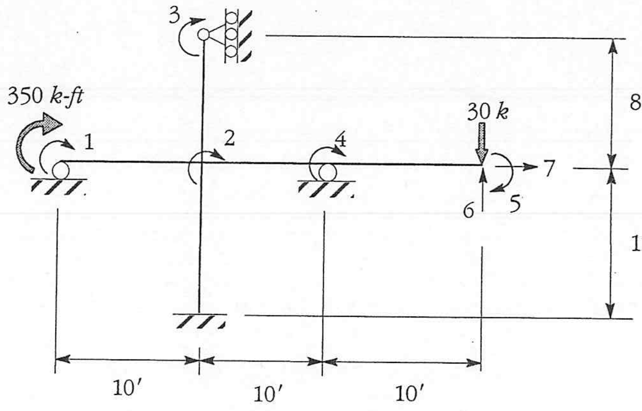
CE 363 Advanced Structural Analysis

Homework 8

Due: 17 NOV

1. The frame shown in the sketch is kinematically indeterminate to the seventh degree. The degrees of freedom are numbered as shown. Assuming that all members have the same properties, perform the following computations:

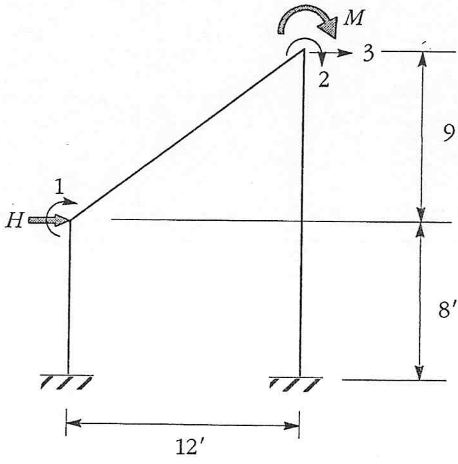
   (a). Develop the stiffness matrix ***K***.
   (b). Develop the load vector ***F***.
   (c). Determine the displacement vector ***u***.
   (d). Draw the shear force, axial force, and bending moment diagrams for the structure.



$E = 29,000\ ksi \qquad I = 288\ in^4$

2. Assuming all members are axially rigid, the frame shown to the right is kinematically indeterminate to the third degree. Using the degrees of freedom shown on the sketch, perform the following computations:

   (a). Develop the stiffness matrix ***K***.
   (b). Develop the load vector ***F***.



CE363: STRUCT. ANALYSIS

CATHERINE HOVELL
29 November 05

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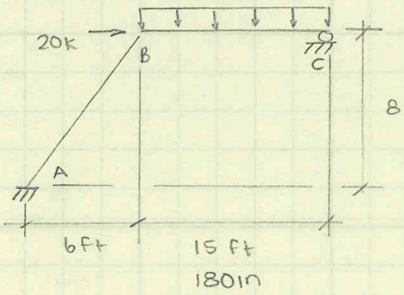
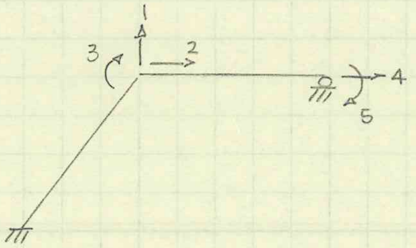
HOMEWORK #9

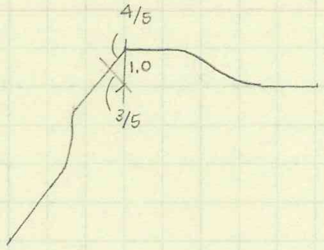
unconstrained case

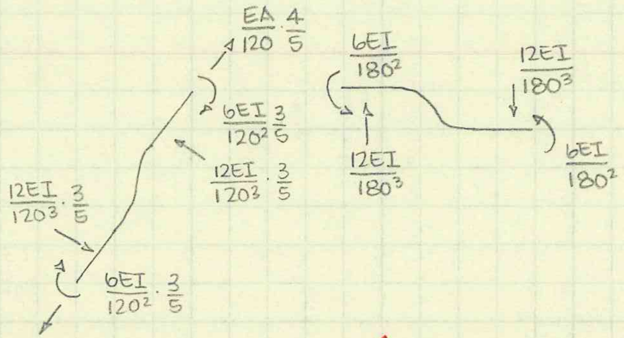




E=29000 ksi
I=288 in⁴
A=24 in²

$u_1 = 1.0$





$$k_{11} = \frac{12EI}{180^3}(1) + \frac{12EI}{120^3}\cdot\frac{3}{5}\cdot\frac{3}{5} + \frac{EA}{120}\cdot\frac{4}{5}\cdot\frac{4}{5}$$

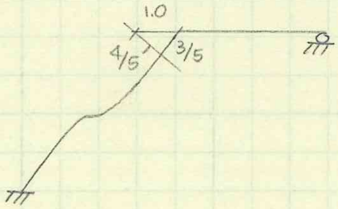
$$k_{12} = \frac{EA}{120}\cdot\frac{4}{5}\cdot\frac{3}{5} - \frac{12EI}{120^3}\cdot\frac{3}{5}\cdot\frac{4}{5}$$

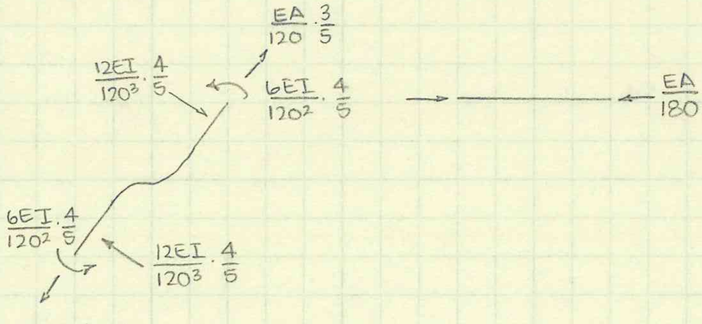
$$k_{13} = \frac{6EI}{120^2}\cdot\frac{3}{5} - \frac{6EI}{180^2}(1)$$

$$k_{14} = 0$$

$$k_{15} = \frac{-6EI}{180^2}$$

$u_2 = 1.0$





$$k_{21} = \frac{EA}{120}\cdot\frac{3}{5}\cdot\frac{4}{5} - \frac{12EI}{120^3}\cdot\frac{4}{5}\cdot\frac{3}{5}$$

$$k_{22} = \frac{EA}{120}\cdot\frac{3}{5}\cdot\frac{3}{5} + \frac{EA}{180} + \frac{12EI}{120^3}\cdot\frac{4}{5}\cdot\frac{4}{5}$$

$$k_{23} = \frac{-6EI}{120^2}\cdot\frac{4}{5}$$

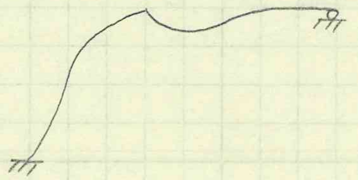
$$k_{24} = \frac{-EA}{180}$$

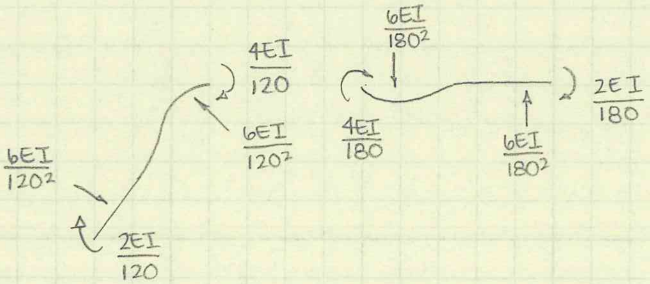
$$k_{25} = 0$$

HOMEWORK #9

Unconstrained case

$u_3 = 1.0$





$$k_{31} = \frac{6EI}{120^2}\cdot\frac{3}{5} - \frac{6EI}{180^2}$$

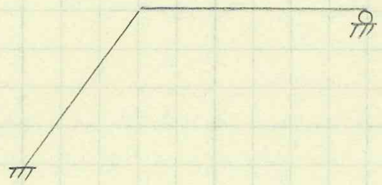
$$k_{32} = \frac{-6EI}{120^2}\cdot\frac{4}{5}$$

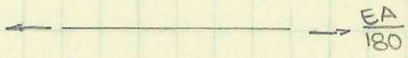
$$k_{33} = \frac{4EI}{120} + \frac{4EI}{180}$$

$$k_{34} = 0$$

$$k_{35} = \frac{2EI}{180}$$

$u_4 = 1.0$





$$k_{41} = 0$$

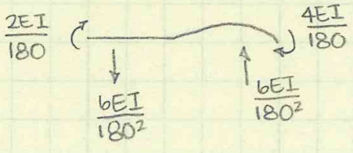
$$k_{42} = -\frac{EA}{180}$$

$$k_{43} = 0$$

$$k_{44} = \frac{EA}{180}$$

$$k_{45} = 0$$

$u_5 = 1.0$



$$k_{51} = \frac{-6EI}{180^2}$$

$$k_{52} = 0$$

$$k_{53} = \frac{2EI}{180}$$

$$k_{54} = 0$$

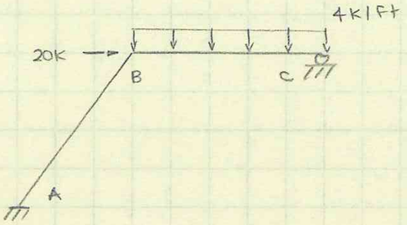
$$k_{55} = \frac{4EI}{180}$$

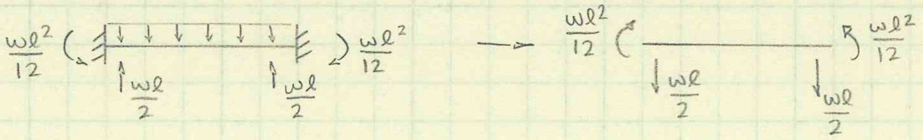
HOMEWORK #9

Stiffness Matrix $\underset{\sim}{K}$ (a)

$$\underset{\sim}{K} = \begin{bmatrix} 3750.1 & 2756.2 & 541.33 & 0 & -1546.7 \\ 2756.2 & 5991.8 & -2784 & -3866.7 & 0 \\ 541.33 & -2784 & 464000 & 0 & 92800 \\ 0 & -3866.7 & 0 & 3866.7 & 0 \\ -1547 & 0 & 92800 & 0 & 185600 \end{bmatrix}$$

Force vector calcs.





$$\frac{wl}{2} = \frac{1}{2}(4k/ft)(15ft) = 30k$$

$$\frac{wl^2}{12} = \frac{1}{12}(4k/ft)(15ft)^2(12in/ft) = 900k \cdot in$$

Force & Displacement vectors (b)(c)

$$\underset{\sim}{F} = \begin{bmatrix} -30 \\ 20 \\ 900 \\ 0 \\ -900 \end{bmatrix} \begin{matrix} k \\ k \\ k \cdot in \\ k \\ k \cdot in \end{matrix} \qquad \underset{\sim}{K}\underset{\sim}{u} = \underset{\sim}{F} \qquad \underset{\sim}{u} = \begin{bmatrix} -0.7675 \\ 1.02115 \\ 0.0125 \\ 1.02115 \\ -0.0175 \end{bmatrix} \begin{matrix} in \\ in \\ rad \\ in \\ rad \end{matrix}$$

HOMEWORK #9

Forces

A-B

$$F_{AX}= \frac{EA}{120}\cdot\frac{4}{5}\,u_1 + \frac{EA}{120}\cdot\frac{3}{5}\,u_2 = \frac{(29000\,\text{KSI})(24\,\text{in}^2)}{5(120\,\text{in})}\left[(4)(-0.7675\,\text{in})+3(1.02115\,\text{in})\right]$$

$$= 7.66\ \text{K} \quad \text{comp.}$$

$$M_A=\left[\frac{6EI}{120^2}\cdot\frac{3}{5}\right]u_1+\left[\frac{-6EI}{120^2}\cdot\frac{4}{5}\right]u_2+\left[\frac{2EI}{120}\right]u_3=-2711.5\ \text{k·in}$$

$$M_A = 226.0\ \text{k·ft}\ ↺$$ ✓

$$F_{VA}=\left[-\frac{12EI}{120^3}\cdot\frac{3}{5}\right]u_1+\left[\frac{12EI}{120^3}\cdot\frac{4}{5}\right]u_2+\left[-\frac{6EI}{120^2}\right]u_3$$

$$F_{VA}= 30.74\ \text{K}\ ↖$$ ✓

$$M_B=\left[\frac{6EI}{120^2}\cdot\frac{3}{5}\right]u_1+\left[\frac{-6EI}{120^2}\cdot\frac{4}{5}\right]u_2+\left[\frac{4EI}{120}\right]u_3$$

$$M_B=-977.5\ \text{k·in} = 81.46\ \text{k·ft}\ ↻$$ ✓

$$F_{VB}=\left[\frac{12EI}{120^3}\cdot\frac{3}{5}\right]u_1+\left[\frac{-12EI}{120^3}\cdot\frac{4}{5}\right]u_2+\left[\frac{6EI}{120^2}\right]u_3$$

$$F_{VB}=30.74\ \text{K}\ ↘$$ ✓

B-C

$$F_{AX}=\frac{-EA}{180}u_2+\frac{EA}{180}u_4=0$$

$$M_B=\left[\frac{-6EI}{180^2}\right]u_1+\left[\frac{4EI}{180}\right]u_3+\left[\frac{2EI}{180}\right]u_5-900\ \text{k·in}=977.51\ \text{k·in}=81.46\ \text{k·ft}\ ↷$$

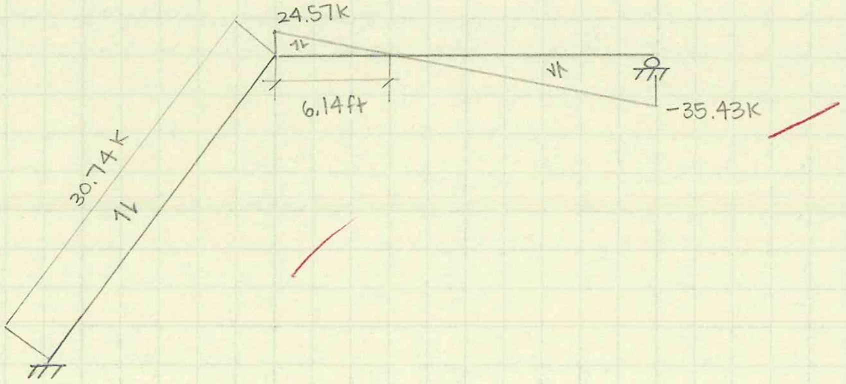
$$M_c=\left[\frac{-6EI}{180^2}\right]u_1+\left[\frac{2EI}{180}\right]u_3+\left[\frac{4EI}{180}\right]u_5+900\ \text{k·in}=0$$ ✓

$$F_{VB}=\left[\frac{12EI}{180^3}\right]u_1+\left[\frac{-6EI}{180^2}\right]u_3+\left[-\frac{6EI}{180^2}\right]u_5+30\ \text{K}=24.57\ \text{K UP}$$ ✓

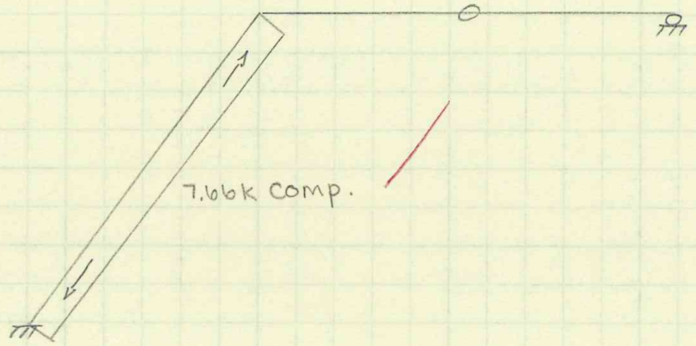
$$F_{VC}=\left[\frac{-12EI}{180^3}\right]u_1+\left[\frac{6EI}{180^2}\right]u_3+\left[\frac{6EI}{180^2}\right]u_5+30\ \text{K}=35.43\ \text{K UP}$$ ✓

HOMEWORK #9

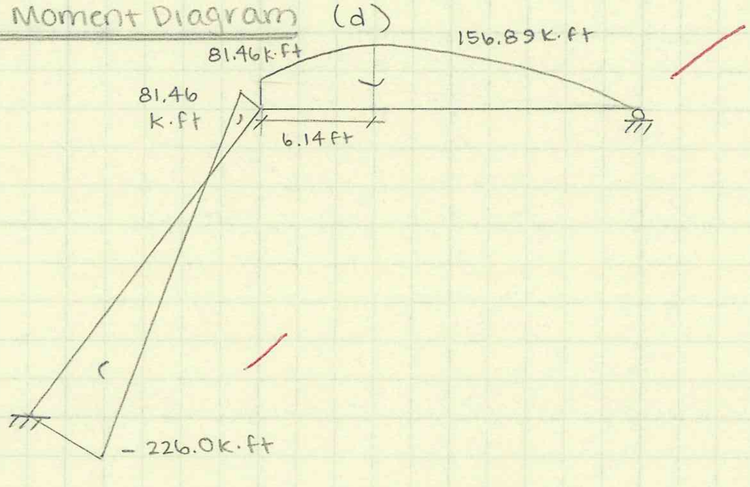
Shear Force Diagram (d)



Axial Force Diagram (d)



Bending Moment Diagram (d)



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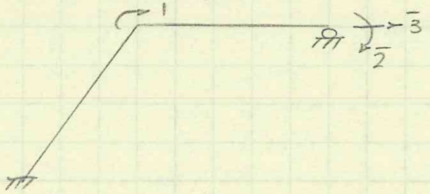
HOMEWORK #9

Gamma matrix (e)

$$\underset{\sim}{\Gamma} = \begin{bmatrix} 0 & 0 & -3/4 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

considering displaced shapes in constrained case, shown on the next page



$$\underset{\sim}{\bar{K}} = \underset{\sim}{\Gamma}^T \underset{\sim}{K} \underset{\sim}{\Gamma} \quad , \quad \underset{\sim}{\bar{F}} = \underset{\sim}{\Gamma}^T \underset{\sim}{F}$$

$$\underset{\sim}{\Gamma}^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -3/4 & 1 & 0 & 1 & 0 \end{bmatrix}$$

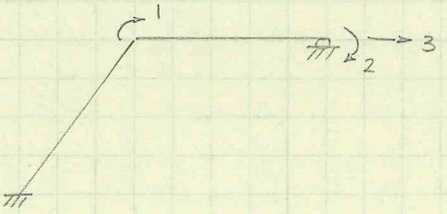
Calculated $\underset{\sim}{\bar{K}}, \underset{\sim}{\bar{F}}$ (f)

$$\underset{\sim}{\bar{K}} = \begin{bmatrix} 464000 & 92800 & -3190 \\ 92800 & 185600 & 1160 \\ -3190 & 1160 & 100.29 \end{bmatrix}$$

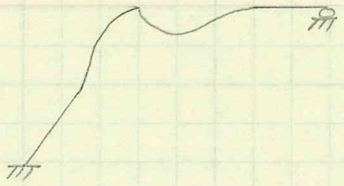
$$\underset{\sim}{\bar{F}} = \begin{bmatrix} 900 \\ -900 \\ 42.5 \end{bmatrix} \begin{matrix} k\cdot in \\ k\cdot in \\ k \end{matrix}$$

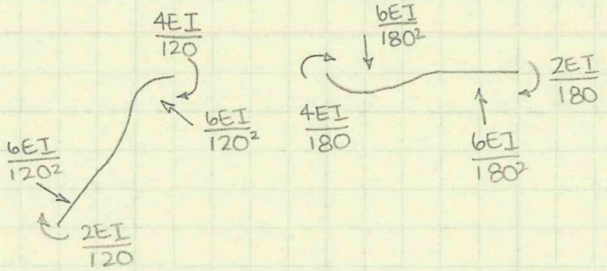
HOMEWORK #9

constrained case



$\bar{u}_1 = 1.0$



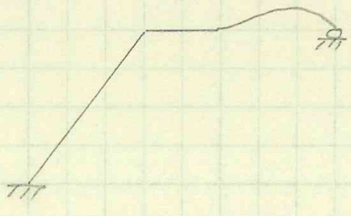


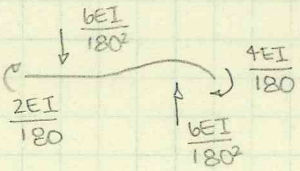
$$\bar{k}_{11} = \frac{4EI}{120} + \frac{4EI}{180}$$

$$\bar{k}_{21} = \frac{2EI}{180}$$

$$\bar{k}_{31} = \frac{6EI}{180^2} \cdot \frac{3}{4} - \frac{6EI}{120^2} \cdot \frac{5}{4}$$

$\bar{u}_2 = 1.0$



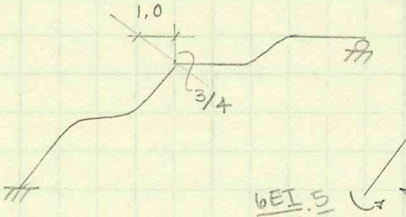


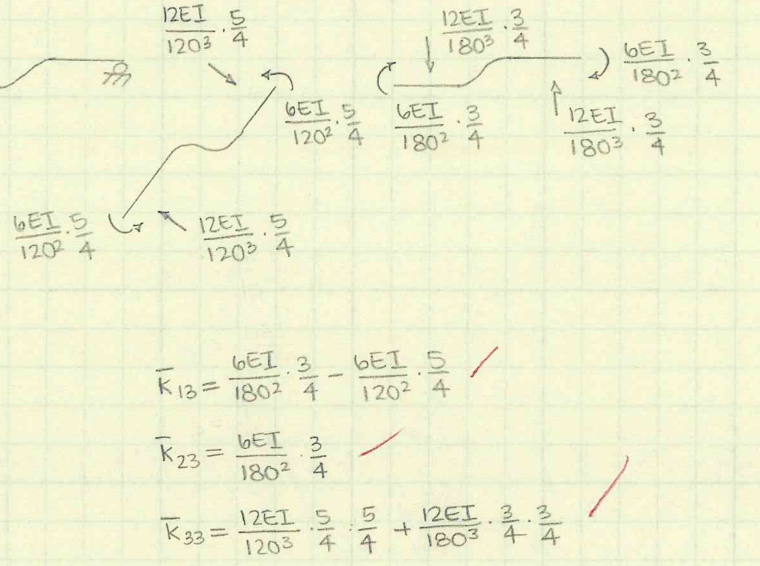
$$\bar{k}_{12} = \frac{2EI}{180}$$

$$\bar{k}_{22} = \frac{4EI}{180}$$

$$\bar{k}_{32} = \frac{6EI}{180^2} \cdot \frac{3}{4}$$

$\bar{u}_3 = 1.0$





$$\bar{k}_{13} = \frac{6EI}{180^2} \cdot \frac{3}{4} - \frac{6EI}{120^2} \cdot \frac{5}{4}$$

$$\bar{k}_{23} = \frac{6EI}{180^2} \cdot \frac{3}{4}$$

$$\bar{k}_{33} = \frac{12EI}{120^3} \cdot \frac{5}{4} \cdot \frac{5}{4} + \frac{12EI}{180^3} \cdot \frac{3}{4} \cdot \frac{3}{4}$$

$F_3 = 20k(1.0) + 30k(3/4) = 42.5k \rightarrow$
(from distributed load)

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HOMEWORK #9

constrained Stiffness matrix (g)

$$\bar{\underset{\sim}{K}} = \begin{bmatrix} 464000 & 92800 & -3190 \\ 92800 & 185600 & 1160 \\ -3190 & 1160 & 100.29 \end{bmatrix}$$

constrained Load vector (g)

$$\bar{\underset{\sim}{F}} = \begin{bmatrix} 900 \\ -900 \\ 42.5 \end{bmatrix} \begin{matrix} \text{K·in} \\ \text{K·in} \\ \text{K} \end{matrix}$$

* compare (g) and (f): they match, which they should ✓

University of Texas
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. Williamson

CE 363 Advanced Structural Analysis

Homework 9

Due: 29 NOV

Accounting for both axial and flexural deformations, the frame shown in Fig. 1 has five degrees of freedom. The structure is subjected to a uniformly distributed load of 4 *k/ft* over member *bc*, and a lateral force of 20*k* at joint *b*. All of the members have area *A*, moment of inertia *I*, and modulus of elasticity *E*, with numerical values as indicated.

(a). Develop the stiffness matrix **K**.
(b). Develop the load vector **F**.
(c). Determine the displacement vector **u**.
(d). Draw the shear force, axial force, and bending moment diagrams for the structure.

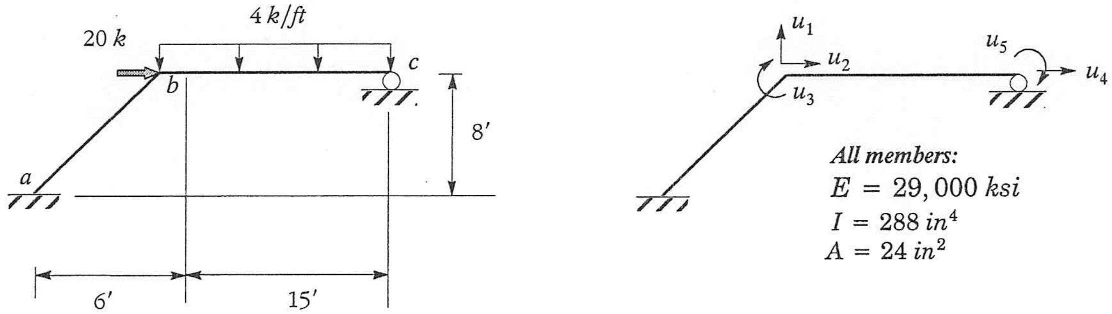

Fig. 1. Description of structure and loading

If we assume that both members are axially rigid, the number of degrees of freedom for the structure is reduced from five to three.

(e). Determine the matrix **Γ** that relates the unconstrained degrees of freedom to the constrained degrees of freedom.

should be the same

(f). Using the matrix **Γ** developed in *(e)*, compute the stiffness matrix and load vector for the constrained system.
(g). Develop the stiffness matrix and load vector directly using the motions associated with the constrained degrees of freedom. Your answer should match the solution obtained in *(f)*.

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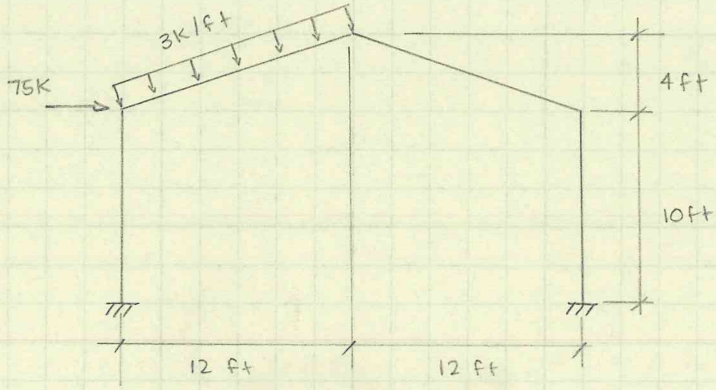
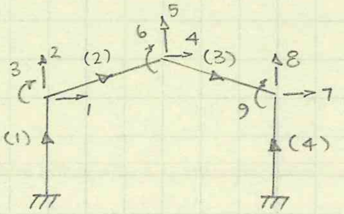
HOMEWORK #10





$b = 1.5\,in$, $d = 15\,in$ > $A = 22.5\,in^2$, $I = 421.875\,in^4$

$E = 29000\,ksi$

General forms/equations:

For each member:

$$k_{local}(E,A,L,I) := \begin{pmatrix} \frac{E\cdot A}{L} & 0 & 0 & \frac{-E\cdot A}{L} & 0 & 0 \\ 0 & \frac{12\cdot E\cdot I}{L^3} & \frac{-6\cdot E\cdot I}{L^2} & 0 & \frac{-12\cdot E\cdot I}{L^3} & \frac{-6\cdot E\cdot I}{L^2} \\ 0 & \frac{-6\cdot E\cdot I}{L^2} & \frac{4\cdot E\cdot I}{L} & 0 & \frac{6\cdot E\cdot I}{L^2} & \frac{2\cdot E\cdot I}{L} \\ \frac{-E\cdot A}{L} & 0 & 0 & \frac{E\cdot A}{L} & 0 & 0 \\ 0 & \frac{-12\cdot E\cdot I}{L^3} & \frac{6\cdot E\cdot I}{L^2} & 0 & \frac{12\cdot E\cdot I}{L^3} & \frac{6\cdot E\cdot I}{L^2} \\ 0 & \frac{-6\cdot E\cdot I}{L^2} & \frac{2\cdot E\cdot I}{L} & 0 & \frac{6\cdot E\cdot I}{L^2} & \frac{4\cdot E\cdot I}{L} \end{pmatrix}$$

$$T = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$ where $c = \cos\theta = \frac{x_j - x_i}{L}$, $s = \sin\theta = \frac{y_j - y_i}{L}$

& $L = \left[(x_i - x_j)^2 + (y_i - y_j)^2\right]^{1/2}$

HOMEWORK #10

Member (1), (4) – different x, y values, assembly points, but the same otherwise.

x_I^1	0	y_I^1	0	L =	120 in
x_J^1	0	y_J^1	120		

x_I^4	288	y_I^4	0	L =	120 in
x_J^4	288	y_J^4	120		

$$k^1_{local} = \begin{vmatrix} 3750 & 0 & 0 & -3750 & 0 & 0 \\ 0 & 58.59 & -3515.63 & 0 & -58.59 & -3515.625 \\ 0 & -3515.63 & 281250 & 0 & 3515.63 & 140625 \\ -3750 & 0 & 0 & 3750 & 0 & 0 \\ 0 & -58.59 & 3515.63 & 0 & 58.59 & 3515.63 \\ 0 & -3515.63 & 140625 & 0 & 3515.63 & 281250 \end{vmatrix} = k^4_{local}$$

$$T_1 = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = T_4$$

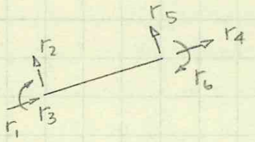
$$T_1^T = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = T_4^T$$

$$k^1_{global} = \begin{array}{|cccccc|c} - & - & - & 1 & 2 & 3 & \\ 58.59 & 0 & 3515.63 & -58.59 & 0 & 3515.63 & - \\ 0 & 3750 & 0 & 0 & -3750 & 0 & - \\ 3515.63 & 0 & 281250 & -3515.63 & 0 & 140625 & - \\ -58.59 & 0 & -3515.63 & 58.59 & 0 & -3515.63 & 1 \\ 0 & -3750 & 0 & 0 & 3750 & 0 & 2 \\ 3515.63 & 0 & 140625 & -3515.63 & 0 & 281250 & 3 \end{array}$$

$$k^4_{global} = \begin{array}{|cccccc|c} - & - & - & 7 & 8 & 9 & \\ 58.59 & 0 & 3515.63 & -58.59 & 0 & 3515.63 & - \\ 0 & 3750 & 0 & 0 & -3750 & 0 & - \\ 3515.63 & 0 & 281250 & -3515.63 & 0 & 140625 & - \\ -58.59 & 0 & -3515.63 & 58.59 & 0 & -3515.63 & 7 \\ 0 & -3750 & 0 & 0 & 3750 & 0 & 8 \\ 3515.63 & 0 & 140625 & -3515.63 & 0 & 281250 & 9 \end{array}$$

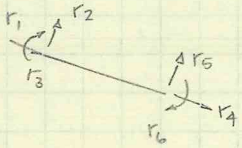
HOMEWORK #10

Member (2), (3)



x_I^2	0	y_I^2	120
x_J^2	144	y_J^2	168

L = 151.7893 in



x_I^3	144	y_I^3	168
x_J^3	288	y_J^3	120

L = 151.7893 in

$$k^2_{local}=\begin{vmatrix} 2964.64 & 0 & 0 & -2964.64 & 0 & 0 \\ 0 & 28.95 & -2197.27 & 0 & -28.95 & -2197.27 \\ 0 & -2197.27 & 222347.6 & 0 & 2197.27 & 111173.8 \\ -2964.64 & 0 & 0 & 2964.635 & 0 & 0 \\ 0 & -28.95 & 2197.27 & 0 & 28.95 & 2197.27 \\ 0 & -2197.27 & 111173.8 & 0 & 2197.27 & 222347.6 \end{vmatrix} = k^3_{local}$$

$$T_2=\begin{vmatrix} 0.949 & 0.316 & 0 & 0 & 0 & 0 \\ -0.316 & 0.949 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.949 & 0.316 & 0 \\ 0 & 0 & 0 & -0.316 & 0.949 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = T_3^T$$

$$T_2^T=\begin{vmatrix} 0.949 & -0.316 & 0 & 0 & 0 & 0 \\ 0.316 & 0.949 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.949 & -0.316 & 0 \\ 0 & 0 & 0 & 0.316 & 0.949 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = T_3$$

$$k^2_{global}=\begin{array}{|cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ 2671.1 & 880.7 & 694.8 & -2671.1 & -880.7 & 694.8 & 1 \\ 880.7 & 322.5 & -2084.5 & -880.7 & -322.5 & -2084.5 & 2 \\ 694.8 & -2084.5 & 222347.6 & -694.8 & 2084.5 & 111173.8 & 3 \\ -2671.1 & -880.7 & -694.8 & 2671.1 & 880.7 & -694.8 & 4 \\ -880.7 & -322.5 & 2084.5 & 880.7 & 322.5 & 2084.5 & 5 \\ 694.8 & -2084.5 & 111173.8 & -694.8 & 2084.5 & 222347.6 & 6 \end{array}$$

$$k^3_{global}=\begin{array}{|cccccc|c} 4 & 5 & 6 & 7 & 8 & 9 & \\ 2671.1 & -880.7 & -694.8 & -2671.1 & 880.7 & -694.8 & 4 \\ -880.7 & 322.5 & -2084.5 & 880.7 & -322.5 & -2084.5 & 5 \\ -694.8 & -2084.5 & 222347.6 & 694.8 & 2084.5 & 111173.8 & 6 \\ -2671.1 & 880.7 & 694.8 & 2671.1 & -880.7 & 694.8 & 7 \\ 880.7 & -322.5 & 2084.5 & -880.7 & 322.5 & 2084.5 & 8 \\ -694.8 & -2084.5 & 111173.8 & 694.8 & 2084.5 & 222347.6 & 9 \end{array}$$

CE 363: STRUCT. ANALYSIS

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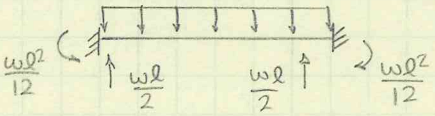
HOMEWORK #10

Load vector



$w = 3k/ft$

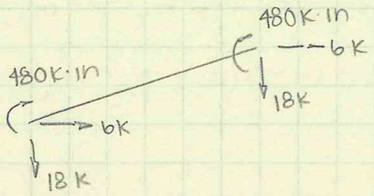
$\ell = 151.8\ in$

$$\frac{w\ell}{2} = \frac{(0.25k/in)(151.8in)}{2} = 18.97k$$

$$\frac{w\ell^2}{12} = \frac{(0.25k/in)}{12}(151.8in) = 480k\cdot in$$

$$f^2_{local} = \begin{bmatrix} 0 \\ -18.97 \\ 480 \\ 0 \\ -18.97 \\ -480 \end{bmatrix}$$

$$F^{(2)}_{glob} = T^T_2 f^2_{loc} = \begin{array}{cc} \begin{bmatrix} 6 \\ -18 \\ 480 \\ 6 \\ -18 \\ 480 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$



$$F_{global} = \begin{bmatrix} 6+75 \\ -18 \\ 480 \\ 6 \\ -18 \\ 480 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Assemble Global K matrix

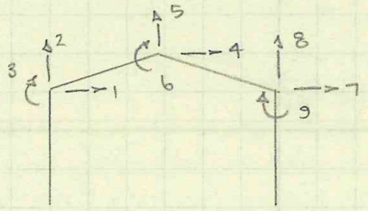
$$K_{global} = \begin{vmatrix} 2729.7 & 880.7 & -2820.8 & -2671.1 & -880.7 & 694.8 & 0 & 0 & 0 \\ 880.7 & 4072.5 & -2084.5 & -880.7 & -322.5 & -2084.5 & 0 & 0 & 0 \\ -2820.8 & -2084.5 & 503597.6 & -694.8 & 2084.5 & 111173.8 & 0 & 0 & 0 \\ -2671.1 & -880.7 & -694.8 & 5342.1 & 0.0 & -1389.7 & -2671.1 & 880.7 & -694.8 \\ -880.7 & -322.5 & 2084.5 & 0.0 & 645.0 & 0.0 & 880.7 & -322.5 & -2084.5 \\ 694.8 & -2084.5 & 111173.8 & -1389.7 & 0.0 & 444695.3 & 694.8 & 2084.5 & 111173.8 \\ 0 & 0 & 0 & -2671.1 & 880.7 & 694.8 & 2729.7 & -880.7 & -2820.8 \\ 0 & 0 & 0 & 880.7 & -322.5 & 2084.5 & -880.7 & 4072.5 & 2084.5 \\ 0 & 0 & 0 & -694.8 & -2084.5 & 111173.8 & -2820.8 & 2084.5 & 503597.6 \end{vmatrix}$$

HOMEWORK #10

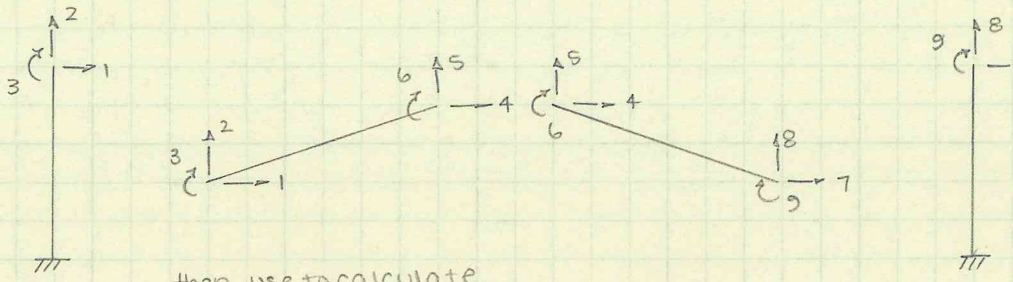
Deflection / Rotation Calcs

$\underset{\sim}{K}\,\underset{\sim}{u}=\underset{\sim}{F}$, so:

$$u=\left|\begin{array}{r|l}1.540641 & \text{in}\\ -0.003732 & \text{in}\\ 0.012875 & \text{rad}\\ 1.497427 & \text{in}\\ 0.090839 & \text{in}\\ -0.007329 & \text{rad}\\ 1.448553 & \text{in}\\ -0.005868 & \text{in}\\ 0.012198 & \text{rad}\end{array}\right.$$



Disassemble from global displacements back to local



then, use to calculate

$R=k_{glob}\cdot r$: global member forces

$Q=T\cdot R$: local member forces

$$r^1=\left|\begin{array}{r}0\\0\\0\\1.540641\\-0.003732\\0.012875\end{array}\right|\begin{array}{l}\text{in}\\\text{in}\\\text{rad}\\\text{in}\\\text{in}\\\text{rad}\ \ (\text{typ})\end{array}\qquad R^1=\left|\begin{array}{r}-45.0075\\13.99639\\-3605.739\\45.0075\\-13.99639\\-1795.161\end{array}\right|\begin{array}{l}\text{k}\\\text{k}\\\text{k}\cdot\text{in}\\\text{k}\\\text{k}\\\text{k}\cdot\text{in}\ \ (\text{typ})\end{array}\qquad Q^1=\left|\begin{array}{r}13.99639\\45.0075\\-3605.739\\-13.99639\\-45.0075\\-1795.161\end{array}\right|\begin{array}{l}\text{axial}\\\text{shear}\\\text{moment}\\ \\ \\ \end{array}$$

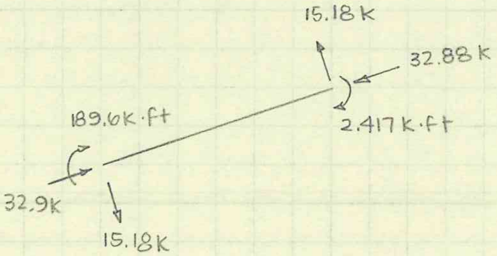
$$r^2=\left|\begin{array}{r}1.540641\\-0.003732\\0.012875\\1.497427\\0.090839\\-0.007329\end{array}\right|\qquad R^2=\left|\begin{array}{r}35.9925\\-4.003606\\2275.161\\-35.9925\\4.003606\\28.99805\end{array}\right|\qquad Q^2=\left|\begin{array}{r}32.87943\\-15.17998\\2275.161\\-32.87943\\15.17998\\28.99805\end{array}\right|$$

$$r^3=\left|\begin{array}{r}1.497427\\0.090839\\-0.007329\\1.448553\\-0.005868\\0.012198\end{array}\right|\qquad R^3=\left|\begin{array}{r}41.9925\\-22.00361\\-508.9981\\-41.9925\\22.00361\\1661.877\end{array}\right|\qquad Q^3=\left|\begin{array}{r}46.79573\\-7.595259\\-508.9981\\-46.79573\\7.595259\\1661.877\end{array}\right|$$

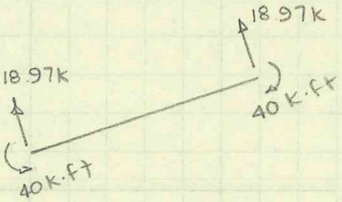
$$r^4=\left|\begin{array}{r}0\\0\\0\\1.448553\\-0.005868\\0.012198\end{array}\right|\qquad R^4=\left|\begin{array}{r}-41.9925\\22.00361\\-3377.223\\41.9925\\-22.00361\\-1661.877\end{array}\right|\qquad Q^4=\left|\begin{array}{r}22.00361\\41.9925\\-3377.223\\-22.00361\\-41.9925\\-1661.877\end{array}\right|$$

HOMEWORK #10

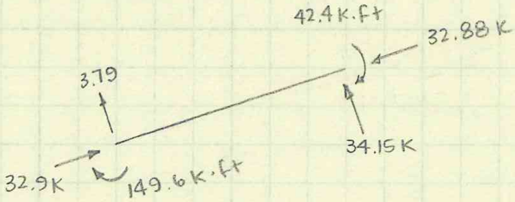
Beam (2) Force resolution



Fixed·end forces



combination



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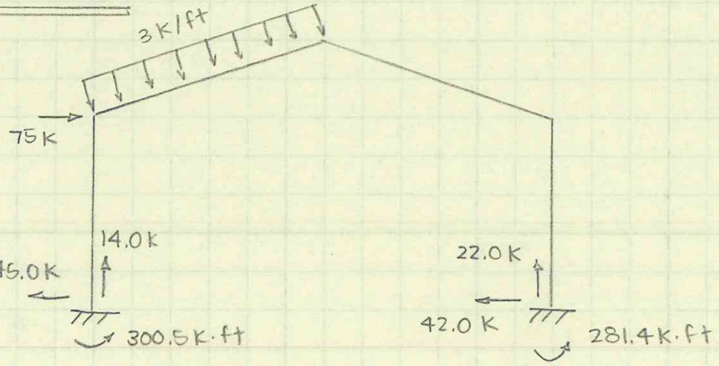
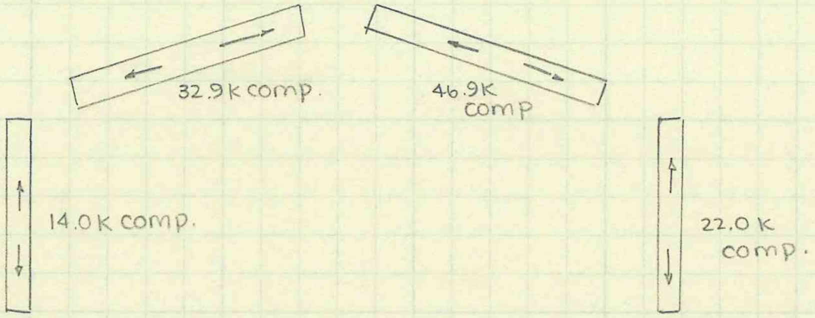
HOMEWORK #10

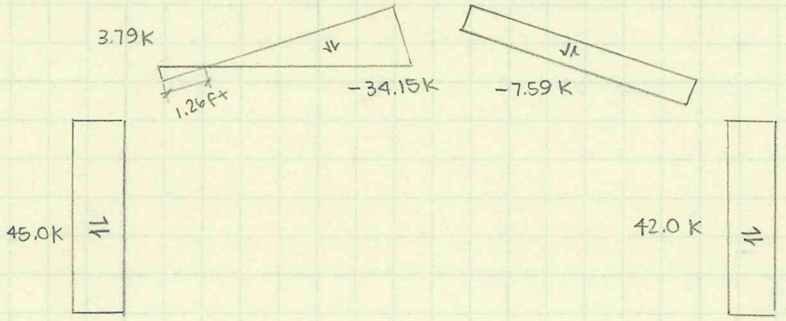
Reaction Forces



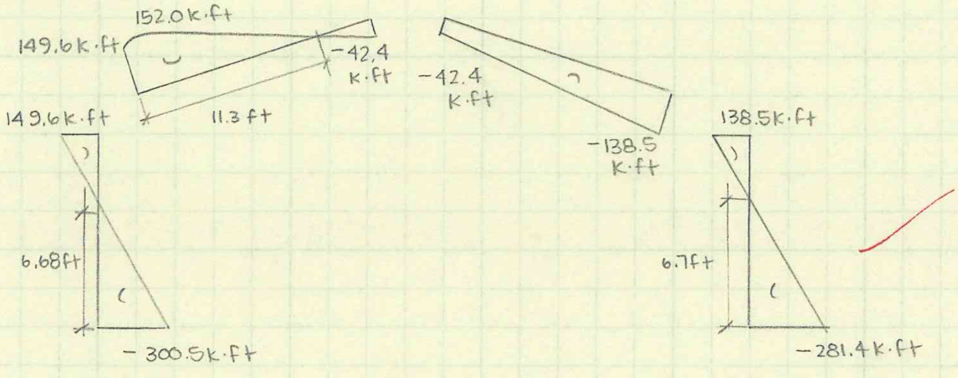
Axial Force



Shear Force

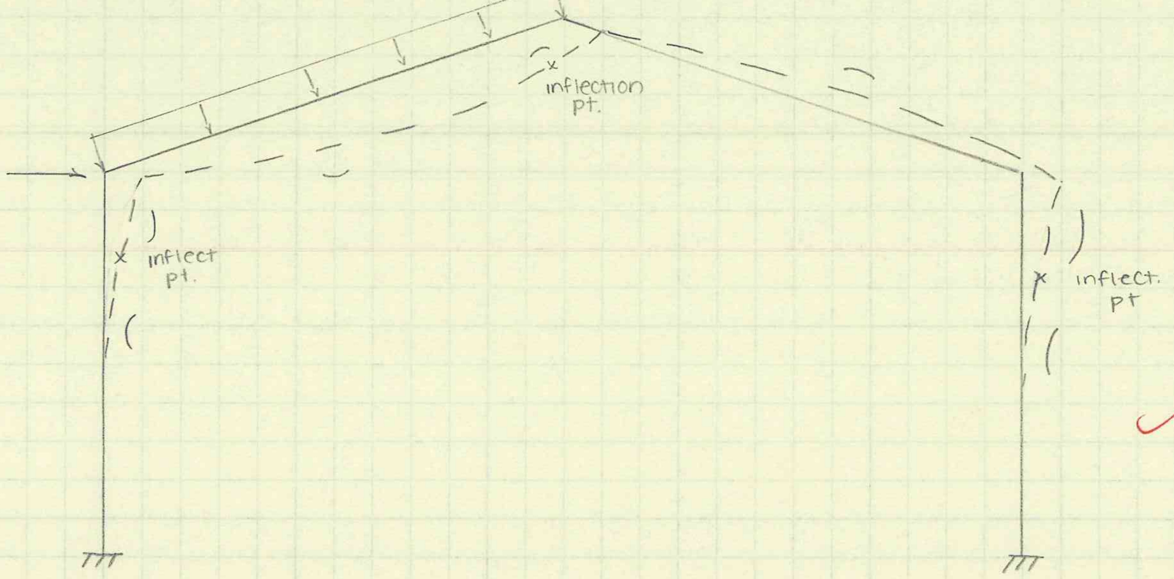


Bending Moment



HOMEWORK #10

Deflected Shape



University of Texas
Dept. of Civil, Arch. & Env. Eng.

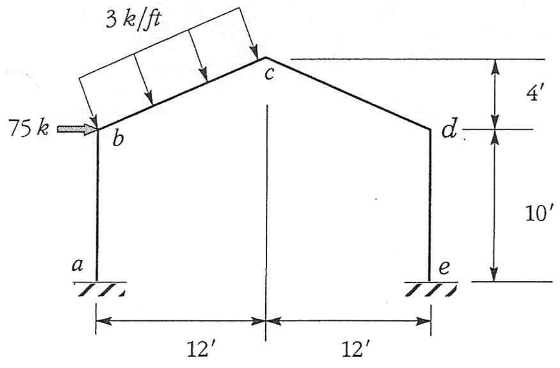
Instructor: E. Williamson

CE 363 Advanced Structural Analysis

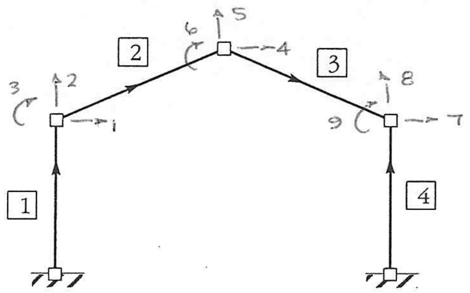
Homework 10 (OPTIONAL)

Due: 8 DEC

Accounting for axial and flexural deformations, the structure shown below has nine degrees of freedom. The member numbers and orientation relative to the global X-Y coordinate system are shown on the accompanying sketch. All members have the same rectangular cross-section with width $b = 1.5$ inches and depth $d = 15$ inches. In addition, all members are comprised of the same material with $E = 20{,}000\ ksi$. The structure is subjected to a uniform load (acting perpendicular to the member axis) and a concentrated horizontal force. Set up and solve the equilibrium equations $\boldsymbol{Ku} = \boldsymbol{F}$ to determine the rotations and displacements at the ends of each of the members. Find the reaction forces and plot the axial force, shear force, and bending moment diagrams. Sketch the deflected shape of the structure.



Description of structure and loading



Member Definitions

FUNDAMENTALS OF ANALYSIS

3 components to all problems

- compatability
- equilibrium
- constitution (material props)
  - assume linear elastic behavior; ignore #3

Equilibrium

forces = 0 or = ma

in this class, assume static response

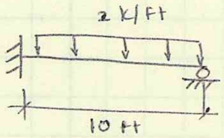
$\underset{\sim}{\Sigma F} = \underset{\sim}{0}$ - forces in 3 directions each = 0

$\underset{\sim}{\Sigma M} = \underset{\sim}{0}$

Determinacy (static —)

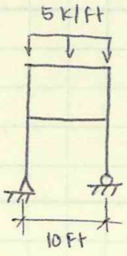
equilibrium equations are sufficient to determine all reactions and internal forces

Ex. 1

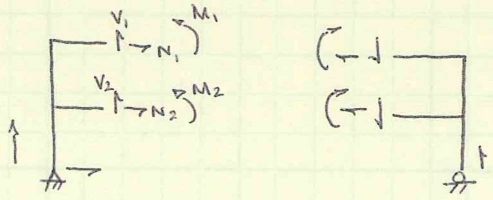


statically indeterminant to the 1st degree

Ex. 2



$R_A = R_B = 25$ k



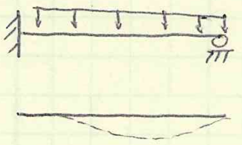
9 unknowns

6 equilibrium equations (3 eqns per body)

Compatibility (kinematics)

- motion of the structure needs to be a certain way
- all parts of the structure must stay connected correctly and be consistent with support conditions
  - ex. for a beam, plane sections remain plane
- internal compatibility satisfied if consistent with structural behavior model
  - eg. beam behavior
    - v = transferse displacement
    - θ = rotation = v'

Ex.



need to address all parts to get correct solution!

PROBLEM-SOLVING APPROACHES

1. Establish Equilibrium, force compatibility
   - flexibility method
     - assume reaction isn't there to find equilibrium
     - force-based approach

       redundant forces are primary unknowns

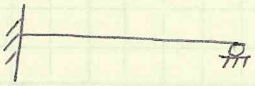
2. Establish compatibility, enforce equilibrium
   - stiffness (or displacement) based method
     - unknown: displacements

       ↳ rotations included (twist, too)

Kinematic determinancy

motion is completely known

consider all the possible ways it could move



how many ways can it move?

1° if beam cannot elongate

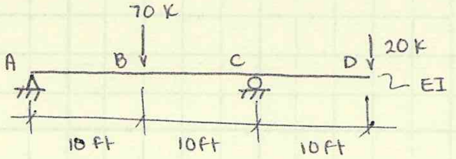
2° if it can — slope at rt. end

movement left / right

how many degrees of freedom?

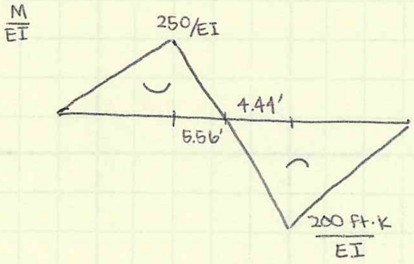
... number of unknowns

Moment Area Deflection Calculation example

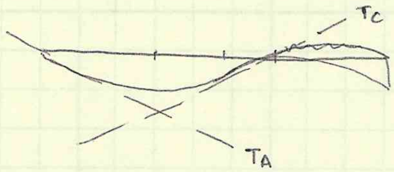


Find Δ at B and D, and rotation, θ_D

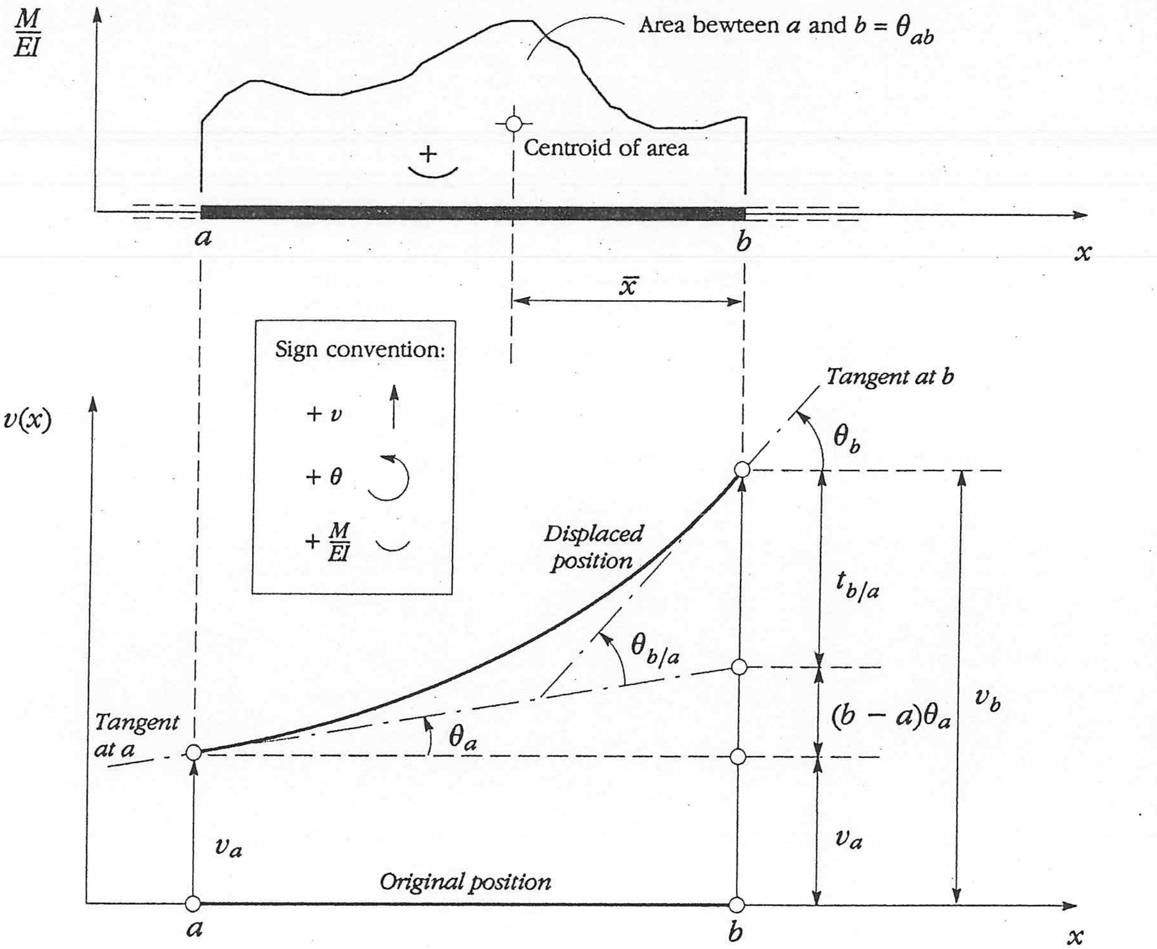
Moment-area: relates distance between 2 tangents and the angle between them



DEFLECTED SHAPE:



The Moment-Area Equations



Area under M/EI diagram:

$$\theta_{b/a} = \int_a^b \frac{M(x)}{EI(x)} dx$$

First moment of area under M/EI diagram about *b*:

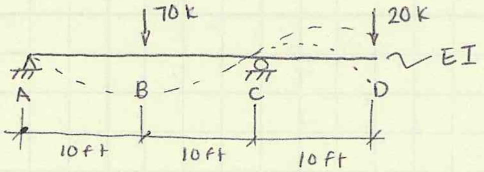
$$t_{b/a} = \int_a^b (b - x) \frac{M(x)}{EI(x)} dx = \bar{x} \cdot \theta_{b/a}$$

Angle change and deflection equations:

$$\theta_b = \theta_a + \int_a^b \frac{M(x)}{EI(x)} dx \qquad \theta_b = \theta_a + \theta_{b/a}$$

MOMENT · AREA METHOD

Example from last class

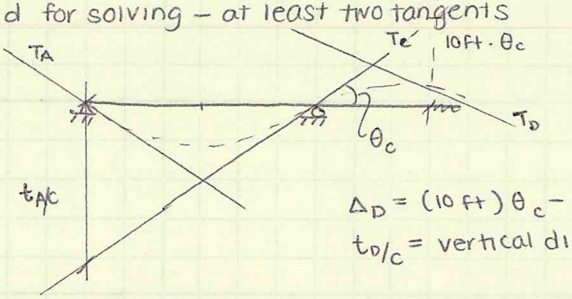


Moment diagram on Thursday's notes

- Is right end above or below unbent line?
- find $\Delta_B, \Delta_D, \theta_D$

Solve for Δ_D:

Need for solving – at least two tangents

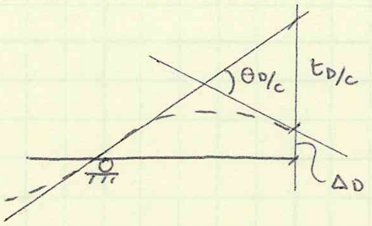


T_C, T_D are tangent lines

$$\theta_C = \frac{t_{A/C}}{20\text{ ft}}$$

$$\Delta_D = (10\text{ ft})\,\theta_C - t_{D/C}$$

$t_{D/C}$ = vertical distance between T_D and T_C at pt. D



$$\Delta_D = \tfrac{1}{2}\, t_{A/C} - |t_{D/C}|$$

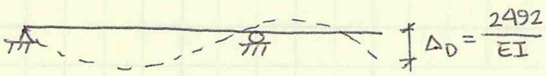
$$t_{A/C} = \frac{8349}{EI}$$

$$t_{D/C} = \frac{-6667}{EI}$$

(negative sign then ignored)

$$\boxed{\Delta_D = \frac{-2492}{EI}}$$

(–) signs means deflected shape goes below unbent line – go back and modify deflected shape



$$\theta_D = \theta_C + \theta_{D/C}$$

$\theta_{D/C}$ = area under M/EI curve between C and D

if $\theta_{D/C} > 0$ → going from tan C to tan D is counterclockwise

$$t_{A/C} = \tfrac{2}{3}(10\text{ft})\tfrac{1}{2}(10\text{ ft})\frac{250}{EI} + (10 + \tfrac{1}{3} \cdot 5.56\text{ ft})\tfrac{1}{2} \cdot (5.56\text{ft})\frac{250}{EI} + (15.56 + \tfrac{2}{3} \cdot 4.44\text{ft})\tfrac{1}{2}(4.44)\frac{-200}{EI}$$

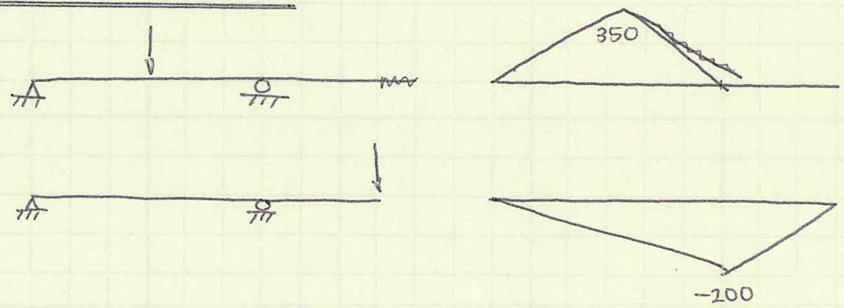
$t_{A/C}$ = (area of curve from A to C) · (moment arm from A)

$$\Delta_B = \|\theta_C (10\text{ft})\| - \|t_{B/C}\|$$

– draw tangent at B

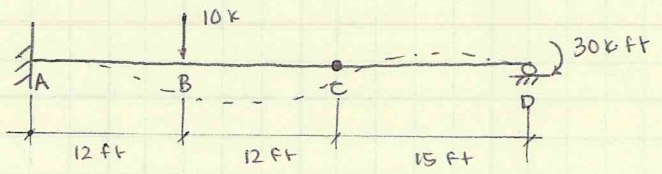
$$\theta_A = \frac{t_{C/A}}{20\text{ ft}}$$

USING SUPERPOSITION

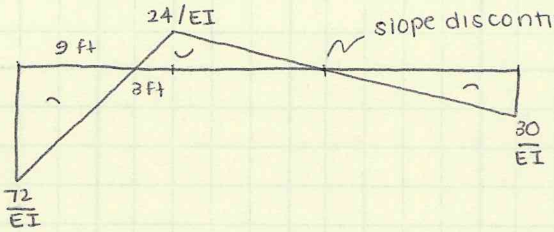


NEED:
- linear material response
- small deflections

Example #2 – with hinge



FIND: Δ_C, θ_{C_L}, θ_{C_R}



slope discontinuity – prevents calculating deflections or slopes across the hinge ($t_{D/A}$ = nothing)

$\Delta_C = t_{C/A}$

known:

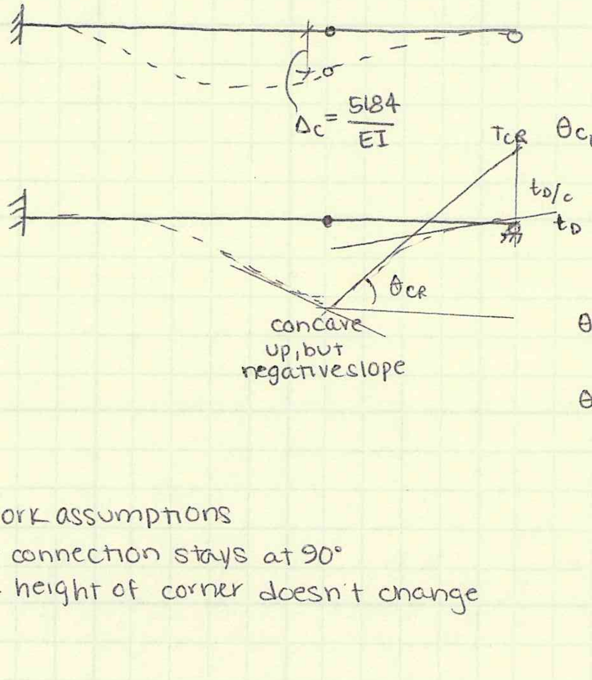
$\theta_A = 0$

$\Delta_A = 0$

$\Delta_D = 0$

$$\Delta_C = t_{C/A} = \tfrac{1}{2}(9\,ft)(-72/EI)(21\,ft) + \tfrac{1}{2}(3\,ft)(24/EI)(13\,ft) + \tfrac{1}{2}(12\,ft)(24/EI)(8\,ft) = -\frac{5184}{EI}$$

what does (−) mean? Point C moved down.



$\theta_{C_L} = \theta_{C_L/A} = \overset{-144}{\cancel{504}}/EI$

(−) number: rotation from A to C_L is clockwise NOT AS DRAWN.

$\theta_{CR} = \theta_D + \theta_{C/D}$, $\theta_D = \dfrac{\Delta_C - t_{C/D}}{15\,ft}$

OR

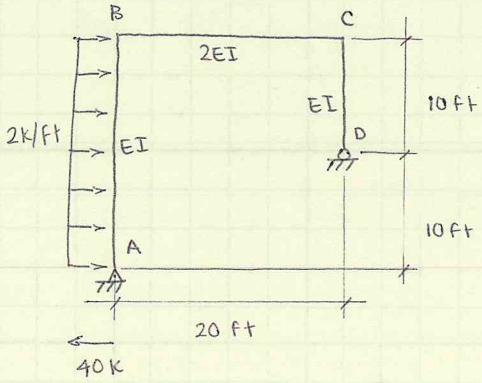
$\theta_{CR} = (\Delta_C + t_{D/C})/15\,ft$

keep as $\theta_{D/C}$, from left to right, because then we know something about sign vs. shape

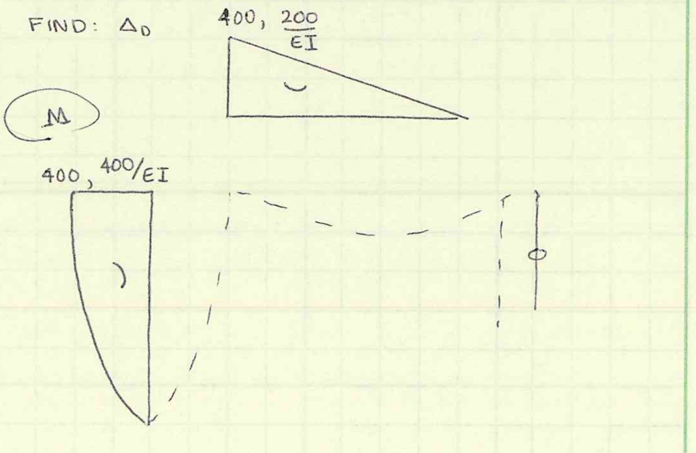
Homework assumptions
- connection stays at 90°
- height of corner doesn't change

MOMENT · AREA METHOD

Frame example



FIND: Δ_D

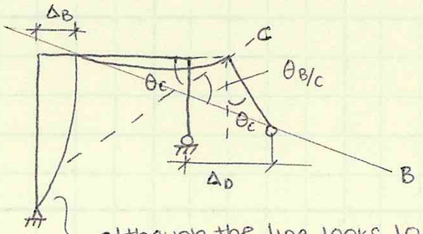


Assumptions:

- angle of intersection of the members stays the same before and after loading
- axial deformations can be ignored
- magnitude of deflections are "small"

AREA OF A PARABOLA $= 2/3\, ab$, if it includes the apex

Deflected shape



$\Delta_D = \Delta_B + \theta_C\,(10\text{ ft})$

although the line looks longer, we say it's not, and that $\Delta_y = 0$ at the top (for ease in calculations)

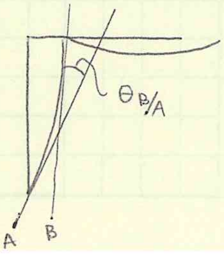
$$\theta_B = \frac{t_{C/B}}{20\text{ ft}}, \quad \theta_C = \theta_{B/C} - \theta_B, \quad \theta_C = \frac{t_{B/C}}{20\text{ ft}} = \frac{1}{20\text{ft}}\left[\frac{200}{EI}\left(\tfrac{1}{2}\right)\left(20\text{ ft}\right)^2\left(\tfrac{1}{3}\right)\right]$$

$\theta_{A/B}$ - axis goes to the right - get negative

USE: $\theta_{B/A} = \int_A^B f(x)\,dx$ from B to C, (+) means counterclockwise

$$\Delta_B = \theta_B\,(20\text{ ft}) + t_{A/B}, \quad \Delta_B = \theta_A\,(20\text{ ft}) + t_{B/A}$$

$$\theta_A = \theta_{B/A} + \theta_B$$



BE CAREFUL WITH SIGNAGE!

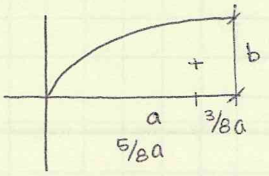
INTEGRALS AND GRAPHS

Using Numbers in Example

$\theta_{B/A} = \frac{1}{EI}\int_0^{20}\left(40x - \frac{2x^2}{2}\right)dx$, $\theta_{B/A} = 2/3\,(20ft)\left(\frac{400}{EI}\right)$

$t_{A/B} = (2/3)\left(\frac{400}{EI}\right)(20\,ft)(5/8)(20\,ft)$

$\llcorner\!\!\rightarrow$ for $t_{B/A}$, use 3/8 instead



$t_{A/B} = \frac{1}{EI}\int_0^{20} x\cdot(40x - x^2)\,dx$

$t_{B/A} = \frac{1}{EI}\int_0^{20}(20-x)(40x-x^2)\,dx$

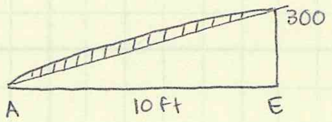
centroid located 5/8 the distance towards the apex of parabola

An example of where integrals are easier...

Δ_E (E = a point along AB) $= \theta_A(10\,ft) - t_{E/A} = \Delta_B - \left[10ft\cdot\theta_B + t_{E/B}\right]$

$t_{E/A} = \int_0^{10}(10-x)(40x-x^2)\frac{dx}{EI}$

Graphical Method Trick



E is not at the apex, can't use $A = 2/3ab$

break it into a triangle and a "sliver"

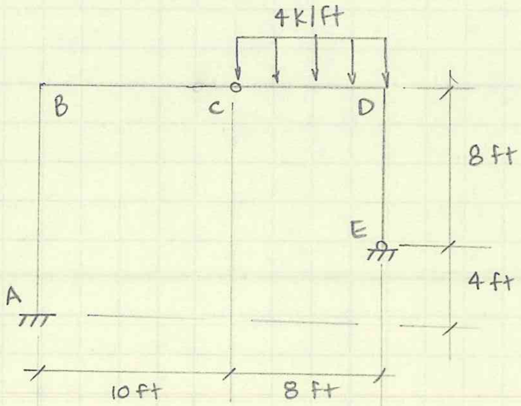
$A_{sliver} = \frac{\omega x^3}{12}$, with the centroid right in the middle

$A_{triangle} = \frac{1}{2}LM_{max}$

only works for a parabola (distributed load along beam) but will work at any point (even apex)

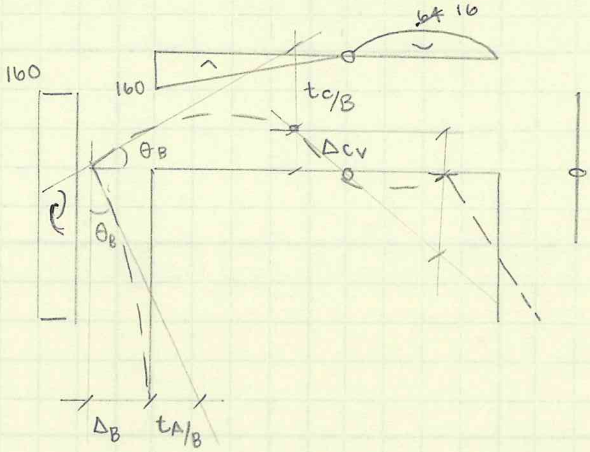
$\frac{1}{2}ab + \ldots$ crap, too lazy

PROBLEM IN·CLASS

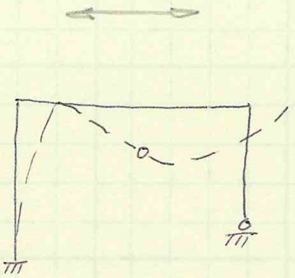


FIND: Δ_C – horizontal & vertical
θ_C – left and right
Δ_E

Δ_C = (115,20 left, 13,867 up)
θ_C = (1120 ↻, 1904 ↻)
Δ_E = 6443



B, C, D all move the same amount



$$\theta_B = \theta_{B/A} = \frac{160}{EI}(12) = \frac{1920}{EI}$$

$$\Delta_B = \theta_B(12\text{ft}) - t_{A/B} = \frac{1920}{EI}(12\text{ft}) - \frac{160}{EI}(12\text{ft})(6\text{ft})$$

$$\underline{\Delta_B = \frac{11520}{EI}} = \Delta_{C_H} = \Delta_D$$
left right

$$\Delta_{CV} = \theta_B(10\text{ft}) - t_{C/B} = \frac{1920}{EI}(10\text{ft}) - \frac{160}{EI}(10\text{ft})(1/2)\frac{2}{3}(10\text{ft})$$

$$\underline{\Delta_{CV} = \frac{13867}{EI} \text{ up}}$$

$\theta_{C_L} = \theta_{C/B} + \theta_B$ (negative)

$\theta_{C/B} = -800/EI$

$$\underline{\theta_{CL} = \frac{1120}{EI}}$$

$\Delta_{CV} + t_{D/C} = \theta_{CR}(8\text{ ft})$

$t_{D/C} = 2/3(8\text{ft})(64/EI)(1/2)(8\text{ft}) = \frac{1365}{EI}$

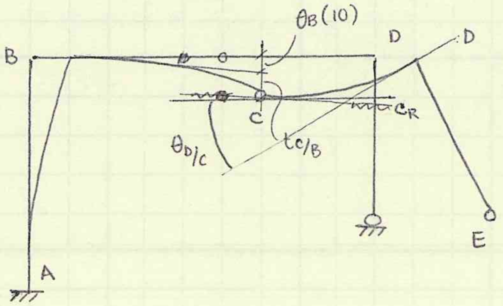
$$\underline{\theta_{CR} = \frac{1904}{EI}}$$

$$\theta_D = \theta_{CR} + \theta_{D/C} = \frac{1}{EI}(1904 + 341) = \frac{2245}{EI}$$

$$\underline{\Delta_E = \theta_D(8\text{ft}) - \Delta_B = \frac{6443}{EI}}$$

LOTS OF PROBLEMS.

MOMENT AREA WRAP-UP

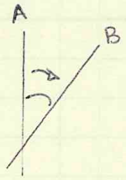


$\Delta_{C_H} = \Delta_B = t_{B/A} = \frac{11520}{EI}$

$\Delta_{CV} = \theta_B(10) + t_{C/B}$

$\theta_B = \theta_{B/A}$ — is this (+) or (−)?
negative bending moment,
so negative $\theta_{B/A}$

Negative $\theta_{B/A}$ means



from A to B is
clockwise!

however, use (+) number
in further calculations

$\theta_{CR} = \theta_D - \theta_{D/C}$
always put letter to
the right on top, to maintain
sign convention ⟶

$\theta_D > \theta_{D/C}$ C to D: counterclockwise,
so $\theta_{D/C}$ should be (+)
– moment is (+) ✓

(8) $\theta_D = t_{C/D} + \Delta_{CV}$

$\theta_{CL} = \frac{1}{10}(t_{B/C} + \Delta_{CV})$, $\theta_{CL} = \theta_B + \theta_{C/B}$ calculate both ways and check

$\Delta_E = \Delta_B + \theta_D(8)$

Some parts done right; others,
not so much. Key step: good
and clear moment diagram,
and deflected shape

VIRTUAL WORK

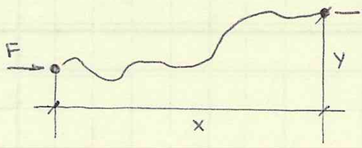
Deflection analysis

- relate internal virtual work to external virtual work
- use it to compute beam deflections

Virtual work

imagined or fake work

$W = Fd$ (force times distance)



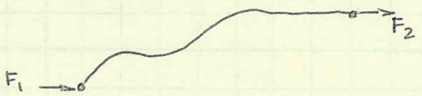
$W = Fx$ (force not in y-direction)

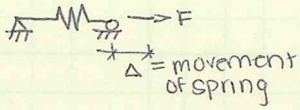
Needs:

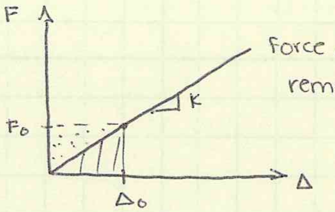
- constant force

or, use more rigorous math (integrate over path)

$W = \int_0^x F(x)\,dx$







Force does not remain constant

from $\Delta = 0$ to $\Delta = \Delta_0$, what is the work done?

Area under the curve

$W = \frac{1}{2} F_0 \Delta_0$ or $W = \int_0^{\Delta_0} kx\,dx$

$W = \frac{1}{2} k \Delta_0^2$

work: scalar quantity

(force & displacement are both vectors)

complimentary work, W^*

- area above the curve
- indicated with ⁙ on graph

$W^* = \int_0^{F_0} \Delta(F)\,dF = \int \frac{F}{K}\,dF = \frac{1}{2}\frac{F_0^2}{K}$ (very similar equation)

in this case (straight line), $W = W^*$

$F_0 = K\Delta_0$, $W^* = \frac{1}{2}(K\Delta_0)^2/K = \frac{1}{2} K \Delta_0^2$

what if we add a little?

$\Delta = \Delta_0 + \delta x$, $F = F_0 + \delta F$ what is the change in work done?

$$\delta W = \int_{\Delta_0}^{\Delta_0+\delta x} kx\,dx = \frac{1}{2} kx^2 \Big|_{\Delta_0}^{\Delta_0+\delta x}$$

solve geometrically: $\delta W = F_0 \delta x + \frac{1}{2} \delta F \cdot \delta x$

or $= F_0 \delta x + \frac{1}{2} k \delta x^2$

if δx is very small, second term goes to zero

$$\boxed{\delta W \approx F_0 \delta x}$$

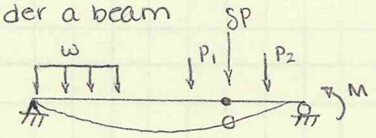
thought experiment: load remains constant through some displacement, how does the work change?

The change in W^* can be approximated (or imagined as) a system in equilibrium, moving to an new equil. position under small changes in applied load (displacements held constant)

$$\boxed{\delta W^* = \Delta_0 \delta F}$$

APPLICATION OF V.W.

consider a beam



goal: find deflection at point O

1. pretend there is an additional load, δP ~~ΔP~~ , acting at the location and in the direction of displacement you want to calculate
   - does not create any more deflection

2. For a valid solution

$$\delta W^*_{int} = \delta W^*_{ext}$$

internal and external work must be equal (or, complimentary virtual work)

$$\delta W^*_{ext} = \delta P \cdot \Delta_o$$

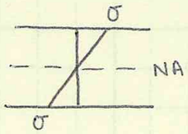
$$\delta W^*_{int} = \int_{vol} \delta\sigma \cdot \varepsilon \, dvol$$

$\delta\sigma$ = virtual stress quantity
= internal stresses that would result if δP were actually applied

ε = strains that develop in beam due to the actual applied load
→ real strains

Assume all the same things as for beam theory

$$\sigma = \frac{MC}{I}$$



$$\varepsilon = \frac{My}{EI} = \underbrace{\frac{M}{EI}}_{\text{curvature, } K \text{ (kappa)}} y$$

$$\varepsilon = Ky$$

but... this is real stress.

Virtual stress

$\delta M(x)$ = bending moment distribution in the beam if δP were actually applied

$$\delta\sigma = \frac{\delta My}{I}$$

Putting it all together

$M(x)$ = "real" moment distribution

$$\delta W^*_{int} = \int_{vol} \frac{\delta My}{I} \cdot \frac{My}{EI} \, dvol$$

$$= \int_0^L \int_A \left(\delta M \cdot \frac{M}{EI}\right) \frac{y^2}{I} \, dA \, dx$$

$I = \int_A y^2 \, dA$, so that simplifies kinda nice

$$\boxed{\delta W^*_{int} = \int_0^L \delta M \frac{M}{EI} dx}$$

or, more generally, $\delta W^*_{int} = \int_0^L \delta M \underset{\text{curvature}}{K(x)} dx$

VIRTUAL WORK

Building from Tuesday

$$\delta W^*_{int} = \int_0^L \delta M \cdot \frac{M}{EI}\, dx$$

moment(x) caused by virtual load δP

curvature due to applied load effects

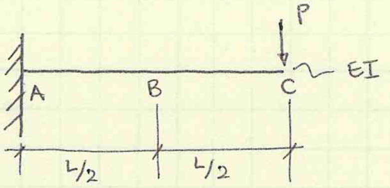
complimentary work — measure of strain energy internally, work done by forces externally

energy stored in system must equal work/energy applied

$$\delta W^*_{ext} = \delta P \cdot \Delta$$

Application of VW principles

Ex 1:

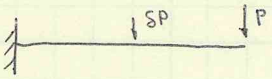


FIND: $\Delta_B, \Delta_C, \theta_C$

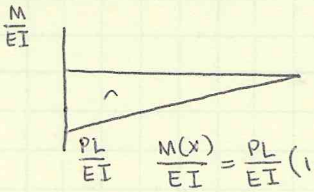
Be careful with signs — moment equations here are negative of actual. You must keep track!

★ Force and deflection in same direction!

for Δ_B: 1.

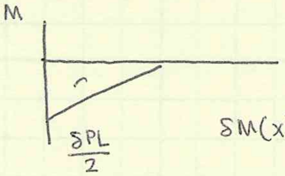


2. construct M/EI



$$\frac{M(x)}{EI} = \frac{PL}{EI}\left(1 - \frac{x}{L}\right)$$

3. construct δM



$$\delta M(x) = \frac{\delta PL}{2}\left(1 - \frac{2x}{L}\right)$$

$(0 < x \le L/2)$

$= 0, \ (L/2 \le x \le L)$

4. write expression for complimentary virtual work

$$\delta W^*_{int} = \int_0^{L/2} \frac{\delta PL}{2}\left(1 - \frac{2x}{L}\right)\left[\frac{PL}{EI}\left(1 - \frac{x}{L}\right)\right] dx + \int_{L/2}^{L} \cancel{(0)}\, dx$$

$$\delta W^*_{int} = \frac{\delta P \cdot P \cdot L^2}{2EI} \int_0^{L/2} \left(1 - \frac{2x}{L}\right)\left(1 - \frac{x}{L}\right) dx$$

$$\text{or} \int_0^{L/2} 1 - \frac{3x}{L} + \frac{2x^2}{L^2}\, dx$$

$$= \frac{\delta P \cdot PL^2}{2EI}\left[x - \frac{3x^2}{2L} + \frac{2x^3}{3L^2}\Big|_0^{L/2}\right]$$

$$\delta W^*_{int} = \frac{\delta P \cdot PL^2}{2EI}\left[\ell/2 - \frac{3\ell}{8} + \ell/12\right] = \frac{\delta P \cdot PL^3}{EI} \cdot \frac{5}{48}$$

5. $\delta W^*_{ext} = \delta P \cdot \Delta_B$

6. $\delta W^*_{ext} = \delta W^*_{int}$, $\dfrac{5\cancel{\delta P} \cdot PL^3}{48EI} = \cancel{\delta P} \cdot \Delta B$,

$$\Delta_B = \frac{5PL^3}{48EI}$$

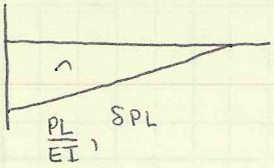
VIRTUAL WORK

More examples

(+) answer for deflection: virtual force and real displacement are in the same direction (in Ex 1, down = positive)

Example 1, cont'd.

Δ_c: M/EI, δM graphs look the same



$$\frac{M(x)}{EI} = \frac{PL}{EI}\left(1-\frac{x}{L}\right)$$

$$\delta M(x) = \delta P \cdot L\left(1-\frac{x}{L}\right)$$

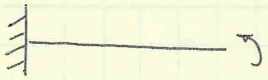
$$\delta W^*_{int} = \int_0^L \delta P \cdot L(1-x/L)\frac{\overset{PL}{\cancel{M}}}{EI}(1-x/L)\,dx = \frac{\delta P \cdot L^2 \cdot P}{EI}\int_0^L 1 - \frac{2x^{\cancel{2}}}{L^{\cancel{2}}} + \frac{x^2}{L^2}\,dx$$

$$= \frac{\delta P \cdot PL^2}{EI}\left[L - L + \frac{L}{3}\right] = \frac{\delta P \cdot PL^3}{3EI}$$

$$\delta W^*_{int} = \delta W^*_{ext} = \frac{\cancel{\delta P} \cdot PL^3}{3EI} = \cancel{\delta P}\,\Delta_c$$

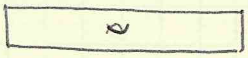
$$\underline{\Delta_c = \frac{PL^3}{3EI}}$$

θ_c:



(drawn backwards to show how signs work out)

- real moment diagram doesn't change
- virtual one does



moments are opposite one another

$$\delta W^*_{int} = \int_0^L -\frac{PL}{EI}\left(1-\frac{x}{L}\right)\delta M\,dx = \frac{-PL}{EI}\cdot\delta M\left[x - \frac{x^2}{2L}\Big|_0^L\right]$$

$$-\frac{PL}{EI}\cdot\delta M\left(L - \tfrac{1}{2}L\right) = \frac{-PL^2}{2EI}\cancel{\delta M} = \cancel{\delta M}\cdot\theta_c$$

$$\theta_c = \frac{-PL^2}{2EI}$$

negative sign indicates that drawn direction is wrong — δM and actual rotation are in opposite directions.

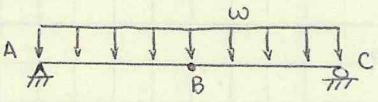
NO DEFLECTED SHAPE DRAWINGS!

also, (-) and (+) work out easily

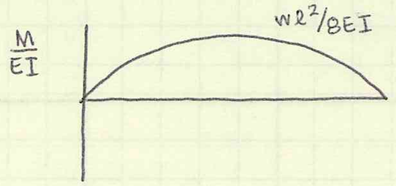
disadvantage: gives you one answer at a specific point — no general solutions. Eg, no idea where max. deflection is

MORE VIRTUAL WORK

Example 2

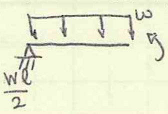


FIND: Δ_B, θ_A



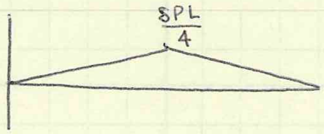
$$\frac{M}{EI}(x) = \frac{wl}{2}x - \frac{wx^2}{2}$$

(find using FBD)



$M(x) = \frac{wl}{2}x - \frac{w}{2}x^2$

Δ_B:



Option: assume $\delta P = 1$, so it drops out on both sides at the beginning

Unit Dummy Load Method

$$\delta P = \begin{cases} \frac{\delta P}{2}x & 0 < x \le L/2 \\ \cancel{\frac{\delta P L}{2}} - \cancel{\frac{\delta P}{2}x} & L/2 \le x < L \end{cases}$$

or $-\delta P\left(x - \frac{L}{2}\right) + \frac{\delta P}{2}x$

$$\delta W^*_{int} = \int_0^{L/2} (\qquad) \, dx + \int_{L/2}^{L} (\qquad) \, dx \qquad \text{UGLY.}$$

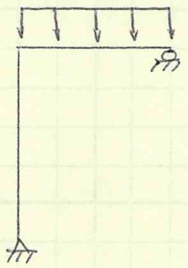
However, note: symmetric.

So, take 2X first integral, ~~ing~~ ignore the second.

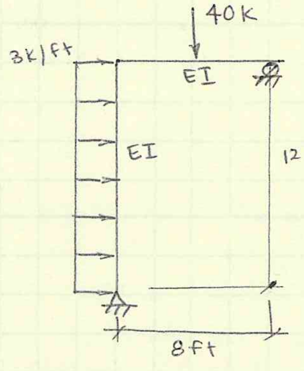
$$= \frac{\delta P}{2EI} \cdot 2 \int_0^{L/2} x \left(\frac{wlx}{2} - \frac{wx^2}{2}\right) dx$$

$$\ldots \quad \cancel{\delta P}\frac{5wl^4}{384EI} = \cancel{\delta P} \cdot \Delta_B \qquad \Delta_B = \frac{5wl^4}{384EI}$$

Frames and hinges



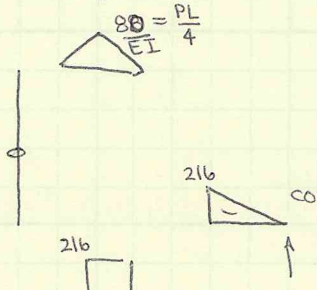
OR

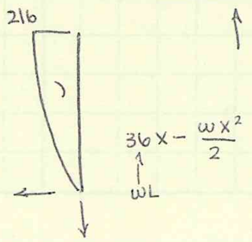


Superposition:

add just point load to just distributed load

$\frac{80}{EI} = \frac{PL}{4}$





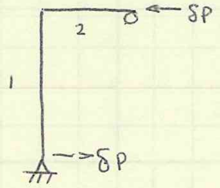
FRAMES WITH V.W.

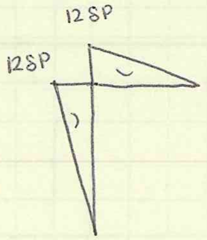
Example (cont'd)

- If point moves horizontally and vertically, you must use two fake loads, one in each direction

calculating Δ_c

δM from δP





Need to add parts from both column and beam

$$\delta W^*_{int} = \int_0^{12} \cancel{0 \cdot \delta P \cdot x_1 \, dx_1} + \int_0^4 (20x_2)/EI \left(12\delta P\left(1-\frac{x_2}{8}\right)\right] dx_2 + \int_4^8 \left(\frac{20x_2 - 40(x_2-4)}{EI}\right)\left[12\delta P\left(1-\frac{x_2}{8}\right)\right] dx_2$$

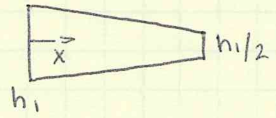
$$+ \int_0^{12} \left(\frac{36x_1 - \frac{3x_1^2}{2}}{EI}\right)(\delta P \cdot x_1)\, dx_1 + \int_0^8 \left[\frac{216}{EI}\left(1-\frac{x_2}{8}\right)\right]\left(12\delta P\left(1-{}^{x_2}/_8\right)\right) dx_2$$

$\delta W^*_{ext} = \delta P \cdot \Delta_c$, δPs cancel

$$\Delta_c = \frac{1}{EI}\left[\text{big equation}\right] = \text{(hopefully) answer from HW \#1}$$

Specific cases

Beam whose area varies along length



$h(x) = h_1(1 - {}^{x}/_{2l})$

$I\text{(rectangle)} = {}^1/_{12}\, bh^3$

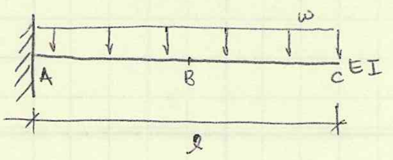
$I_x(x) = {}^1/_{12}\, b\left[h_1(1 - {}^{x}/_{2l})\right]^3$ ugly integral addition

- use calculator
- Simpson's Rule

VIRTUAL WORK

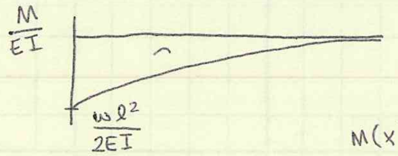
Graphical methods of solving stuff

Example (no graph)

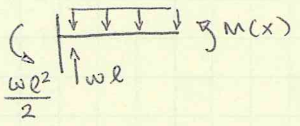


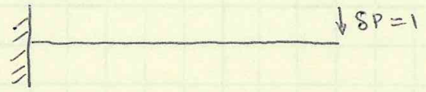
Δ_c

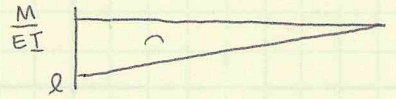
1. get reaction force
2. draw bending moment



$$M(x) = \omega \ell x - \frac{\omega x^2}{2} - \frac{\omega \ell^2}{2}$$







$$\delta M(x) = -L\left[1 - x/\ell\right]$$

$$\Delta_c = \frac{1}{EI}\int_0^\ell -\ell\left[1 - x/\ell\right]\omega\left[\ell x - \frac{x^2}{2} - \frac{\ell^2}{2}\right] dx$$

solve through to get Δ_c value

$$\Delta_c = \frac{\omega \ell 4}{8EI}$$

(+) deflection means deflection is in same direction as applied virtual load

[illegible]. overview of method to evaluate virtual work integrals

- at most, virtual moment [illegible] diagram is linear

$$\delta M(x) = ax + b$$

General case

$$1 \cdot \Delta = \int_0^\ell (ax+b)\frac{M(x)}{EI}dx = a\int_0^\ell x \cdot \frac{M}{EI}dx + b\int_0^\ell \frac{M}{EI}dx$$

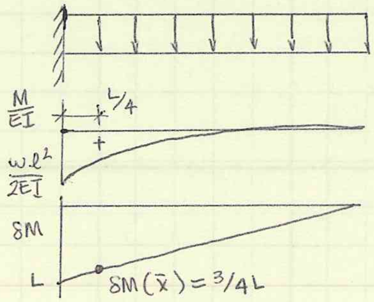
$$= \underbrace{a\int_0^\ell x f(x)\,dx}_{\text{centroid}} + b\int_0^\ell f(x)\,dx$$

$$= a \cdot \bar{x}\int_0^\ell f(x)\,dx + b\int_0^\ell f(x)\,dx$$

$$= (a\bar{x} + b)\int_0^\ell \frac{M(x)}{EI}dx$$ → virtual moment evaluated at the centroid (of the real moment diagram) multiplied by the area under that diagram.

GRAPHICAL VIRTUAL WORK

Example again, graphically

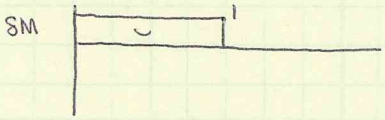


1. find reactions
2. draw real moment diagram
3. apply virtual load, draw moment diagram
4. calculate area under real moment diagram
5. find location of centroid on real M.D.
6. find value of δM at centroid
7. multiply 4 and 6

$$\Delta_c = \tfrac{1}{3}L\left(\frac{-w\ell^2}{2EI}\right)\left(-\tfrac{3}{4}\ell\right)$$

$$\Delta_c = \frac{w\ell^4}{8EI}$$ = same value calculated using long, complex integral

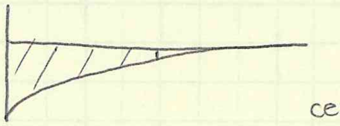
calculate θ_B



for ↻ added at point B

NOTE: you need to split in half to account for discontinuity in moment diagram

- need area of M/EI from A to B
- need centroid of that area



shaded area = total − Area from B to C

$$\text{centroid} = \frac{1}{\text{Area of shaded}}\left[\bar{x}_{tot} A_{tot} - \bar{x}_{tail} \cdot A_{tail}\right]$$

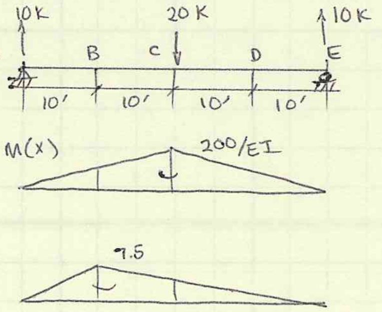
tail properties are easy to calculate because it includes the apex of the parabola

$A = \frac{1}{3}bh$, $\bar{x} = \frac{1}{4}b$ from left side

just need b at Pt. B – graphically, or by evaluating equation (found earlier)

VW: OUR OWN EXAMPLE

Example



Find Δ_B

~~$A_{M(x)} = \frac{1}{2}(200)\frac{1}{EI}(40) = 4000/EI$~~

~~$\bar{x} = 20$ ft~~

~~$\delta M(\bar{x}) = 5$~~ ~~$\Delta_B = \frac{4000}{EI}$.~~

CRAP, NO.

As I pointed out earlier – need to consider discontinuity

$A_{M_1} = \frac{100}{EI} \cdot \frac{1}{2} \cdot 10 = 500/EI$

$\bar{x}_{M_1} = \frac{2}{3}(10) = 6.67$

$\delta M(\bar{x}_{M_1}) = 5$ $\Delta_{B_1} = \frac{500}{EI} \cdot 5 = \frac{2500}{EI}$

ANYWHERE you'd break up the integral, break up these equations.

$A_{M_2} = \frac{150}{EI} \cdot 10 = \frac{1500}{EI}$, $\bar{x}_{M_2} = 5.56$

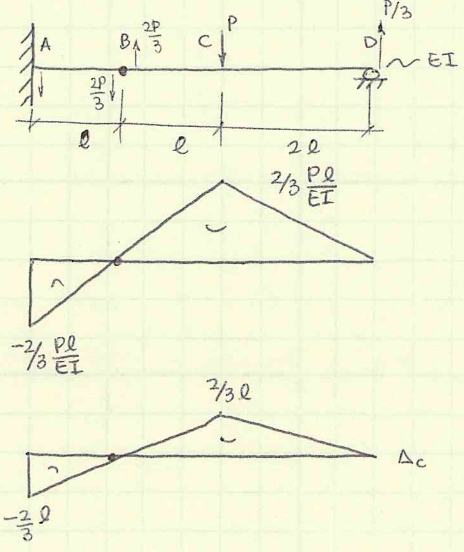
$\delta M(\bar{x}_{M_2}) = 6.11$ $\Delta_{B_2} = \frac{1500}{EI} \cdot 6.11 = \frac{9167}{EI}$

$A_{M_3} = \frac{1}{2}(20)\frac{200}{EI} = \frac{2000}{EI}$, $\bar{x}_{M_3} = 6.67$

$\delta M(\bar{x}_{M_3}) = 3.33$ $\Delta_{B_3} = \frac{2000}{EI} \cdot 3.33 = \frac{6666.7}{EI}$ $\underline{\Delta_B = \frac{18334}{EI}} \downarrow$

So – easier than the integral?

Example considering a hinge



Find Δ_c, θ_{BR}

$$\left[-\frac{2}{3}\frac{PL}{EI}\right]\frac{1}{2}L\left[\frac{-2}{3}L\right]\frac{2}{3} + \left[\frac{2}{3}\frac{PL}{EI}\right]\frac{1}{2}L\left[\frac{2}{3}L\right]\frac{2}{3}$$

$$+\left[\frac{2}{3}\frac{PL}{EI}\right]\frac{1}{2}\cdot 2L\left[\frac{2}{3}L\right]\cancel{\frac{2}{3}}\frac{2}{3}\cdot = \frac{16}{27}\frac{PL^3}{EI}$$

VIRTUAL WORK: TRUSSES

Keys of Structural Analysis

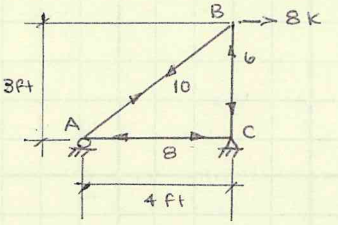
- equilibrium
- compatibility
- constitutive relations

Principle of virtual work (compatibility)

external VW = internal VW

$$\underbrace{1}_{\text{virtual}}\cdot\Delta = \sum_i \underbrace{n_i \cdot \Delta L_i}_{\text{real}}$$

VW on trusses

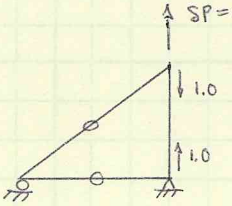


E = 29,000 ksi

A = 4 in²

$\Delta_{BV} = ?$

Member	Length	n Virtual	N Real	NLn
AB	5	0	+10	0
BC	3	+1	− 6	−18
AC	4	0	+8	0
	ft		kip	



$\Sigma n_i \cdot \Delta L_i$

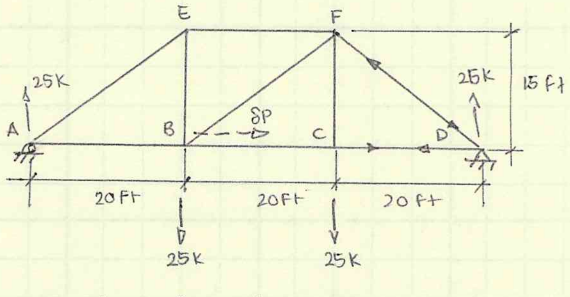
$\hookrightarrow \Delta L = \frac{NL}{EA}$

$$\Delta_{BV} = \frac{-18\text{ kip}\cdot\text{ft}\,(12\text{ in/ft})}{29{,}000\text{ ksi}\,(4\text{ in}^2)}$$

$\Delta_{BV} = -0.062$ in

compressive forces are considered to be negative

Another example



A = 4 in²

except BE, CF → A = 3 in²

$\Delta_{BH} = ?$

Member	L	n	N	nLN
AB	20	0	–	0
BC	20	-1.0	100/3	-2000/3 [12/AE]
CD	20	-1.0	100/3	-2000/3 [12/AE]
AE	25	0	–	0
BE	15	0	–	0
EF	20	0	–	0
BF	25	0	–	0
CF	15	0	–	0
DF	25	0	–	0

virtual reactions: $A_y = 0$, $D_y = 0$

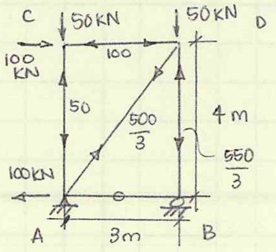
– draw and find zero force members before making the table

$F_{CD} = \frac{100}{3} = F_{BC}$ tension

$$\Delta_{BH} = \frac{(-4000/3)(12\text{ in/ft})}{29000\text{ksi}\,(4\text{in}^2)} = -0.138\text{ in},\ 0.138\text{ in left}$$

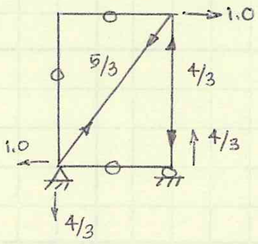
TRUSS CALCULATIONS

Another example



smallest area for all bars so that $\Delta_{DH} \leq 10 \text{mm}$

$E = 70 \text{ GPa} = 70 \times 10^6 \text{ KN/m}^2$



so, consider AD and BD

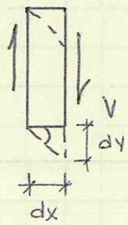
$$\begin{array}{c} L_{AD} = 5 \text{ m} \\ n_{AD} = 5/3 \\ N_{AD} = 500/3 \\ \hline 12500/9 \end{array} + \begin{array}{c} L_{BD} = 4 \text{ m} \\ n_{BD} = -4/3 \\ N_{BD} = -550/3 \\ \hline 8800/9 \end{array} = \frac{21300}{9} \text{ kN} \cdot \text{m}$$

$$\Delta_{DH} = \frac{21300}{9} \text{ kN} \cdot \text{m} \left[(70 \times 10^6 \text{ kN/m}^2) A \right]^{-1} = 0.010 \text{ m}$$

$$A = 0.0034 \text{ m}^2$$

SHEAR DEFORMATIONS

In Beams – computed using VW



- additional aspect of displacement to account for
- angle indicated = γ

$$\gamma = \frac{dy}{dx} \qquad \gamma = \text{shear strain}$$

- want expressions for shear stress, shear strain

Assumptions:

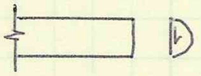
- linear elastic material

$$\tau = G\gamma$$

shear stress = shear modulus · shear strain

$$G = \frac{E}{2(1+\nu)}, \quad \nu = \text{Poisson's ratio}$$

Shear stresses



non-linear (quadratic) distribution

Using virtual work

$$\tau = \frac{VQ}{It}$$

V = shear force
Q = first moment of area = $\int y\,dA$
I = moment of inertia
t = thickness/width of section in question

internal δW^*

$$\underline{\delta W^*_{int} = \int \delta\sigma \cdot \varepsilon \, dvol}$$ – virtual stress x real strain through volume

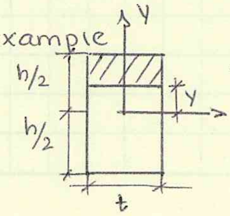
$$\delta W^* = \int_{vol} \frac{\delta V \cdot Q}{It} \cdot \frac{1}{G} \frac{VQ}{It} \, dvol$$

$\delta V = v$ (lowercase v) – virtual shear that would result if virtual load were applied

$$= \int \underbrace{\frac{vV}{G}}_{\text{constant}} \left(\frac{Q^2}{I^2 t^2} \right) dvol$$

$$= \int_0^{l} \frac{vV}{G} \left[\int_A \left(\frac{Q}{It} \right)^2 dA \right] dx$$

Minor example



$I = 1/12 t h^3$

$Q = (h/2 - y)(t)(y/2 + h/4)$

$$\boxed{k \equiv \text{form factor} \equiv A \int_A \left(\frac{Q}{It} \right)^2 dA}$$

SHEAR DEFORMATIONS

Virtual work method
using form factor in equation

$$\delta W^*_{int} = \int_0^\ell k \frac{vV}{GA}\,dx$$

$$\tau = k\frac{V}{A} \rightsquigarrow \frac{V}{A} \equiv \text{average shear stress}$$

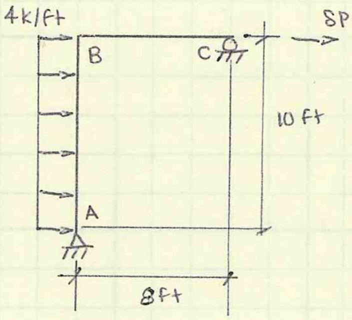
to calculate k for a rectangle,
integrate from $-t/2$ to $t/2$
$-h/2$ to $h/2$

k, form factor, depends on the geometry of the section, tabulated as such

- rectangle = 1.2
- circular x-section = 10/9
- wide flange = 1.0 (through web)

– multiplied by A to make k dimensionless, and able to work on any typical section

Example with shear

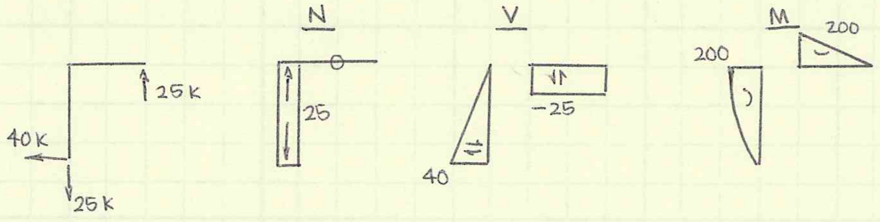


E = 29000 ksi
G = 12,000 ksi
I = 600 in⁴
A = 80 in²
k = 1.2 (rectangle)

goal: Δ_C

$$\Delta_C = \underset{\text{flexure}}{\int_0^\ell \frac{mM}{EI}dx} + \underbrace{\underset{\text{axial}}{\Sigma \frac{nNL}{EA}} + \underset{\text{shear}}{\int_0^\ell \frac{vV}{GA}k\,dx}}_{\text{previously ignored – ok to do?}}$$

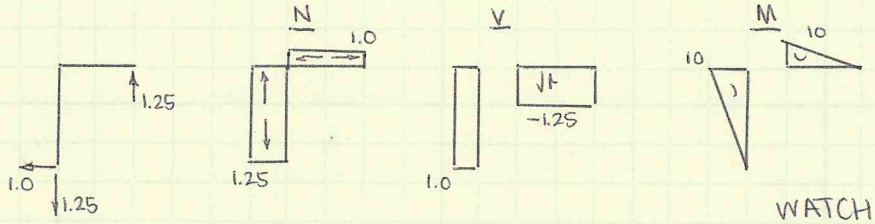
Real loads



Flexure :	99.4%
Axial :	0.1%
Shear :	0.5%

NOT THAT IMPORTANT.

Virtual loads



WATCH UNITS!

Δ_C: flexural $= \frac{2}{3}(200)(10)\frac{1}{EI}(5/8)(10) + \frac{1}{2}\left(\frac{200}{EI}\right)(8)\left(\frac{2}{3}\right)(10) = 1.357$ in →

shear $= 1.2\left[\frac{1}{2}\left(\frac{40}{GA}\right)(10)(1.0) + \cancel{\tfrac{1}{2}}\left(\frac{-25}{EA}\right)(8)(-1.25)\right] = 0.00675$ in →

axial $= (25)(1.25)(10)\frac{1}{EA} + 0 = 0.00161$ in →

$\underline{\Delta_C = 1.365}$

SHEAR DEFORMATIONS

When are they important?

1. short beam
2. deep beam

(3. short & deep beam)

however, the majority of structural members are not affected greatly by shear or axial deformations

Axial deformations

- really, really short beam
- generally not a large contributing factor

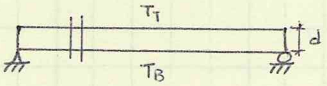
likely short enough to not be called a beam

Up next: thermal deformations / effects on beams

THERMAL EFFECTS

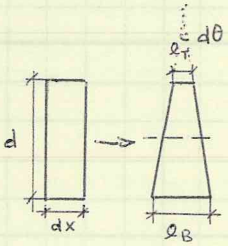
in flexural members

$T_M \equiv$ median temp (original temp. of beam)



Assume (to start): T_B ; T_T - temp at top and bottom - are changed ; linear variation through depth
$T_B > T_T$

what happens to surfaces?
- hotter side expands
- midheight does not change
  if T_T, T_B are not evenly spaced around T_M, beam will change lengths - axial component



- for ~~tot~~ truss behavior, temperature must be constant through depth

$$\ell_B = \left[1 + \alpha(T_B - T_M)\right]dx$$

$$\ell_T = \left[1 + \alpha(T_T - T_M)\right]dx$$

goal: find curvature of beam

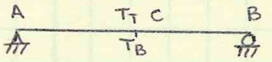
$$d\theta = \frac{\ell_B - \ell_T}{d} = \frac{dx\left[1 + \alpha(T_B - T_M) - 1 - \alpha(T_T - T_M)\right]}{d} = \frac{\alpha(T_B - T_T)}{d}\,dx$$

curvature $= d\theta/dx$

$$\frac{d\theta}{dx} = \frac{\alpha(T_B - T_T)}{d}$$

we expect:
$T_B > T_T$, (+) curvature
for positive ⤻ curvature, bottom temp must be hotter than top temp.

Using this information

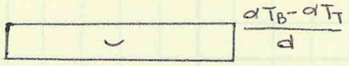


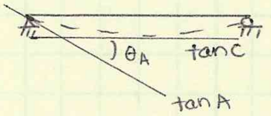
solve for θ_A

$$\boxed{\theta_B - \theta_A = \int_A^B \frac{\alpha(T_B - T_T)}{d}\,dx}$$

form of equation is just like first moment-area equation

General Approach:
1. draw curvature diagram
   - constant T along beam, constant $d\theta/dx$
2. sketch deflected shape
3. apply moment-area theorem to get θ_A





$$\theta_{C/A} = \frac{\alpha(T_B - T_T)}{d} \cdot L \cdot 1/2$$

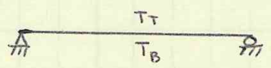
(area from A to C)

$$\theta_A = \frac{t_{B/A}}{L} = \alpha(T_B - T_T) \cdot \frac{L}{d} \cdot \frac{L}{2} \cdot \frac{1}{L}$$

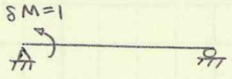
(+) value, A to C is counterclockwise ✓

THERMAL EFEECTS

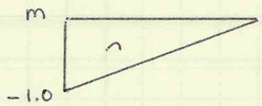
Virtual Work Approach



virtual load case



equation for curvature:



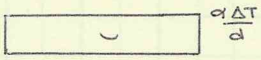
$K \equiv$ thermal curvature

$T_B - T_T \equiv \Delta F$

$$K = \alpha \Delta T / d$$

$$\delta W^*_{int} = \int_0^\ell m K_T \, dx \quad , \quad \delta W^*_{ex} = \cancel{\int_0^\ell} \; 1.0 \cdot \theta_A$$

curvature:



graphically

$$\delta W^*_{ext} = \delta W^*_{int}$$

$$\theta_A = \int_0^\ell m(x) \frac{\alpha \Delta T}{d} dx$$

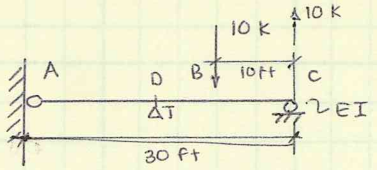
$$\theta_A = \underbrace{\left(\frac{\alpha \Delta T}{d}\right) L}_{\text{area under curvature dia.}} \left(-\frac{1}{2}\right) = \frac{-\alpha \Delta T}{2d} L$$

↳ value of virtual moment at centroid of area under K diagram

(−) sign means assumed direction of virtual moment is incorrect.

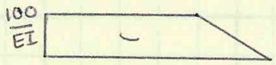
$$\theta_A = \frac{\alpha \Delta T}{2d} L \circlearrowright$$

In-class Problem

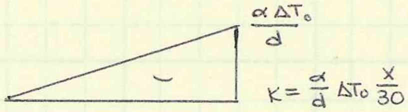


$\Delta_T = \Delta T_0 \frac{x}{30}$ temperature varies, left to right

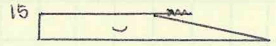
solve for ~~Δ~~ Δ_D - ignore axial and shear

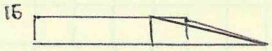


M load



$K = \frac{\alpha}{d} \Delta T_0 \frac{x}{30}$





$\Delta_D = t_{C/A} - t_{D/A}$
$= 15\theta_{C/A} - t_{D/C}$

$$\Delta_{D load} = \frac{100}{EI}(15)(15) + \frac{100}{EI}(5)(12.5) + \frac{1}{2} \cdot \frac{100}{EI}(10) \cdot \frac{2}{3}(10) = \frac{32083}{EI} \checkmark$$

$$\Delta_{D temp} = \frac{\alpha}{d} \Delta T_0 \cdot \frac{1}{2} \cdot \frac{1}{2}(15)(15) + \left\{ \underset{(7.5)}{\frac{1}{2} \frac{\alpha}{d} \Delta T_0 (15)} + \frac{1}{2}\frac{\alpha}{d} \Delta T_0 \cdot \frac{1}{2}(15) \right\}(5) = \overset{131.25}{\cancel{158}} \frac{\alpha}{d} \Delta T_0$$

131.25

strain = distance to N.A. × curvature

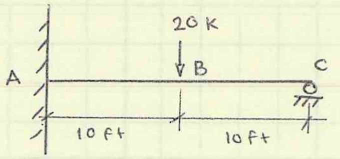
statically determinant structures loaded thermally, no stress occurs, because it's not restrained.

movement occurs; stress only if movement is restrained

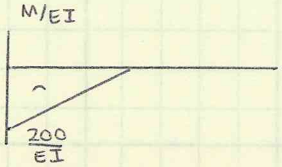
STATICALLY INDETERMINANCY

S.1. Structures
- often good, although thermal loads and support settlements are bad
- two approaches to analysis
  - force·based – forces are unknown (establish equilibrium, force compatibility)
  - stiffness·based – displacements are unknown (establish compatibility, force equilibrium)

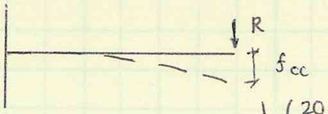
Force·based approach



statically indeterminant to the first degree
- three ~~unknowns~~ equations
- four unknowns



1. establish degree of indeterminancy
2. chose redundant force
   need to leave structure as a stable structure
   C_y, A_y, M_A are okay
3. compute response of the "primary structure"
   - structure without chosen force, F_{cy}
   - in this case, cantilever beam is left
4. solve for Δ_c
   $\Delta_c = 1/2\left(\frac{200}{EI}\right)(10)\left[10 + 2/3(10)\right] \; (= t_{c/A})$
5. compute structural response for secondary structure(s)



solve for $\Delta_c = \frac{1}{2}\left(\frac{20}{EI}\right)(20)\frac{2}{3}(20)R = f_{cc}$

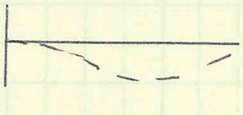
terminology:
$f_{\alpha\beta}$ = flexibility coefficient
displacement at α, due to a unit load at β.

6. enforce compatibility
   $-f_{cc} = \Delta_c \quad \therefore \quad R = C_y = 6.25$ k up
7. reapply redundant force to original structure, and enforce equilibrium (as you now know one force)
8. draw shear, moment diagrams

using virtual work to calc Δ_B: same total M diagram, but values divided by 20 (P).

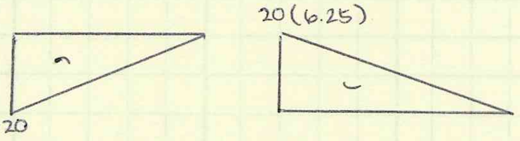
Finding Δ_B:

Moment·Area



$t_{B/A} = \Delta_B$
$t_{B/A}$ = (Moment Area)(Moment Arm)

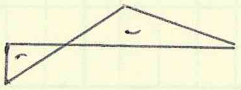
(virtual work) moment diagrams



- changing from incorrect unit load to actual response load
- superimpose with moment diagram

Virtual work
- make SP at B moment diagram (same shape, 1/20th)

$\Delta_B = \frac{1}{2}(5.45)\left(\frac{-75}{EI}\right)\frac{2}{3}\left(\frac{-75}{20}\right) + \ldots = \frac{1458}{EI}$



STATIC INDETERMINANCY

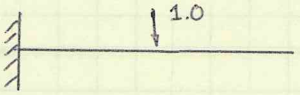
continue example

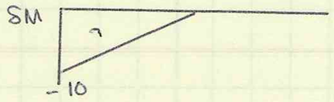
using virtual work to find Δ at quarter point

- seems like you would have to start over to get δM diagram
- however, not true

Primary structure

(pretending moment diagram doesn't scale)





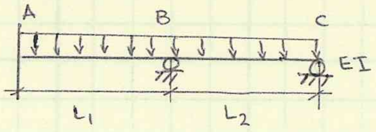
use real moment diagram, this δM

$$\Delta_B = -\tfrac{1}{2}(5.44)\left(\frac{-75}{EI}\right)\left[4.55 + \tfrac{2}{3}(5.45)\right] +$$
$$-\tfrac{1}{2}(4.55)\left(\frac{62.5}{EI}\right)\left[4.55\left(\tfrac{1}{3}\right)\right]$$
$$= \frac{1458.3}{EI}$$

- virtual work premise says the structure must be in equilibrium
  - does not need to satisfy compatibility
- thus, as long as structure is still in equilibrium, virtual work will work
- consistently use same primary structure to make your life easier
- able to assume $F_{cy}=0$, just... cause. crazy.

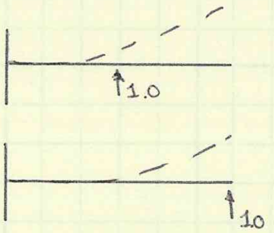
More degrees of indeterminancy

2nd degree:



solving for response of system

1. choose to ignore B_y, C_y - primary structure is a cantilever with a uniform load
2. make two secondary structures



- compatability ~~equations~~ values must be scaled due to load

  $\delta P \neq 1.0$ (or, ≠ actual load)

3. establish ~~compat~~ compatibility equations using $f_{\alpha\beta}$ format

$$\Delta_B + \overset{B_y}{f_{BB}} + \overset{C_y}{f_{BC}} = 0$$
$$\Delta_C + \overset{B_y}{f_{CB}} + \overset{C_y}{f_{CC}} = 0$$

- $\underset{\sim}{f}\,\underset{\sim}{R} = \underset{\sim}{\Delta}$

$$\begin{bmatrix} f_{BB} & f_{BC} \\ f_{CB} & f_{CC} \end{bmatrix}\begin{bmatrix} B_y \\ C_y \end{bmatrix} = \begin{bmatrix} \Delta_B \\ \Delta_C \end{bmatrix}$$

↑
flexibility matrix

- symmetric, so $f_{BC} = f_{CB}$

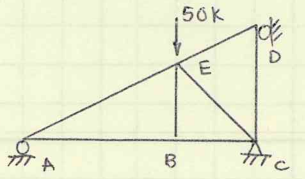
ok, clearer now:

$$\Delta_B \downarrow + B_y f_{BB} \uparrow + C_y f_{BC} \uparrow = 0$$
$$\Delta_C \downarrow + B_y f_{CB} \uparrow + C_y f_{CC} \uparrow = 0$$

USING SUPERPOSITION

Spring support –
roller with infinite stiffness

Statically indeterminant truss example



EA, L = 10ft typ. dimension (AB, BC, CD)

is truss statically determinant?

$2j = m + r$

number of ~~members~~ joints ↵ | ↳ number of reactions | ↳ number of members

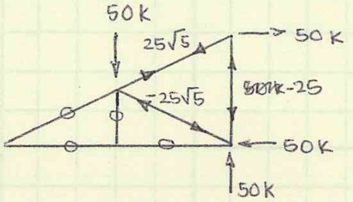
$10 \neq 11$

also check to see if structure is stable

- choose redundant

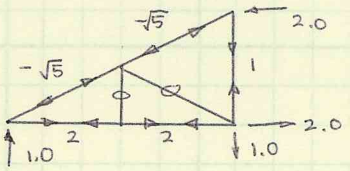
  A_y

- primary structure



only know how to use virtual work for a truss
need – virtual member forces
– real member forces
(including misfit, temps.)

- secondary structure



secondary structure is identical to virtual force diagram

- compatibility ↑

$$\Delta_A \downarrow + A_y f_{aa} = 0$$

$$\Delta_i = \underbrace{\sum \frac{n_i N_i L_i}{(EA)_i}}_{\text{load effect}} + \underbrace{\sum n_i \alpha \Delta T L_i}_{\text{temperature change}} + \underbrace{\sum n_i \Delta L_i}_{\text{initial misfit}}$$

$$f_{aa} = \sum_i \frac{(n_i)^2 L_i}{EA}$$

- table of values

	EA	L	N	n	nNL/EA	n^2L/EA
AB	↓	10	0	2	20	40
BC		10	0	2	20	40
CD		10	−25	1	−250	10
DE		$5\sqrt{5}$	$25\sqrt{5}$	$-\sqrt{5}$	$-25^2\sqrt{5}$	$25\sqrt{5}$
AE		$5\sqrt{5}$	0	$-\sqrt{5}$	0	$25\sqrt{5}$
BE		5	0	0	0	0
CE		$5\sqrt{5}$	$-25\sqrt{5}$	0	0	0

$$\Delta_a = \frac{1647.5}{EA} \downarrow \qquad f_{aa} = \frac{201.8}{EA} \uparrow$$

- use in compatability equation

$$1647.5 + A_y(-201.8) = 0$$

$$A_y = 8.16\text{K up}$$

- if support A settles, change compatibility to equal 3" down
- if support C settles, A would move in primary structure

MEMBER FORCES

Using calculations

Now that we have A_y, how do we use it?

member forces $= N_i + A_y n_i$

└> scaling of secondary structure

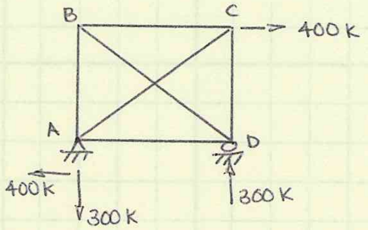
include this value in table

reaction forces

$R_i + A_y R_i$; eg. $C_x = 50k - 8.16k(2)$

Truss with extra member

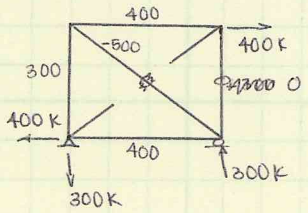
internally indeterminant



AB, CD = 6 ft
BC, AD = 8 ft
} EA

static indeterminancy comes from extra truss member, not support

- select redundant: force in member AC
- primary structure

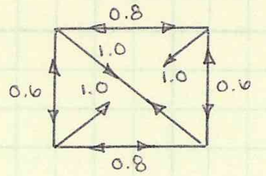


draw member as cut

compatibility comes from intersection of end

(+) = tension

- secondary structure



- assume removed tension force
- support reactions = 0 (internal force)

- ~~compat~~ compatibility

$\Delta_{AC} + F_{AC} f_{AC/AC} = 0$

$\underline{\Delta_{AC} = \sum_i \frac{n_i N_i L_i}{EA}}$

$\underline{f_{AC/AC} = \sum_i \frac{n_i^2 L_i}{EA}}$ → remember to include force on cut member, as $n \neq 0$ (although $N = 0$)

if calculated...

$\Delta_{AC} = \frac{1}{EA}(-11{,}200)$

$f_{...} = \frac{1}{EA}(34.56)$

> $F_{AC} = 324$ kip

positive value means member is what we thought — in tension

→ (-) value

says gap opened up, which goes against virtual load case, which would have caused overlap

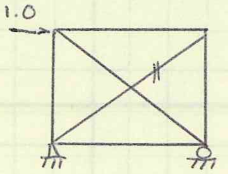
VIRTUAL WORK–TRUSSES

Using calculations

we have F_{AC}, can calculate all other member forces ($N_{Final_i} = N_i + F_{AC} n_i$)

Asked to find Δ_B

- in drawing virtual case, left with statically indeterminant truss



- for virtual work, system must be in equilibrium
- assume $F_{AC} = 0$

$$\Delta_B = \sum \frac{n_{i_2} (N_{Final})_i L_i}{EA}$$

n_{i_2} = virtual member forces from virtual case above, with δP at B, F_{AC} assumed to equal zero

Second degree of indeterminancy

add in effect of second structure

ex: roller at B

$$\Delta_{AC} + F_{AC} f_{AC/AC} + F_{BX} f_{AC/BX} = 0$$

each term leads to a deflection along member AC

$$\Delta_{BX} + F_{AC} f_{BX/AC} + F_{BX} f_{BX/BX} = 0$$

again, could ~~b~~ draw as a matrix

$$\begin{bmatrix} f_{AC/AC} & f_{AC/BX} \\ f_{BX/AC} & f_{BX/BX} \end{bmatrix} \begin{bmatrix} F_{AC} \\ F_{BX} \end{bmatrix} = \begin{bmatrix} \Delta_{AC} \\ \Delta_{BX} \end{bmatrix}$$

General form

- secondary forces from cut member = n_b
- from load at B = n_a

$$f_{AC/BX} = \sum \frac{n_a n_b L}{EA}$$

could have lots of letters rotating through

$n_a n_c$, $n_b n_c$, $n_a n_d$, $n_b n_d$, $n_c n_d$, etc...

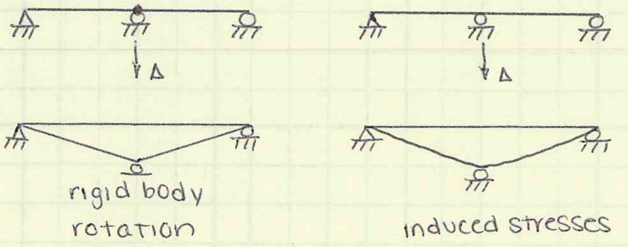
FORCES IN MEMBERS

Dammit. Supplement with notes from Katie.

Self-Straining Problems

- support settlement
- temperature change
- misfits in length

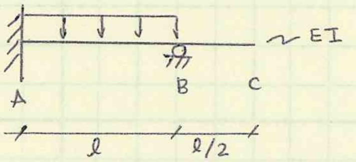
Example:



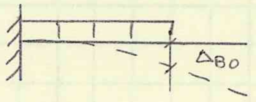
settling of center support

If statically determinant, these forces will not induce stress/strain. If indeterminant, they will form.

Support settlement

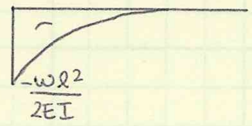


Primary structure
- get rid of By



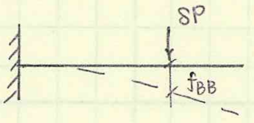
B settles, $\overline{\Delta_B} = \frac{w\ell^4}{24EI}$

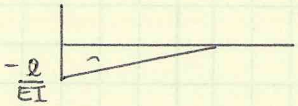
solve for reaction forces



only virtual load case doesn't get divided by EI. Secondary M does.

Secondary structure





load deflections   actual deflection (=0, usually)

$$f_{BB} By \downarrow + \Delta_B \downarrow = \overline{\Delta}_B \downarrow$$

$$\Delta_B = 1/3 \frac{-w\ell^2}{2EI} \ell \underbrace{(3/4)\left(\frac{-\ell}{\cancel{\quad}}\right)} = 1/8 \frac{w\ell^4}{EI}$$

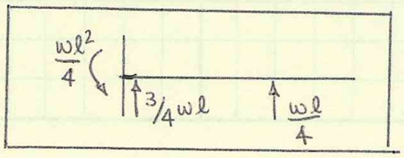
virtual load case is the same as secondary structure · EI

$$f_{BB} = 1/2 \frac{-\ell}{EI} \ell \frac{2}{3}(-\ell) = 1/3 \, \ell^3/EI$$

use compatibility

$$\underset{\text{loads}}{\frac{w\ell^4}{8EI}} + B_y \frac{\ell^3}{3EI} = \underset{\text{support}}{\frac{w\ell^4}{24EI}}, \quad B_y = -w\ell/4$$

(-) means By is going against δP, or, is going ↑

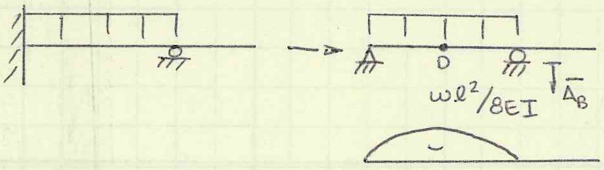


Support settlement reduces reaction load. If $\overline{\Delta}_B = w\ell^4/8EI$, $B_y = 0$.

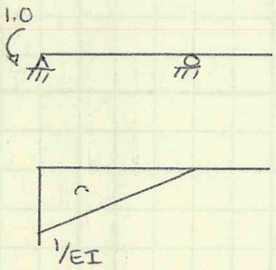
SELF-STRAINING PROBLEMS

Same problem, new primary structure

P.S.



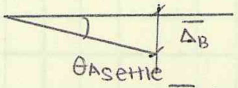
S.S.



enforce capatibility at A

$$\theta_A = \theta_{D/A} = \frac{2}{3}\left(\frac{w\ell^2}{EI8}\right)\frac{\ell}{2} = \frac{w\ell^3}{24EI} \circlearrowright$$

support at B:



$$\theta_{AS} = \bar{\Delta}_B/\ell = \frac{w\ell^4}{24EI}\cdot\frac{1}{\ell} = \frac{w\ell^3}{24EI} \circlearrowright$$

$$\theta_{A_{TOT}} = \frac{w\ell^3}{24EI}(2) = \frac{w\ell^3}{12EI} \circlearrowright$$

$$f_{AA} = \frac{t_{B/A}}{\ell} = \frac{1}{2}\left(\frac{-1}{EI}\right)(\ell)\frac{2}{3}(\ell)\cdot\frac{1}{\ell} = \frac{\ell}{3EI} \circlearrowleft$$

negative value: B is below A

compatibility equation

$$\theta_{A_{TOT}} \circlearrowright + M_A f_{AA} \circlearrowleft = 0$$

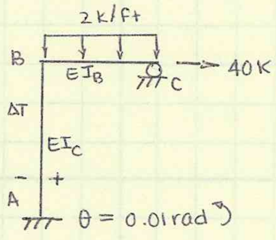
$$\frac{w\ell^3}{12EI} - \frac{\ell}{3EI}M_A = 0$$

$$M_A = \frac{w\ell^2}{4} \circlearrowleft$$

positive ∴ secondary drawn correctly.

Point: need not select the point that settles as your redundant

Frame Problem



$\bar{\Delta}_C = 2$ in

$\alpha = 6.5\times10^{-6}/°F$

$E = 29000$ ksi

$I_B = 1000$ in^4

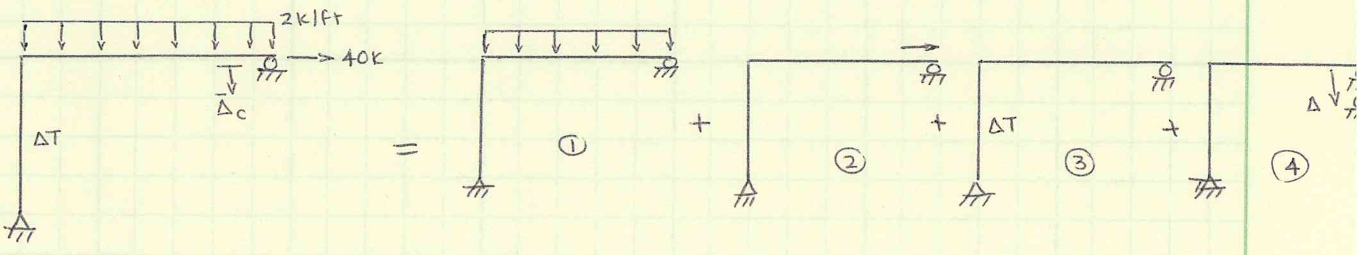
$I_c = 600$ in^4, $d_c = 6$ in

$\Delta T_o = 100°F$, 0°F at A

$\Delta T = \Delta T_o(x/\ell)$

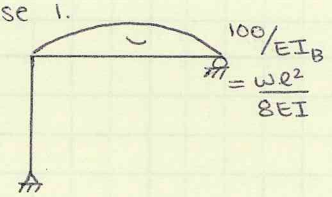
$L_B = 20$ ft, $L_c = 12$ ft

Redundant: Moment at A

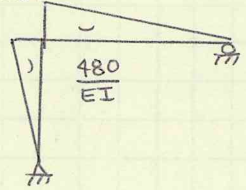


TRUSS PROBLEM (or, FRAME...)

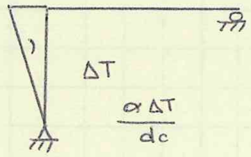
Case 1.



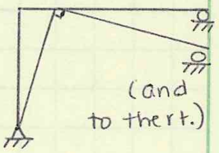
Case 2.



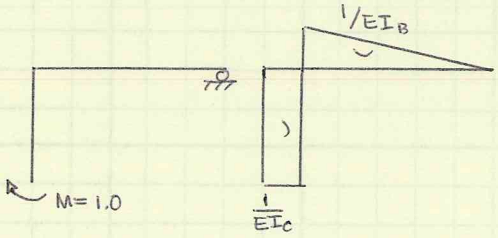
Case 3.



Case 4.



Solve for θ_A ; need secondary structures – only need one (one level of indeterminancy)



$\theta_{A_1}: \frac{2}{3}\left(\frac{100}{EI_B}\right)\frac{1}{2}(1.0)(20\,ft) = \frac{1}{3}\,\frac{2000}{EI_B} = 0.00331$ ↻

$\theta_{A_2}: \frac{1}{2}\,\frac{480}{EI_c}(12)(1) + \frac{1}{2}\left(\frac{480}{EI_B}\right)(20)\frac{2}{3}(1.0) = 0.03972$ ↻

$\theta_{A_3}: \frac{1}{2}\left(\alpha/d_c\,\Delta T\right)(12)(1.0) = 0.0039\,rad$ ↻

$\Delta \times (12)^3,\ \theta \times (12)^2$

$\theta_{A_4}: \bar{\Delta}_c / 20(12) = 0.0083\,rad$ ↻

$\Sigma\theta_A = \theta_{A0} = 0.055263\,rad$ ↻ watch for rounding errors!

$f_{AA}: \left[\frac{1}{EI_c}(12)(1.0) + \frac{1}{2}\left(\frac{1}{EI_B}\right)(2/3)(1.0)(20)\right](12)$

$f_{AA} = 0.000011034\,rad$ ↻

compatibility

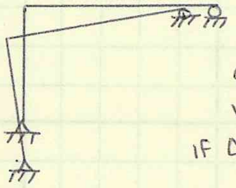
θ_{A0} ↻ $+ M_A f_{AA}$ ↻ $= 0.01\,rad$ ↺

$\underline{M_A = 492.9\,k\cdot ft}$ ↺ (negative number)

Use M_A to solve for ~~BM~~, Cy, Ay

Other cases:

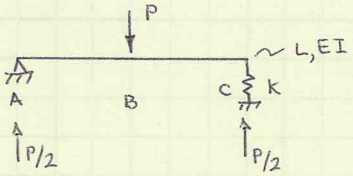
- support at A moves down



angle is the other way (in this case)
if $\Delta_A = 2in$, $\theta = 0.0083\,rad$ ↺
cancels out θ_{A4}.

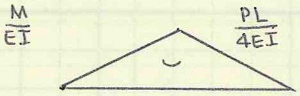
FLEXIBLE SUPPORTS

Statically determinant case (determinate?)

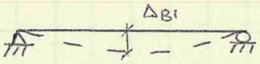


spring only compresses axially

all are the same

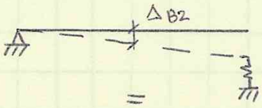


– treat spring as roller to calculate reactions



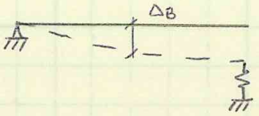
load effect

+



spring movement effect

=



combination

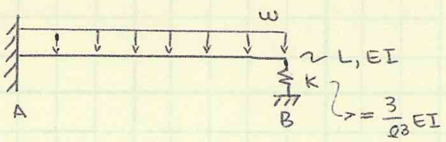
$$\Delta_B = \Delta_{B1} + \Delta_{B2}$$

$$\Delta_{B1} = \frac{PL}{4EI}\,\frac{1}{2}L \cdot \frac{1}{4}L = \frac{PL^3}{48EI}$$

$$\Delta_{B2} = \frac{P}{2k}\cdot\frac{1}{2}$$

$$\boxed{\Delta_B = \frac{PL^3}{48EI} + \frac{P}{4k}}$$

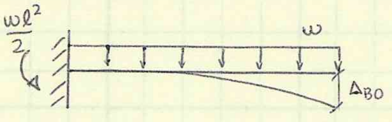
Statically indeterminate case

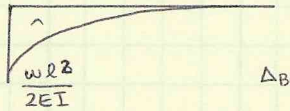


redundant: force in the spring at B

- similar to removing a member from a truss
- gap / overlap in compatibility

Primary structure





$$\Delta_{BO} = \frac{w\ell^4}{8EI}$$

Compatibility:

$$\Delta_{BO}\downarrow + F_B f_{BB}\uparrow = F_B / k \downarrow$$

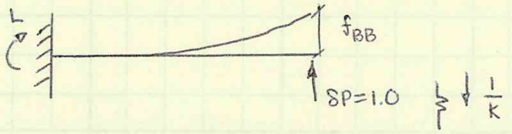
$$(+1/k)$$

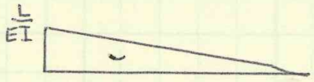
or...

$$\frac{w\ell^4}{8EI} + F_B\left(\frac{-L^3}{3EI}\right) - F_B\left(1/k\right) = 0$$

$$F_B = \frac{w\ell^4}{8EI}\cdot\left[\frac{L^3}{3EI} + \frac{1}{k}\right]^{-1}$$

Secondary structure





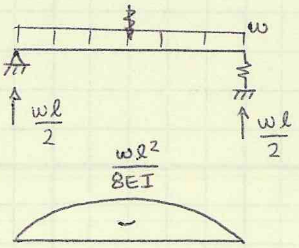
$$f_{BB} = \frac{L^3}{3EI}$$

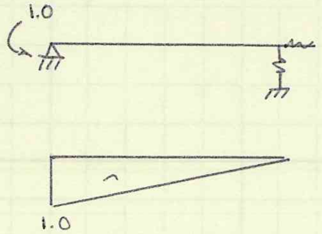
$$\text{if } k = \frac{3EI}{\ell^3}$$

$$F_B = \frac{3}{16}w\ell$$

FLEXIBLE SUPPORTS

Same problem, new redundant
remove M_A





$$\theta_A = \frac{w\ell^3}{24EI}$$

$$\theta_{A_S} = \frac{w\ell^3}{6EI} = \frac{w\ell}{2} \cdot \frac{1}{LK} = \frac{w\ell}{2} \cdot \frac{\ell^3}{3EI} \cdot \frac{1}{L}$$

$$\theta_{A_{TOT}} = \frac{5w\ell^3}{24EI} \circlearrowright$$

$$f_{AA} = \frac{1}{2}\left(\frac{-1.0}{EI}\right)\ell \cdot \frac{2}{3}(-1.0)$$

$$f_{AA} = \frac{\ell}{3EI}$$

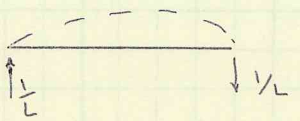
$$+ f_{AA_S} = \frac{1}{L} \cdot \frac{1}{LK} = \frac{\ell}{3EI}$$

$$f_{AA_T} = \frac{2\ell}{3EI} \circlearrowleft$$

compatibility

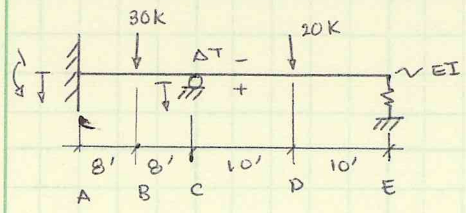
$$\theta_{A_T} \circlearrowright + M_A f_{AA_T} \circlearrowleft = 0$$

$$M_A = \frac{5w\ell^2}{16} \circlearrowleft$$



to get θ, not Δ, divide by L

More complex example

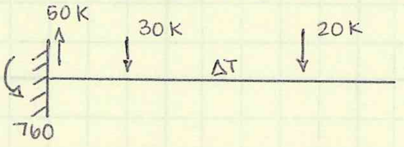


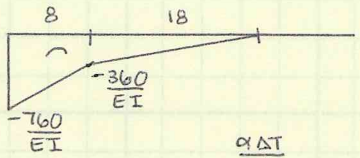
$\bar{\Delta}_C \downarrow = 5.0$ in
$\bar{\Delta}_A \downarrow = 3.0$ in
$\bar{\theta}_A \circlearrowleft = 0.02$ rad

$\alpha = 6.5 \times 10^{-6}/°F$
$\Delta T = 200°F$
$I = 1000\ in^4$
$d = 12\ in$

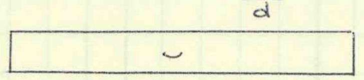
$k = 75\ k/in$

indeterminate to 2 degrees

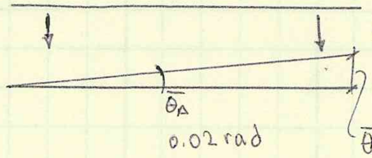




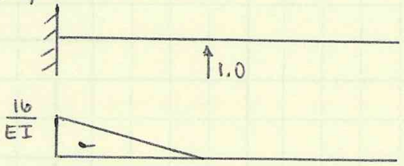
$\frac{M}{EI}$ load



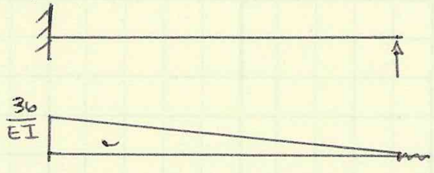
K_T temp



Secondary #1



Secondary #2



cont'd on handout.

University of Texas
Department of Civil Engineering

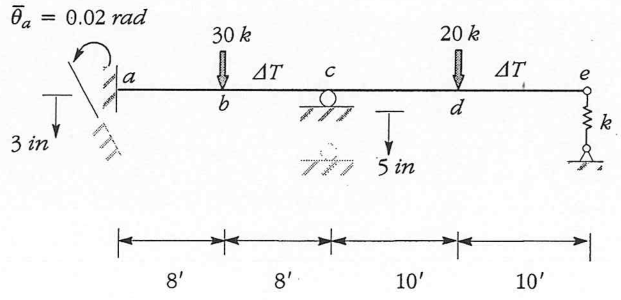
Instructor: E. Williamson
CE 363 Advanced Structural Analysis

Example Problem

The structure shown in the sketch below is statically indeterminate to the second degree. It is fixed at *a*, supported by roller at *c*, and supported by a spring with stiffness *k* at *e*. In addition to the applied loads, the support at *a* settles downward by 3 inches and rotates counterclockwise by 0.02 *rad*. The support at *c* settles downward by 5 inches. The beam is also subjected to a temperature change that varies linearly through the depth and is uniform over the length. The temperature differential between the top and bottom of the beam is 200 degrees F with the bottom hotter than the top. The modulus and moment of inertia are uniform over the entire beam.

Draw the shear and moment diagrams for the structure and determine the deflection at *b*.

Primary, secondary structures in notes



$E = 29{,}000\ ksi$ $I = 1000\ in^4$ $k = 75\ kips/in$ $d = 12\ in$ $\alpha = 6.5 \times 10^{-6}/°F$

$\Delta_{C\,loads} = t_{C/A} = (8)\left(\frac{360}{EI}\right)(12) + \frac{1}{2}(8)\left(\frac{400}{EI}\right)(8 + 2/3 \cdot 8) + \frac{1}{2}\left(\frac{360}{EI}\right)(8)(2/3)(8) + \left(\frac{200}{EI}\right)(8)(4) = 3.9152 \text{ in} \downarrow$

$\Delta_{E\,loads} = t_{E/A} =$ (32) $(28 + 2/3 \cdot 8)$

$+ \frac{1}{2}\left(\frac{360}{EI}\right)(18)(10 + 2/3 \cdot 18) = 12.9167 \text{ in} \downarrow$

$\Delta_{C\,therm} = t_{C/A} = \frac{\alpha \Delta T}{d}(16)(8)$
$= 1.9968$ up

, $\Delta_{E\,therm} = \frac{\alpha \Delta T}{d}(36)(18) = 10.1088$ in up

$\Delta_{C\,settle} = -3$ in (down) , $\Delta_{E\,settle} = 3$ in down

$\Delta_{C\,rot} = (0.02\text{rad})(16) = 3.84$ in up , $\Delta_{E\,rot} = (0.02)(36) = 8.64$ in up

$\Delta_{C\,tot} = 1.0784$ in down, $\Delta_{E\,tot} = 2.8321$ in up

Secondary structures

$f_{CC} = \frac{16}{EI}(16)(1/2)(2/3)(16) = 0.081355$ in up

$f_{EC} = \frac{1}{2}\left(\frac{16}{EI}\right)(16)(20 + 2/3 \cdot 16) = 0.23390$ in up

$\left(\frac{20}{EI}\right)(16)(1/2)(16) + \frac{1}{2}\left(\frac{16}{EI}\right)(16)(2/3)(16)$

$f_{CE} =$ $= 0.23390$ in up

$f_{EE} = \frac{1}{2}\left(\frac{36}{EI}\right)(36)(2/3)(36) = 0.92668$ in up

COMPLEX PROBLEMS

Problem from last class, on handout.

Continued.

$\Delta_{Ctot} = 1.0784$ in down

$\Delta_{Etot} = 2.8321$ in up

compatibility - need secondary structures (past page + handout)

$\Delta_{CTOT} + F_C f_{CC} + F_E f_{CE} = -5$ in (down)

1.0784 in + ~~Fdd~~ $F_c(-0.081355 \text{ in}) + F_E(-0.23390 \text{ in}) = 5$ in

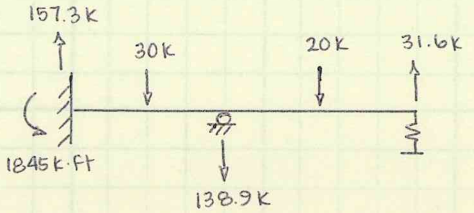
-2.8321 in + $F_c(-0.23390 \text{ in}) + F_E(-0.92668 \text{ in}) = F_E/k$

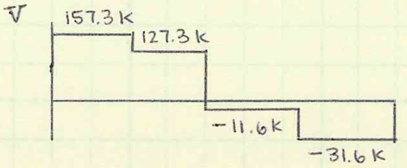
[downward = (+) here]

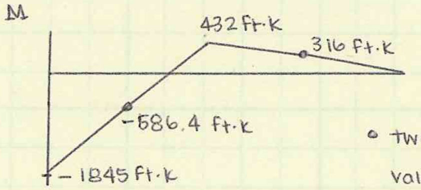
$F_C = 138.9$ k down

$F_E = 31.6$ k up

use reaction forces to draw shear / moment diagrams.







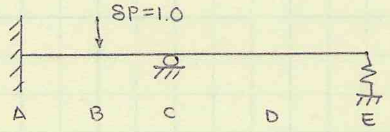
- two very poor slope value changes (in wrong direction, even)

To calculate ΔB

- moment-area
  - $t_{B/A}$ of final diagram - load effect
- virtual work
  - reuse primary structure

CONCLUSION OF TOPIC

Statically indeterminant structure (example)

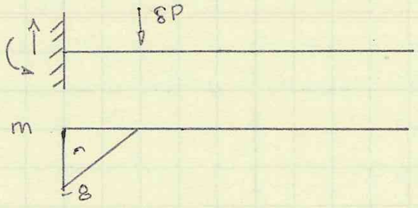


calculating Δ_B

this structure is statically indeterminate. Oy.

but – just needs to satisfy equilibrium

so, use primary structure



using virtual work

$$\delta P \cdot \Delta_B = \int m \frac{M}{EI} dx + \underline{\int m k(T) dx} + \text{rigid body motion}$$

yes, needs to be included again

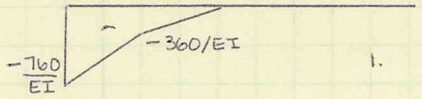
(even though a part of M calculations)

use full formula for displacement, always.

Superposition

Primary

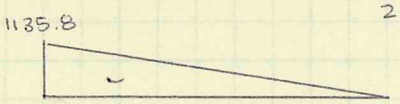
(distances aren't right)



1.

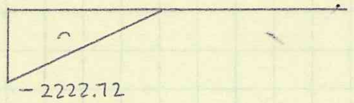
Thermal

2.



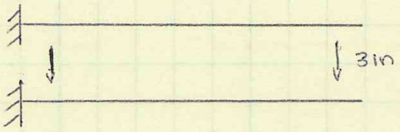
6.

Reactions

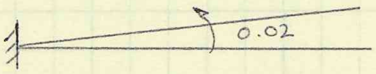


5.

Rigid Body Rotation



3.



4.

use virtual moment diagram above, real diagrams below, add

$$\Delta_{B1} = \tfrac{1}{2}\left(\frac{400}{EI}\right)(8)(2/3)(8) + \left(\frac{360}{EI}\right)(8)(4) \downarrow$$

$$\Delta_{B2} = \frac{\alpha \Delta T}{d}(8)(4) \uparrow$$

$$\Delta_{B3} = 3 \text{ in} \downarrow \qquad \Delta_{B4} = (0.02)(8) = \uparrow$$

$$\Delta_{B5} = \tfrac{1}{2}\left(\frac{2222.72}{EI}\right)(8)(\tfrac{1}{2})(2/3)(8) + \left(\frac{1111.36}{EI}\right)(8)(4) \downarrow$$

$$\Delta_{B6} = \left(\frac{883.4}{EI}\right)(8)(4) + \frac{1}{2}\left(\frac{1135.8-883.4}{EI}\right)(8)(2/3)(8) \uparrow$$

ADD 'EM UP! $\Sigma\Delta_B = 3.30 \text{ in} \downarrow$

don't need to include spring!

SOLVING FOR DEFLECTIONS

Rigid body movements

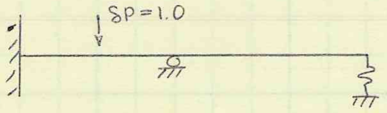
consider either as an addition to the internal side

or, as external work

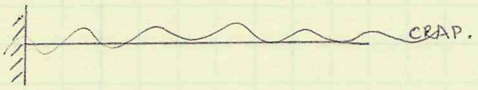
$$\left(\begin{matrix}\text{reactions}\\ \text{from virtual}\end{matrix}\right)\left(\begin{matrix}\text{real support}\\ \text{movement}\end{matrix}\right) + \delta P \cdot \Delta_B = \int m \frac{M}{EI}\,dx + \int m\,K(T)\,dx$$

$$1.0(-3\text{ in}) = -3.0\text{ in}$$
$$8.0(0.02) = 1.92\text{ in}$$

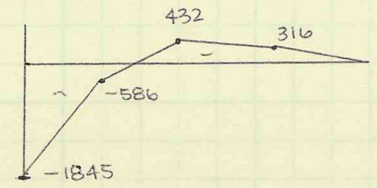
Δ_B, using calculated M diagram

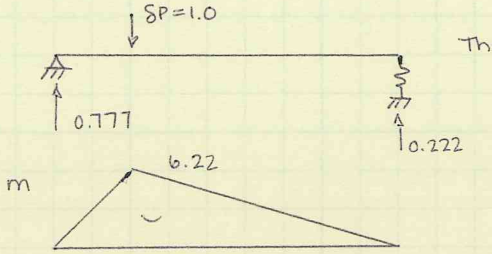


Assume $M_A=0$, $C_y=0$



Real structure





There we go...

$$\Delta_B = \int \frac{M}{EI} m\,dx$$

$$(EI)\Delta_B = (-1845+586)(8)(1/2)(1/3)(6.22) + (-586)(8)(1/2)(6.22) + (-586)(1/2)(4.64)(5.88)$$
$$+ (432)(1/2)(3.36)(4.61) + 1/2(\overset{117}{\cancel{432}})(10)(3.70) + (432)(10)(3.33) + 1/2(316)(10)(1.11)$$

$$\Delta_{B_{load}} = -0.872\text{ in up}$$

$$\Delta_{B_{therm}} = \frac{\alpha\Delta T}{d}(8)(1/2)(6.22) + \frac{\alpha\Delta T}{d}(28)(1/2)(6.22) = 1.75\text{ in down}$$

$$\Delta_{B_{spring}} = (0.222)\left(\frac{31.55}{k}\right) = 0.09\text{ in down}$$

$$\Delta_{B_{settle}} = (3\text{ in down})(0.777) = 2.33\text{ in down}$$

$$\Sigma\Delta_B = 3.30\text{ in down}$$

rotation at A not counted because support is a pin

no deflection change

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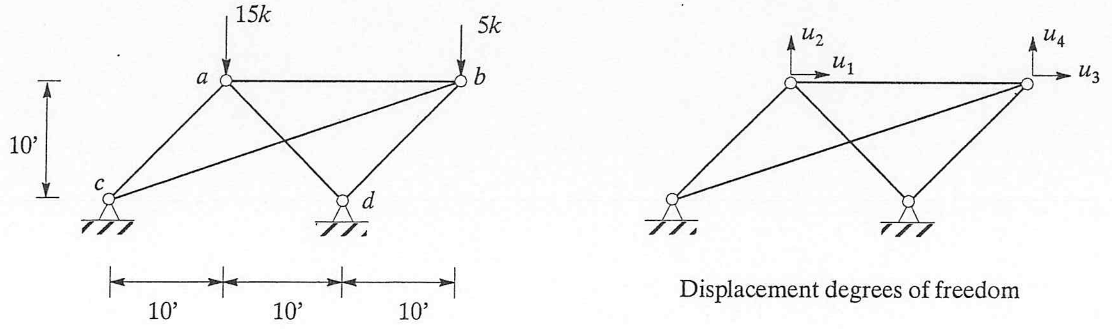
CE 363

Example Problem

The truss shown in Fig. 1 has four displacement degrees of freedom, numbered as shown on the sketch. All of the members have modulus of elasticity $E = 15{,}000$ *ksi*, and cross-sectional areas as follows:

Member	*Area* (in^2)
ab	1
ac	3
ad	2
bc	2
bd	3

The structure is subjected to a downward vertical load at point *a* of 15*k* and a downward vertical load at point *b* of 5*k*. Determine the 4x4 structure stiffness matrix of the structure. Find the joint displacements, member forces, and support reactions induced by these loads.

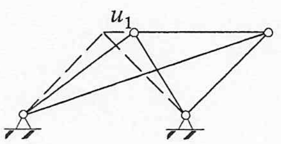


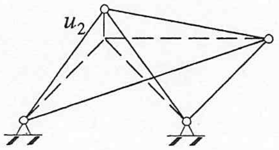
Displacement degrees of freedom

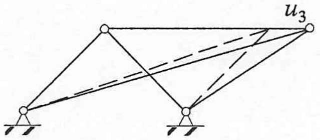
Fig. 1. Description of structure and displacement degrees of freedom

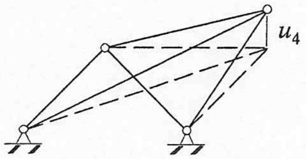
To solve this problem we must first identify the kinematic degrees of freedom. Once the degrees of freedom are identified, we can proceed to determine the relationship between the stretching Δ in each member and the displacements u at these degrees of freedom. We can express the member axial forces in terms of the member stretches by noting that $N = \Delta(EA/L)$. Since we know Δ in terms of the u's we can write expressions for the axial forces in each member in terms of the nodal displacements. Finally, we can establish equilibrium at each joint by setting the sum of forces equal to zero. Since we know the axial forces in terms of the u's, the result is a set of equilibrium equations in terms of the nodal displacements. If we take the equilibrium equations at the degrees of freedom only, this provides us with four equations with four unknown (displacements) for the present problem. This linear systems of equations can be solved to determine the displacements. The displacements can be substituted back into the axial force equations to get the axial forces. The reaction forces can then be determined from equilibrium with the member forces. The following gives two slightly different approaches to executing these steps.

1. Compute member distortions due to nodal displacements





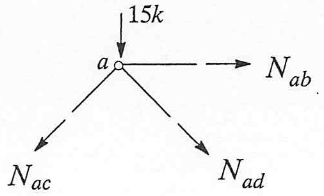




2. Evaluate axial forces in members in terms of nodal displacements

$$\begin{aligned}
\Delta_{ab} &= -u_1 + u_3 & [EA/L]_{ab} &= 750 & N_{ab} &= 750[-u_1 + u_3] \\
\Delta_{ac} &= \tfrac{\sqrt{2}}{2}[u_1 + u_2] & [EA/L]_{ac} &= 2250\sqrt{2} & N_{ac} &= 2250[u_1 + u_2] \\
\Delta_{ad} &= \tfrac{\sqrt{2}}{2}[-u_1 + u_2] & [EA/L]_{ad} &= 1500\sqrt{2} & N_{ad} &= 1500[-u_1 + u_2] \\
\Delta_{cb} &= \tfrac{\sqrt{10}}{10}[3u_3 + u_4] & [EA/L]_{cb} &= 300\sqrt{10} & N_{cb} &= 300[3u_3 + u_4] \\
\Delta_{bd} &= \tfrac{\sqrt{2}}{2}[u_3 + u_4] & [EA/L]_{bd} &= 2250\sqrt{2} & N_{bd} &= 2250[u_3 + u_4]
\end{aligned}$$

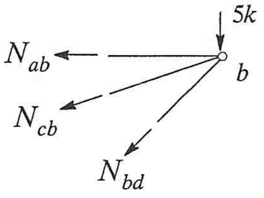
3. Find equations of equilibrium in the direction of each degree of freedom



At point *a*

$$(1) \quad \sum F_{xa} = 0 \qquad -\tfrac{\sqrt{2}}{2}N_{ac} + \tfrac{\sqrt{2}}{2}N_{ad} + N_{ab} = 0$$

$$(2) \quad \sum F_{ya} = 0 \qquad -\tfrac{\sqrt{2}}{2}N_{ac} - \tfrac{\sqrt{2}}{2}N_{ad} - 15 = 0$$



At point *b*

$$(3) \quad \sum F_{xb} = 0 \qquad -\tfrac{\sqrt{2}}{2}N_{bd} - \tfrac{3\sqrt{10}}{10}N_{cb} - N_{ab} = 0$$

$$(4) \quad \sum F_{yb} = 0 \qquad -\tfrac{\sqrt{2}}{2}N_{bd} - \tfrac{\sqrt{10}}{10}N_{cb} - 5 = 0$$

4. Substitute expressions for member forces in terms of nodal displacements

$$(1) \quad \sum F_{xa} = 0 \qquad -1125\sqrt{2}[u_1 + u_2] + 750\sqrt{2}[-u_1 + u_2] + 750[-u_1 + u_3] = 0$$

$$(2) \quad \sum F_{ya} = 0 \qquad -1125\sqrt{2}[u_1 + u_2] - 750\sqrt{2}[-u_1 + u_2] - 15 = 0$$

$$(3) \quad \sum F_{xb} = 0 \qquad -1125\sqrt{2}[u_3 + u_4] - 90\sqrt{10}[3u_3 + u_4] - 750[-u_1 + u_3] = 0$$

$$(4) \quad \sum F_{yb} = 0 \qquad -1125\sqrt{2}[u_3 + u_4] - 30\sqrt{10}[3u_3 + u_4] - 5 = 0$$

5. Gather coefficients of common terms

(1) $\sum F_{xa} = 0 \quad 3401.7u_1 + 530.3u_2 - 750u_3 = 0$

(2) $\sum F_{ya} = 0 \quad 530.3u_1 + 2651.7u_2 = -15$

(3) $\sum F_{xb} = 0 \quad -750u_1 + 3194.8u_3 + 1875.6u_4 = 0$

(4) $\sum F_{yb} = 0 \quad -1875.6u_3 + 1685.9u_4 = -5$

6. Put equations into matrix format **Ku=f**

$$\begin{bmatrix} 3401.7 & 530.3 & -750.0 & 0.0 \\ 530.3 & 2651.7 & 0.0 & 0.0 \\ -750.0 & 0.0 & 3194.8 & 1875.6 \\ 0.0 & 0.0 & 1875.6 & 1685.9 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.0 \\ -15.0 \\ 0.0 \\ -5.0 \end{bmatrix}$$

Units of **K** are *k/ft*

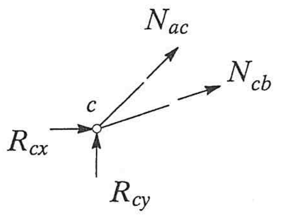
7. Solve equations to get

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.0024263 \\ -0.0061420 \\ 0.0066618 \\ -0.010377 \end{bmatrix} ft$$

8. Substituting the displacement values into the equations for the forces gives:

$$\begin{aligned} N_{ab} &= 750[-u_1 + u_3] = 3.72\,k \\ N_{ac} &= 2250[u_1 + u_2] = -8.36\,k \\ N_{ad} &= 1500[-u_1 + u_2] = -12.85\,k \\ N_{cb} &= 300[3u_3 + u_4] = 2.88\,k \\ N_{bd} &= 2250[u_3 + u_4] = -8.36\,k \end{aligned}$$

9. Find reactions from member forces

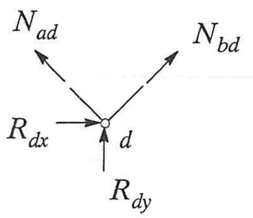


$$\sum F_{xc} = 0 \quad \tfrac{\sqrt{2}}{2}N_{ac} + \tfrac{3\sqrt{10}}{10}N_{cb} + R_{cx} = 0$$

$$\sum F_{yc} = 0 \quad \tfrac{\sqrt{2}}{2}N_{ac} + \tfrac{\sqrt{10}}{10}N_{cb} + R_{cy} = 0$$

$$R_{cx} = 3.179k$$

$$R_{cy} = 5.00k$$



$$\sum F_{xd} = 0 \quad -\tfrac{\sqrt{2}}{2}N_{ad} + \tfrac{\sqrt{2}}{2}N_{bd} + R_{dx} = 0$$

$$\sum F_{yd} = 0 \quad \tfrac{\sqrt{2}}{2}N_{ad} + \tfrac{\sqrt{2}}{2}N_{bd} + R_{dy} = 0$$

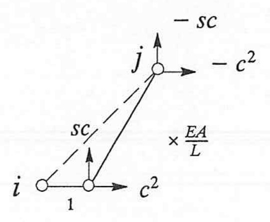
$$R_{dx} = -3.179k$$

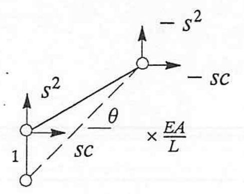
$$R_{dy} = 15.00k$$

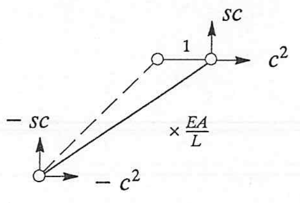
Clearly, the reaction forces satisfy global equilibrium.

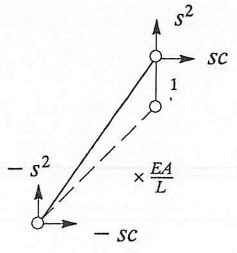
An alternative approach

1. Forces required on a typical member to move unit amount in member degrees of freedom:



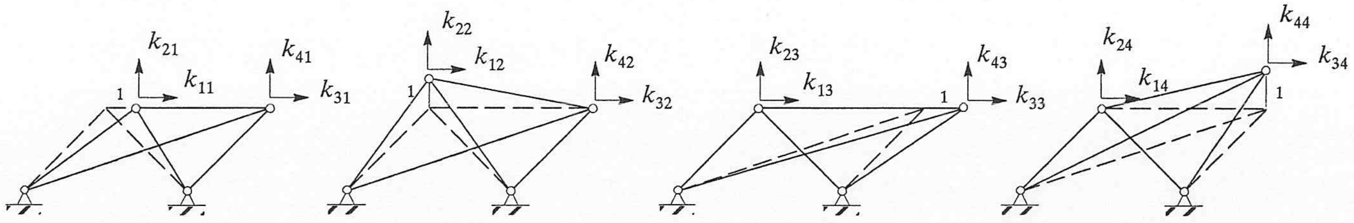






Where $c^2 \equiv \cos^2\theta$, $s^2 \equiv \sin^2\theta$, $sc \equiv \sin\theta\cos\theta$, $\theta \equiv$ angle of inclination of member

2. Forces required on structure to move unit amount in structure degrees of freedom



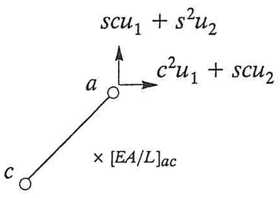
$u_1 = 1$, $u_2 = u_3 = u_4 = 0$

$u_2 = 1$, $u_1 = u_3 = u_4 = 0$

$u_3 = 1$, $u_1 = u_2 = u_4 = 0$

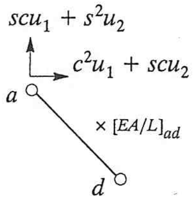
$u_4 = 1$, $u_1 = u_2 = u_3 = 0$

3. Evaluate the forces acting at points a and b due to stretching of the members



$\theta_{ac} = \pi/4$

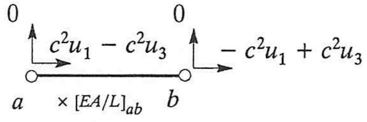
Member *ac*



$\theta_{ad} = 3\pi/4$

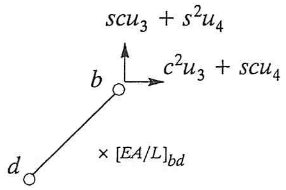
Member *ad*

Noting that sin(0)=0



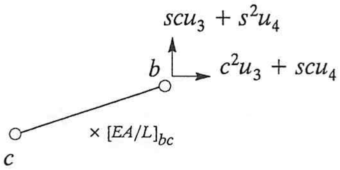
$\theta_{ab} = 0$

Member *ab*



$\theta_{bd} = \pi/4$

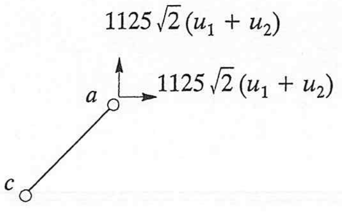
Member *bd*



$\theta_{bc} = \tan^{-1}(1/3)$

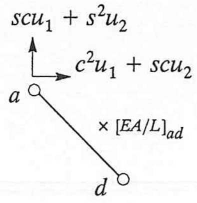
Member *bc*

3. Evaluate the forces acting at points a and b due to stretching of the members



$\theta_{ac} = \pi/4$

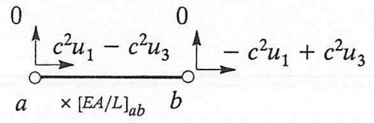
Member ac



$\theta_{ad} = 3\pi/4$

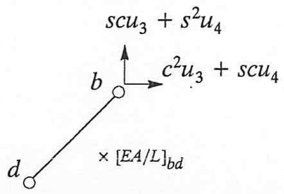
Member ad

Noting that sin(0)=0



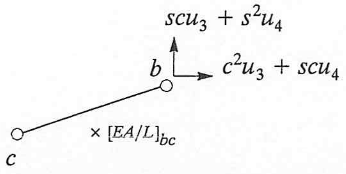
$\theta_{ab} = 0$

Member ab



$\theta_{bd} = \pi/4$

Member bd



$\theta_{bc} = \tan^{-1}(1/3)$

Member bc

4. Sum forces in the direction of each degree of freedom (*i.e.* at nodes a and b):

$$(1) \quad \sum F_{xa} = 0 \quad \overbrace{[(c^2EA/L)_{ac} + (c^2EA/L)_{ad} + (c^2EA/L)_{ab}]}^{k_{11}}u_1 + \underbrace{[(scEA/L)_{ac} + (scEA/L)_{ad}]}_{k_{21}}u_2 - \underbrace{[(c^2EA/L)_{ab}]}_{k_{31}}u_3 = 0$$

$$(2) \quad \sum F_{ya} = 0 \quad \overbrace{[(scEA/L)_{ac} + (scEA/L)_{ad}]}^{k_{12}}u_1 + \overbrace{[(s^2EA/L)_{ac} + (s^2EA/L)_{ad}]}^{k_{22}}u_2 = -15$$

$$(3) \quad \sum F_{xb} = 0 \quad -\overbrace{[(c^2EA/L)_{ab}]}^{k_{13}}u_1 + \overbrace{[(c^2EA/L)_{bc} + (c^2EA/L)_{ab} + (c^2EA/L)_{bd}]}^{k_{33}}u_3 + \underbrace{[(scEA/L)_{bc} + (scEA/L)_{bd}]}_{k_{43}}u_4 = 0$$

$$(4) \quad \sum F_{yb} = 0 \quad \overbrace{[(scEA/L)_{bc} + (scEA/L)_{bd}]}^{k_{34}}u_3 + \overbrace{[(s^2EA/L)_{bc} + (s^2EA/L)_{bd}]}^{k_{44}}u_4 = -5$$

All k's not shown are zero.

5. Substituting values for trigonometric functions and EA/L, and expressing equations in matrix format $\mathbf{Ku}=\mathbf{f}$ gives

$$\begin{bmatrix} 3401.7 & 530.3 & -750.0 & 0.0 \\ 530.3 & 2651.7 & 0.0 & 0.0 \\ -750.0 & 0.0 & 3194.8 & 1875.6 \\ 0.0 & 0.0 & 1875.6 & 1685.9 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.0 \\ -15.0 \\ 0.0 \\ -5.0 \end{bmatrix} \qquad \text{Units of } \mathbf{K} \text{ are } k/ft$$

These equations are exactly the same as those obtained by the other method.

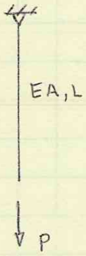
The second approach is nothing more than an abbreviated version of the first approach, wherein we combine the first few steps into one by recognizing that we can solve the problem of determining the relationship between the *x-y* components of the axial force in each member and the nodal displacements once and for all. For specific problems we just use these relationships, adding the contribution that each member makes to the overall nodal equilibrium equations. Study these two approaches carefully until you can see how they are related.

STIFFNESS METHOD

overview

- displacement quantities are unknowns
- kinematically (in)determinate problems } require displacements to be compatible
  - ↳ as opposed to statically
  - we know motion of structure
- enforce equilibrium

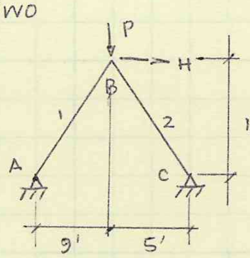
Example



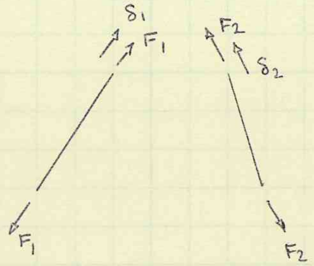
Δ results, $\Delta = \frac{PL}{EA}$, $L/EA = 1/\text{stiffness}$ = flexibility

$P = \frac{EA}{L}\Delta$ EA/L = stiffness

Number Two

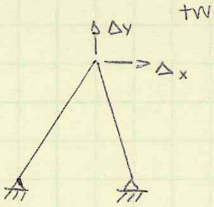


EA constant



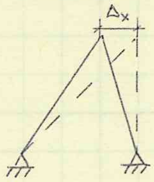
how many degrees of freedom? (unknown displacements)

two — vertical and horizontal movement at B

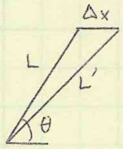


Require compatibility between Δ_x, Δ_y and δ_1, δ_2

consider Δ_x

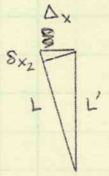


1 elongates, 2 compresses



$\delta x = \Delta_x \cos\theta$

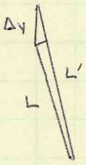
in this case, $\delta_x = 3/5\,\Delta_x$ (member 1)



compresses — negative elongation

$\delta_{x_2} = -5/13\,\Delta_x$

Δy:



$\delta_{y_1} = 4/5\,\Delta y$ $\delta_{y_2} = 12/13\,\Delta y$

$\delta_y = \Delta_y \sin\theta$

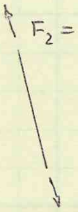
combine δs:

$\delta_1 = 3/5\,\Delta_x + 4/5\,\Delta_y$; $\delta_2 = -5/13\,\Delta_x + 12/13\,\Delta_y$

STIFFNESS METHOD

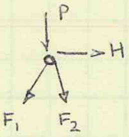
Relating displacements and forces

$F_1 = \frac{EA}{15}(\delta_1)$



$F_2 = \frac{EA}{13}(\delta_2)$

use equilibrium to find solution



Joint B

$\sum F_x: \ 3/5 F_1 - 5/13 F_2 = H$

$\sum F_y: \ 4/5 F_1 + 12/13 F_2 = -P$ subbing in, unknowns are Δ_x, Δ_y

$$3/5\left[\frac{EA}{L_1}\left(3/5\Delta_x + 4/5\Delta_y\right)\right] - 5/13\left[\frac{EA}{L_2}\left(-5/13\,\Delta_x + 12/13\,\Delta_y\right)\right] = H$$

$$4/5\left[\frac{EA}{L_1}\left(3/5\Delta_x + 4/5\Delta_y\right)\right] + 12/13\left[\frac{EA}{L_2}\left(-5/13\,\Delta_x + 12/13\,\Delta_y\right)\right] = -P$$

$$\begin{bmatrix} \left(\frac{3}{5}\cdot\frac{EA}{L_1}\cdot\frac{3}{5} + \frac{5}{13}\cdot\frac{EA}{L_2}\cdot\frac{5}{13}\right) & \left(\frac{3}{5}\cdot\frac{EA}{L_1}\cdot\frac{4}{5} - \frac{5}{13}\cdot\frac{EA}{L_2}\cdot\frac{12}{13}\right) \\ \left[\frac{4}{5}\cdot\frac{EA}{L_1}\cdot\frac{3}{5} + \frac{12}{13}\cdot\frac{EA}{L_2}\left(\frac{-5}{13}\right)\right] & \left(\frac{4}{5}\cdot\frac{EA}{L_1}\cdot\frac{4}{5} + \frac{\overset{12}{\cancel{5}}}{13}\cdot\frac{EA}{L_2}\cdot\frac{12}{13}\right) \end{bmatrix}\begin{bmatrix}\Delta_x \\ \Delta_y\end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

$\underset{\sim}{K}$ matrix:
- symmetric
- positive diagonals

$$\boxed{\underset{\sim}{K}\cdot\underset{\sim}{u} = \underset{\sim}{F}}$$

force vector (known)

structural stiffness matrix (symmetric)

displacement vector (unknown)

More generally...

K_{ij} = force induced at i due to a unit displacement applied at j, with all other DOF = 0. (comparable to f_{ij})

K≡stiffness coefficient

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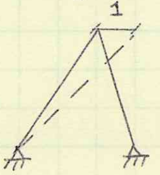
combining initial steps:



kinematically determinate structure—
we prescribe motion, know it exactly

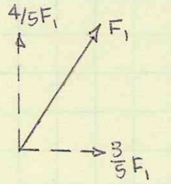
$F_1 = 3/5\,\frac{EA}{L_1}$, $L_2 = \frac{-5}{13}\,\frac{EA}{L_2}$

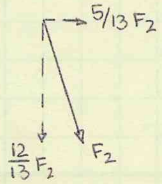
need force in x·direction due to displacement in x·direction

$F_1\cos\theta = 3/5\cdot 3/5\cdot\frac{EA}{L_1}$, $F_2\cos\theta = (-5/13)^2\,\frac{EA}{L_2}$

STIFFNESS METHOD

Resolving Forces





K_{11} = force in the x·direction due to a unit displacement in x·direction

$$= \tfrac{3}{5}F_1 + \tfrac{5}{13}F_2 = \left(\frac{3}{5}\right)^2 \frac{EA}{L_1} + \left(\frac{5}{13}\right)^2 \frac{EA}{L_2}$$

if we use kinematically determinate structures, various cases can be superimposed easily.

$K_{\cancel{12}\,21}$ = force in y due to disp. in x

$$= \tfrac{4}{5}F_1 - \tfrac{12}{13}F_2 = \text{eq. from before}$$

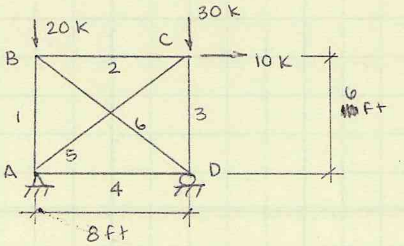
For our own info...

$$\begin{bmatrix} \Delta_x \\ \Delta_y \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 28.429 & -1.232 \\ -1.232 & 9.295 \end{bmatrix} \begin{bmatrix} H \\ -P \end{bmatrix}$$

- then solve for member forces
- check accuracy using equilibrium at B

STIFFNESS METHOD

Example problem



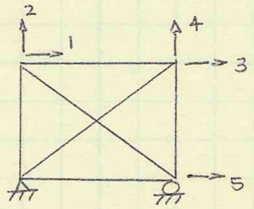
E = 29000 ksi

A = 20 in²

Procedure review for stiffness method

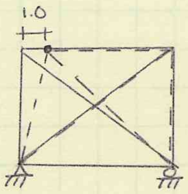
1. How many unknown (indeterminate) kinematic DOF?
2. Analyze kinematically determinate structure(s) to form $\underset{\sim}{K}$ (global stiffness matrix)
3. Set up equilibrium equations
4. Solve for the unknown displacements (potential stopping point)
5. compute actual member deformations (note: not displacements)
6. compute member forces

DOF : 5 (Bx, By; Cx, Cy, Dx)



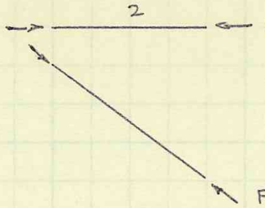
now, move each one a unit amount while holding others = 0.

$u_1 = 1$, all others = 0



members 1,2,6 get stressed

but, 1 just rotates – no stress



$F_2 = \frac{EA}{8ft}(1) = \frac{EA}{L}\Delta$

$F_6 = \frac{4}{5}(1)\frac{EA}{10ft}$

At joint B, what force is needed to cause these forces?

K_{11} = force at 1 due to unit Δ at 1

$= \frac{EA}{8} + \frac{EA}{10}\left(4/5\right)^2$

↳ x-component of F_6

$K_{21} = {}^{-3}/_5 \cdot {}^{4}/_5 \left(\frac{EA}{10}\right)$

↳ (-) because force is in opposite direction of assumed deflection

$K_{31} = \frac{-EA}{8}$

$K_{41} = 0$

$K_{51} = -\frac{4}{5} \cdot \frac{4}{5} \cdot \frac{EA}{10}$

with one diagram and unit Δ, all K values for that column can be calculated.

Additionally, $K_{12} = K_{21}$, etc.

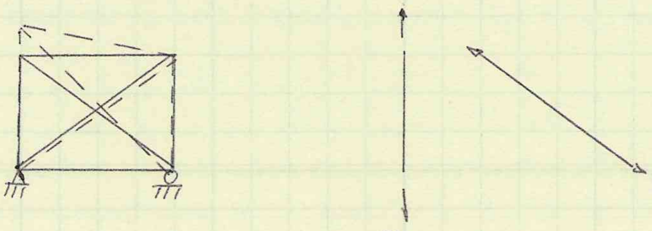
$$K = \begin{bmatrix} 63EA/4000 & \vdots & \\ -EA/250 & \vdots & \\ -EA/96 & \vdots & \cdots \\ 0 & \vdots & \\ 4EA/750 & \vdots & \end{bmatrix}$$

very repetitive, no choices to make – excellent for computers to do.

STIFFNESS METHOD

Example (cont'd)

now set $u_2 = 1$, all others $= 0$



$F_6 = \frac{EA}{120}\,\frac{3}{5}$

$F_1 = \frac{EA}{72}$

$k_{12} = \frac{-4}{5} \cdot 3/5 \, \frac{EA}{120} = k_{21}$

$k_{22} = \frac{EA}{72} + \frac{4}{5} \cdot \frac{3}{5} \, \frac{EA}{120}$

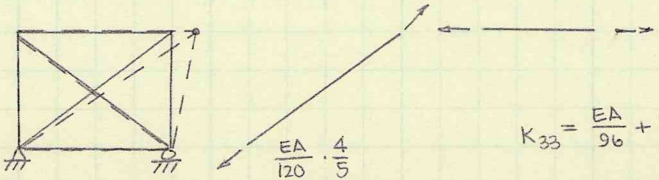
$k_{32} = 0$

$k_{42} = 0$

$k_{52} = \frac{4}{5} \cdot \frac{3}{5} \, \frac{EA}{120}$

$u_3 = 1.0$

SIMILAR TO HOMEWORK PROBLEM



$\frac{EA}{96}$

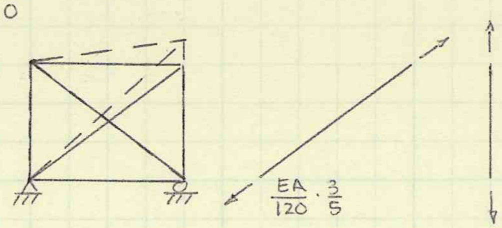
$\frac{EA}{120} \cdot \frac{4}{5}$

$k_{33} = \frac{EA}{96} + \frac{4}{5} \cdot \frac{4}{5} \, \frac{EA}{120}$

$k_{43} = \frac{4}{5} \cdot \frac{3}{5} \, \frac{EA}{120}$

$k_{53} = 0$

$u_4 = 1.0$



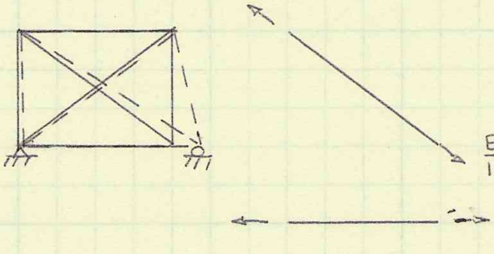
$\frac{EA}{72}$

$\frac{EA}{120} \cdot \frac{3}{5}$

$k_{44} = \frac{EA}{72} + \frac{3}{5} \, \frac{EA}{120} \cdot \frac{3}{5}$

$k_{54} = 0$

$u_5 = 1.0$



$\frac{EA}{120} \cdot \frac{4}{5}$

$\frac{EA}{96}$

$k_{55} = \frac{EA}{96} + \frac{4}{5} \cdot \frac{EA}{120} \cdot \frac{4}{5}$

$$\underset{\sim}{K} = \begin{bmatrix} 9135 & & & & \\ -2320 & 9795\,5/9 & & & \\ -6041\,2/3 & 0 & 9135 & & \\ 0 & 0 & 2320 & 9795\,5/9 & \\ -3093\,1/3 & 2320 & 0 & 0 & 9135 \end{bmatrix} \leftarrow \text{sym.}$$

Note: diagonal terms are larger than the off. diagonals.

STIFFNESS MATRIX

Example (cont'd)

Equilibrium

$$\underset{\sim}{K}\underset{\sim}{u} = \underset{\sim}{F}$$

$$\underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \qquad \underset{\sim}{F} = \begin{bmatrix} 0 \\ -20 \\ 10 \\ -30 \\ 0 \end{bmatrix} \text{K}$$

solve equation
(for tests, be able to invert a 3×3 matrix – calculator use is okay)

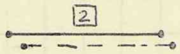
$$\underset{\sim}{u} = \begin{bmatrix} 0.00243 \\ -0.00177 \\ 0.003703 \\ -0.003940 \\ 0.00127 \end{bmatrix} \text{in}$$

keep high precision for future calculations

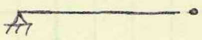
compute member forces

u_1 (movement to the right) does not induce any stresses, only u_2, which puts M1 in compression

$$F_1 = \frac{EA}{L}\Delta_2 = \frac{(29000\text{ ksi})(20\text{ in}^2)}{72\text{ in}}(0.00177\text{ in}) = 14.26\text{ ksi comp.}$$

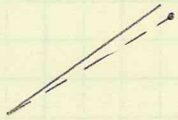


$$F_2 = \frac{EA}{L}(\Delta_3 - \Delta_1) = 7.69\text{ K T}$$

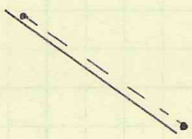


$$F_4 = \frac{EA}{L}\Delta_5 = \frac{(29000\text{ ksi})(20\text{ in}^2)}{96\text{ in}}(0.00127\text{ in}) = 7.67\text{ K T}$$

$$F_3 = \frac{EA}{L}\Delta_4 = 31.74\text{ K C}$$



$$F_5 = \frac{EA}{L}\left[4/5\, u_3 + 3/5\, u_4\right] = 2.89\text{ K T}$$

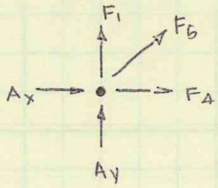


$$F_6 = \frac{EA}{L}\left[\frac{3}{5}(u_2) + \frac{4}{5}(u_5 - u_1)\right] = 9.62\text{ K C}$$

STIFFNESS METHOD

Reaction calculations

use method of joints — if equilibrium cannot be achieved, a problem likely exists.



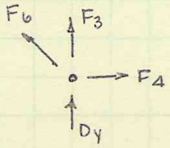
$\Sigma F_x = 0: \; A_x + F_4 + 4/5 \, F_5 = 0$

(hopefully $A_x = 10$!)

although structure is statically determinate, it's nice to verify member forces.

$\Sigma F_y = 0: \; A_y + F_1 + 3/5 \, F_5 = 0$

$A_y = 12.53 k$ up



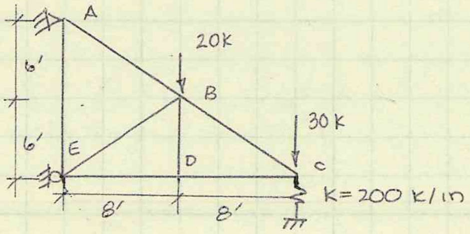
$\Sigma F_x = 0: \; F_4 + 4/5 \, F_6 = 0$

$\Sigma F_y = 0: \; D_y + F_3 + 3/5 \, F_6 = 0$

$D_y = 37.5 k$ up

STIFFNESS METHOD

Truss Problem

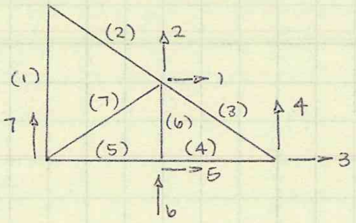


E = 29000 ksi

$A = 20 \text{ in}^2$

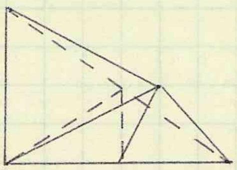
1 degree of indeterminacy in old methods (1 redundant)

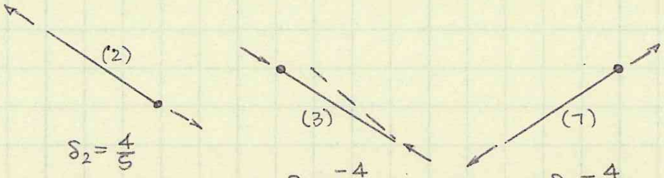
7 degrees of kinematic freedom



developing the K matrix

$u_1 = 1.0$





$\delta_2 = \frac{4}{5}$ $\delta_3 = -\frac{4}{5}$ $\delta_7 = \frac{4}{5}$

$k_{11} = \frac{EA}{L_d} \cdot \frac{4}{5} (1+1+1) \frac{4}{5}$

$k_{21} = \frac{-EA}{L_d} \cdot \frac{4}{5} \cdot \frac{3}{5}$

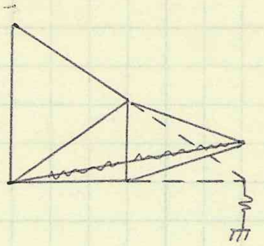
$k_{31} = -\frac{4}{5} \frac{EA}{L_d} \frac{4}{5}$

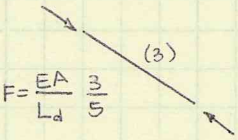
$k_{41} = \frac{EA}{L_d} \frac{4}{5} \cdot \frac{3}{5}$

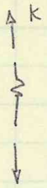
$k_{51} = 0 = k_{61}$

$k_{71} = -\frac{EA}{L} \frac{4}{5} \cdot \frac{3}{5}$

$u_4 = 1.0$



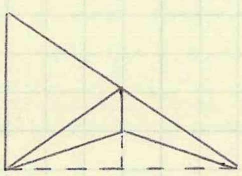


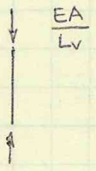


$k_{44} = \frac{3}{5} \frac{EA}{L_d} \cdot \frac{3}{5} + \underline{\underline{K}}$

$k_{45} = 0$ … as do all of them

$u_6 = 0$





$k_{16} = 0$

$k_{26} = -EA/L_v$

$k_{36} = k_{46} = k_{56} = 0$

$k_{66} = \frac{EA}{L_v}$

$k_{76} = 0$

TRUSS CALCULATIONS

Example, cont'd

Force vector – external loads

$$\underset{\sim}{F} = \begin{bmatrix} 0 \\ -20 \\ 0 \\ -30 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{k}$$

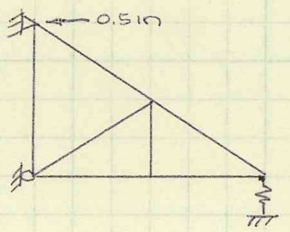
(candy bar worthy ⌣̈)

solve for vector $\underset{\sim}{u} = \begin{bmatrix} 0.003683 \\ -0.01314 \\ -0.00945 \\ -0.04296 \\ -0.004725 \\ -0.0131341 \\ -0.002483 \end{bmatrix}$ in

note: $u_2 = u_6$, $F_6 = 0$ (zero force member)
look at structure – yup, needs to be
– spring: in compression ($-u_4$)
– member (2) is in max tension
$F_2 = 52.3\,K$
– member (5) controls compression
$F_5 = -28.55\,K$

Support movement

same problem + A moves left 1/2"



count it in load vector
or stiffness matrix?
– load vector would require solving problem for forces created by displacement
– so, stiffness matrix
define A_x as another DOF

solving a system of equations using a partitioned matrix

$$\left[\begin{array}{c|c} \underset{\sim}{k}_{00} & \underset{\sim}{k}_{01} \\ \hline \underset{\sim}{k}_{10} & \underset{\sim}{k}_{11} \end{array}\right] \left\{\begin{array}{c} \underset{\sim}{u}_0 \\ \hline \underset{\sim}{u}_1 \end{array}\right\} = \left\{\begin{array}{c} \underset{\sim}{F}_0 \\ \hline \underset{\sim}{F}_1 \end{array}\right\}$$

always square

transpose of each other

$\underset{\sim}{u}_0$ = vector of unknown displacements
$\underset{\sim}{u}_1$ = vector of known displacements

$\underset{\sim}{F}_0$ = vector of known forces
$\underset{\sim}{F}_1$ = vector of unknown forces

(1) $\underset{\sim}{k}_{00}\underset{\sim}{u}_0 + \underset{\sim}{k}_{01}\underset{\sim}{u}_1 = \underset{\sim}{F}_0$

(2) $\underset{\sim}{k}_{11}\underset{\sim}{u}_0 + \underset{\sim}{k}_{11}\underset{\sim}{u}_1 = \underset{\sim}{F}_1$

—— known
- - - unknown

solve Eq(1) for $\boxed{\underset{\sim}{u}_0 = \underset{\sim}{k}_{00}^{-1}(\underset{\sim}{F}_0 - \underset{\sim}{k}_{01}\underset{\sim}{u}_1)}$

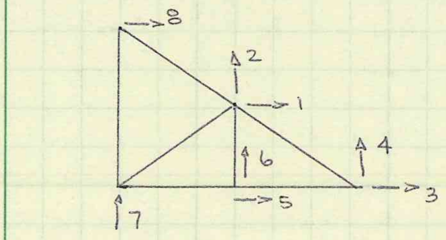
solve Eq(2) using $\underset{\sim}{u}_0$, solve for $\underset{\sim}{F}_1$

$$\underset{\sim}{F}_1 = \left[\underset{\sim}{k}_{10}\left(\underset{\sim}{k}_{00}^{-1}(\underset{\sim}{F}_0 - \underset{\sim}{k}_{01}\underset{\sim}{u}_1)\right) + \underset{\sim}{k}_{11}u_{11}\right]$$

$$\boxed{\underset{\sim}{F}_1 = \left[\underset{\sim}{k}_{11} - \underset{\sim}{k}_{10}\underset{\sim}{k}_{00}^{-1}k_{01}\right]\underset{\sim}{u}_1 + \underset{\sim}{k}_{10}k_{00}^{-1}\underset{\sim}{F}_0}$$

MATRIX MATH

Going back to example



$$\underset{\sim}{K} = \left[\begin{array}{ccccccc|c} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\ k_{21} & & & & & & \vdots & \vdots \\ k_{31} & & & & & & \vdots & \vdots \\ k_{41} & & & \underset{\sim}{K}_{00} & & & \vdots & \vdots \\ k_{51} & & & & & & \vdots & \vdots \\ k_{61} & & & & & & \vdots & \vdots \\ k_{71} & \cdots & \cdots & \cdots & \cdots & \cdots & k_{77} & k_{78} \\ \hline k_{81} & \cdots & \cdots & \cdots & & & k_{87} & k_{88} \end{array}\right]$$

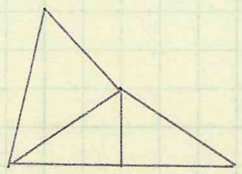
$\underset{\sim}{K}_{01}$

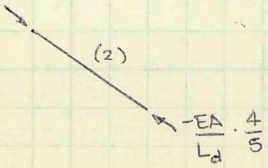
$\underset{\sim}{K}_{11}$

$\underset{\sim}{K}_{10}$

gather unknowns together as low numbers

$u_8 = 1.0$





$$k_{18} = -\frac{4}{5}\,\frac{EA}{L_d}\,\frac{4}{5}$$

$$k_{28} = \frac{EA}{L_d}\,\frac{4}{5} \cdot \frac{3}{5}$$

$$k_{38} = k_{48} = k_{58} = k_{68} = k_{78} = 0$$

$$k_{88} = \frac{EA}{L_d}\,\frac{4}{5} \cdot \frac{4}{5}$$

K matrix is complete

u · vector:

$$u = \left[\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \hline -0.5'' \end{array}\right] \begin{array}{l} \\ \\ \\ \underset{\sim}{u}_0 \\ \\ \\ \\ \underset{\sim}{u}_1 \end{array}$$

F · vector

$$F = \left[\begin{array}{c} 0 \\ -20 \\ 0 \\ -30 \\ 0 \\ 0 \\ 0 \\ \hline A_x \end{array}\right] \begin{array}{l} \\ \\ \\ \underset{\sim}{F}_0 \\ \\ \\ \\ \underset{\sim}{F}_1 \end{array}$$

$$\underset{\sim}{u}_0 = \underset{\sim}{K}_{00}^{-1}\left(\underset{\sim}{F}_0 - \underset{\sim}{K}_{01}\underset{\sim}{u}_1\right)$$

$$\underset{\sim}{F}_1 = \left[\underset{\sim}{K}_{11} - \underset{\sim}{K}_{10}\underset{\sim}{K}_{00}^{-1}\underset{\sim}{K}_{01}\right]\underset{\sim}{u}_1 + \underset{\sim}{K}_{10}\underset{\sim}{K}_{00}^{-1}\underset{\sim}{F}_0$$

now: max tension = 217 k (2)

max. comp = −160 k (5)

Reaction forces

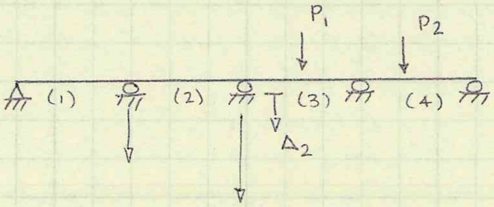
before: equilibrium at joint

now: have DOF at all nodes

use known displacements (=0) and unknown forces, and this method, to calculate reaction forces

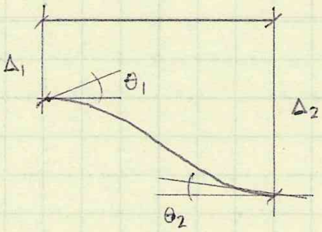
BEAM ANALYSIS

Stiffness-based approach



consider just one span (2)

Kinematically determinant structures:



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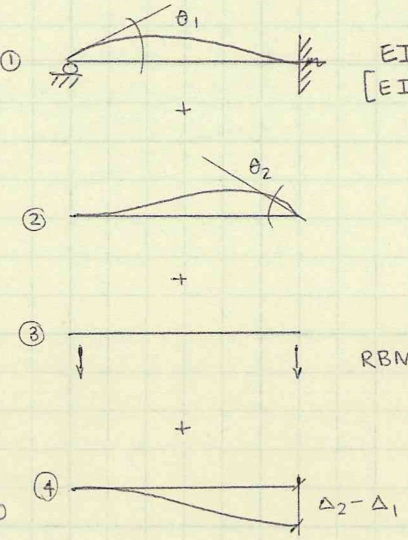


$EI v^{IV} = 0$

$[EI(x)\, v''(x)]'' = 0$

RBM = Δ_1

check superposition:

rotation at L = $\theta_1 + 0 + 0 + 0$

rotation at R = $0 + \theta_2 + 0 + 0$

deflection at L = $0 + 0 + \Delta_1 + 0$

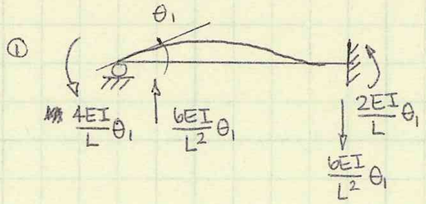
deflection at R = $0 + 0 + \Delta_1 + \Delta_2 - \Delta_1$

what moments and shears are needed to create these shapes?

using diff. EQs:

$EI v^{IV} = 0$, $v = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

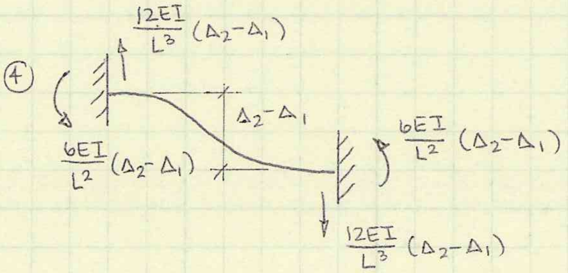
use boundary conditions to solve for constants



$v(0) = 0$ $v'(0) = \theta_1$ $a_0 = 0$, $a_1 = \theta$

$v(L) = 0$ $v'(L) = 0$

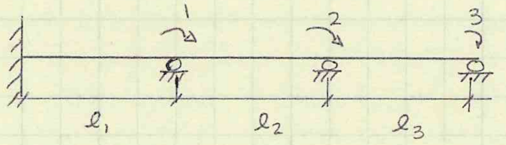
② looks the same, but backwards, with θ_2



foundation for beam stiffness matricies

BEAM STIFFNESS

Example



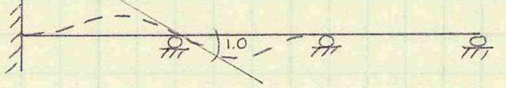
EI uniform

1. Determine degrees of freedom (translation up/down, rotation)

2. Set one DOF = 1, all others = 0

↻ rotation is positive, 3 DOFs

$u_1 = 1$, all others = 0



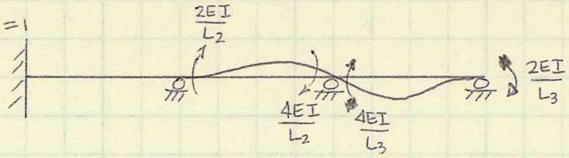
only consider forces/moments where we have DOFs

k_{11} = moment required at 1 due to a unit rotation at 1 (as opposed to force/disp.)

$= \frac{4EI}{L_1} + \frac{4EI}{L_2}$

$k_{12} = \frac{2EI}{L_2}$ $k_{13} = 0 = k_{31}$

$u_2 = 1$



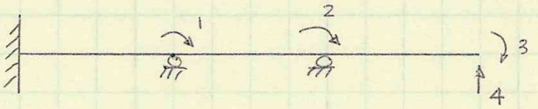
$k_{22} = \frac{4EI}{L_2} + \frac{4EI}{L_3}$

$k_{23} = \frac{2EI}{L_3}$

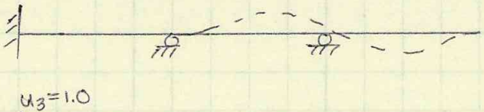
$u_3 = 1.0$

$k_{33} = \frac{4EI}{L_3}$

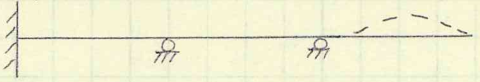
What if cantilevered?



$u_2 = 1.0$



$u_3 = 1.0$



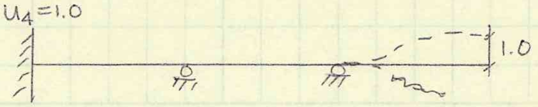
$k_{11} = \frac{4EI}{L_1} + \frac{4EI}{L_2}$ $k_{21} = \frac{2EI}{L_2}$ $k_{31} = k_{41} = 0$

$k_{22} = \frac{4EI}{L_2} + \frac{4EI}{L_3}$ $k_{32} = \frac{2EI}{L_3}$ $k_{42} = \frac{+6EI}{(L_3)^2}$

$k_{33} = \frac{4EI}{L_3}$ $k_{43} = \frac{6EI}{(L_3)^2}$

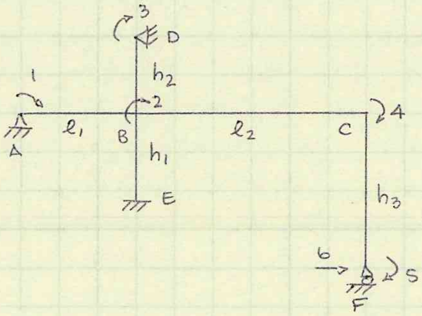
$k_{44} = \frac{12EI}{(L_3)^3}$

$u_4 = 1.0$



BEAM ANALYSIS

Example (in-class)



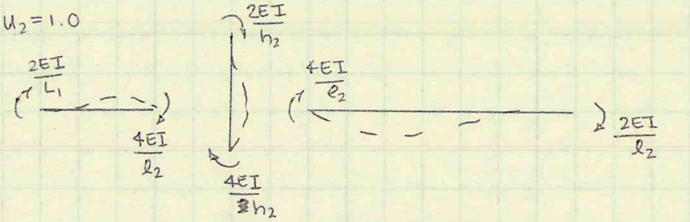
$u_1 = 1.0$

$K_{11} = \frac{4EI}{\ell_1}$

$K_{21} = \frac{2EI}{\ell_1}$

$K_{31}, K_{41}, K_{51}, K_{61}$ all $= 0$

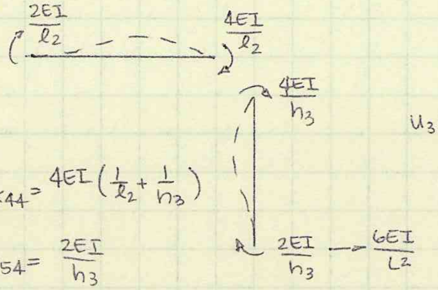
$u_2 = 1.0$



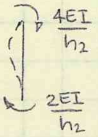
$K_{22} = \frac{4EI}{\ell_2} + \frac{4EI}{h_2} + \frac{4EI}{h_1} + \frac{4EI}{\ell_2}$

$K_{32} = \frac{2EI}{h_2}$ $K_{42} = \frac{2EI}{\ell_2}$ $K_{52} = K_{62} = 0$

$u_4 = 1.0$



$u_3 = 1.0$



$K_{33} = \frac{4EI}{h_2}$

$K_{43} = K_{53} = K_{63} = 0$

$K_{44} = 4EI\left(\frac{1}{\ell_2} + \frac{1}{h_3}\right)$

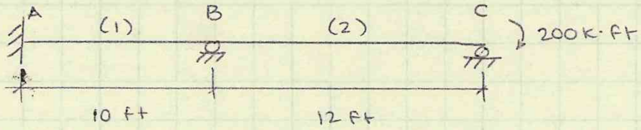
$K_{54} = \frac{2EI}{h_3}$

$K_{64} = \frac{6EI}{h_3^2}$

$K_{55} = \frac{4EI}{h_3}$, $K_{65} = \frac{6EI}{h_3}$, $K_{66} = \frac{12EI}{h_3^2}$

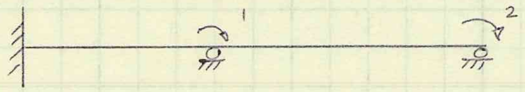
FINISHING A PROBLEM

Example



$E = 15000\text{ ksi}$

$I = 250\text{ in}^4$



$u = 1.0$

$$k_{11} = \frac{4EI}{10\text{ft}} + \frac{4EI}{12\text{ ft}}$$

$$k_{21} = \frac{2EI}{12\text{ ft}}$$

$u_2 = 1.0$

$$k_{22} = \frac{4EI}{12\text{ft}}$$

Equilibrium equations

$$\underset{\sim}{K} = EI \begin{bmatrix} 4/10 + 4/12 & 2/12 \\ 2/12 & 4/12 \end{bmatrix}$$

watch units!

$$EI \begin{bmatrix} 2/77 + 1/30 & 1/77 \\ 1/77 & 1/36 \end{bmatrix}$$

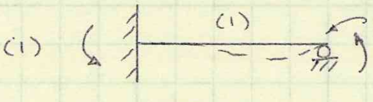
$$\underset{\sim}{F} = \begin{Bmatrix} 0 \\ 200 \end{Bmatrix} \text{k·ft} \qquad \underset{\sim}{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -0.005908 \\ 0.025994 \end{Bmatrix} \text{rad}$$

or, 2400 k·in

So, what can you do with this information?

develop V & M diagrams

- we have displacements, need deformations
- in this case, they're the same



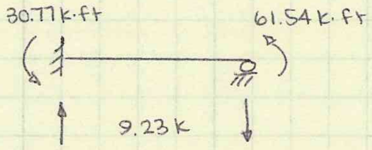
$\theta = 0.005908\text{ rad}$

$\frac{4EI}{10\text{ft}} (0.005908\text{ rad})$

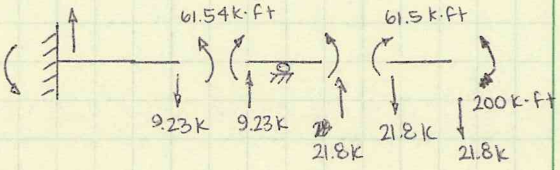
$M = \frac{2EI}{10\text{ft}} \theta$

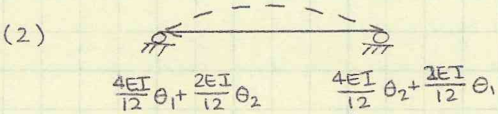
$\downarrow \frac{6EI}{L^2} \theta$

$\uparrow V = \frac{6EI}{L^2} \theta$



* not actually considering supports





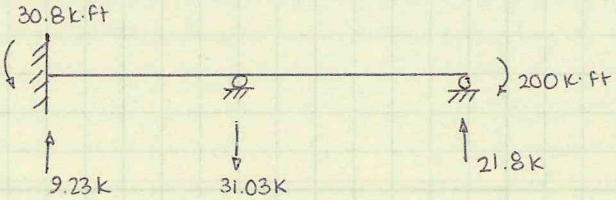
$\frac{4EI}{12} \theta_1 + \frac{2EI}{12} \theta_2$ $\qquad$ $\frac{4EI}{12} \theta_2 + \frac{2EI}{12} \theta_1$

$\uparrow$ 6.4k $\downarrow$ 28.2K $\qquad$ $\downarrow$ 6.4k $\uparrow$ 28.2K

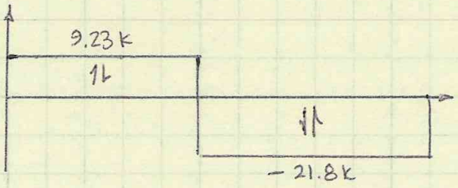
moment equilibrium!

EXAMPLE

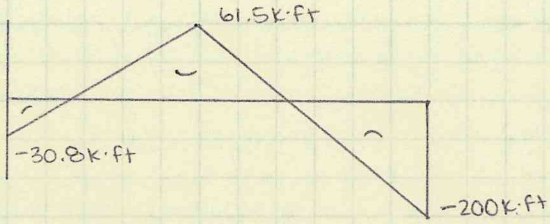
Shear & Moment

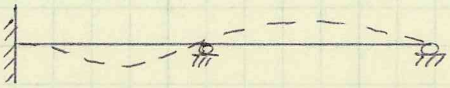


Shear



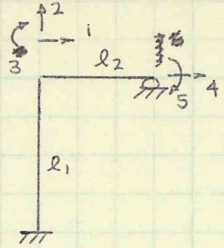
Moment





Frame considerations

Axial deformation capacity



EI, EA

$u_1 = 1.0$

$K_{11} = \frac{12EI}{\ell_1^3} + \frac{EA}{\ell_2}$

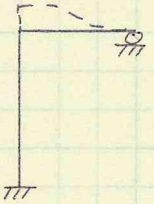
$K_{21} = 0$

$K_{31} = \frac{-6EI}{\ell_1^2}$

$K_{41} = \frac{-EA}{\ell_2}$

$K_{51} =$

$u_2 = 1.0$



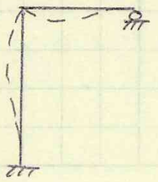
$K_{22} = \frac{EA}{\ell_1} + \frac{12EI}{\ell_2^3}$

$K_{32} = \frac{-6EI}{\ell_2^2}$

$K_{42} = 0$

$K_{52} = \frac{-6EI}{\ell_2^2}$

$u_3 = 1.0$



$K_{33} = \frac{4EI}{\ell_1} + \frac{4EI}{\ell_2}$

$K_{43} = \frac{-6EI}{\ell_1^2}$

$K_{53} = \frac{2EI}{\ell_2}$

$K_{44} = \frac{EA}{\ell_2}$

$K_{54} = 0$

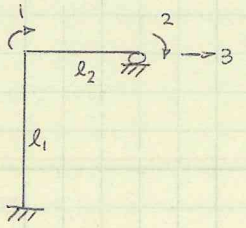
$K_{55} = \frac{4EI}{\ell_2}$

small deflections allow for axial and stiffness(?) movements to be uncoupled (no PΔ effects)

flexural deformations

AXIAL DEFORMATIONS

Now, assume axial values don't matter
- members are axially rigid



3 DOFs – dof #3 can be anywhere along ℓ_2
structure moves as rigid body

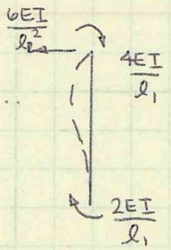
$u_1 = 1.0$ (same as $u_3 = 1.0$ in ~~plan~~ previous problem)

$$k_{11} = \frac{4EI}{\ell_1} + \frac{4EI}{\ell_2}$$

$$k_{21} = \frac{2EI}{\ell_2}$$

$$k_{31} = \frac{-6EI}{\ell_1^2}$$

because...



the force travels axially along ℓ_2

$u_2 = 1.0$

$$k_{22} = \frac{4EI}{\ell_2}$$

$$k_{32} = 0$$

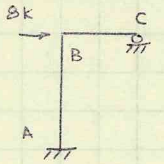
$u_3 = 1.0$

$$k_{33} = \frac{12EI}{\ell_1^3}$$

$$\underset{\sim}{K} = EI \begin{bmatrix} 4/\ell_1 + 4/\ell_2 & 2/\ell_2 & -6/\ell_1^2 \\ 2/\ell_2 & 4/\ell_2 & 0 \\ -6/\ell_1^2 & 0 & 12/\ell_1^3 \end{bmatrix}$$

AXIAL RIGIDITY

Example



$\ell_{AB} = 15$ ft

$\ell_{BC} = 10$ ft

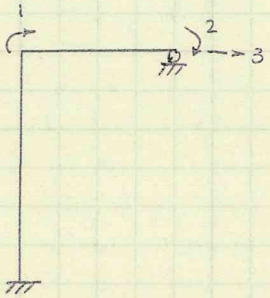
$P = 80$ K

$E = 15000$ ksi

$I = 250$ in^4

$A \rightarrow \infty$ (axially rigid)

Degrees of freedom



develop $\underset{\sim}{K}$

solve for $\underset{\sim}{u}$

interpret

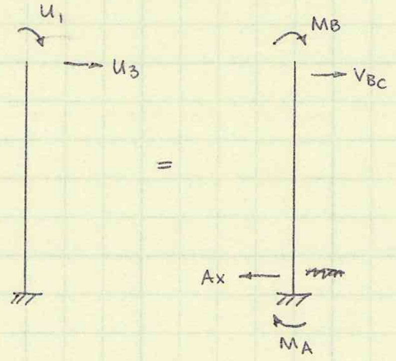
$$\underset{\sim}{K} = \begin{bmatrix} 4EI/\ell_1 + 4EI/\ell_2 & 2EI/\ell_2 & -6EI/\ell_1^2 \\ 2EI/\ell_2 & 4EI/\ell_2 & 0 \\ -6EI/\ell_1^2 & 0 & 12EI/\ell_1^3 \end{bmatrix} = \begin{bmatrix} 1/30 + 1/45 & 1/60 & -6/180^2 \\ 1/60 & 1/30 & 0 \\ -6/180^2 & 0 & 12/180^3 \end{bmatrix} EI$$

$$\underset{\sim}{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \quad \underset{\sim}{F} \begin{Bmatrix} 0 \\ 0 \\ 80 \end{Bmatrix}, \quad \underset{\sim}{u} = \begin{Bmatrix} 0.062836 \\ -0.031418 \\ 16.023273 \end{Bmatrix} \begin{matrix} \text{rad} \\ \text{rad} \\ \text{in} \end{matrix}$$

(mighty big displacement!)

indicates that a linear analysis isn't enough

calculate moments and shears

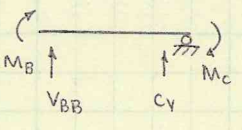


$M_A = EI\left(1/90\,u_1 - 6/180^2\,u_3\right) = -709$ k·ft

$A_x = -6EI/180^2\,u_1 + 12EI/L^3\,u_3 = 80$ K

$M_B = EI\left(1/45\,u_1 - 6/180^2_{u_3}\right) = -491.$ k·ft

$V_{BC} = EI\left(-6/180^2\,u_1 + 12/180^2\,u_3\right) = 80$ K

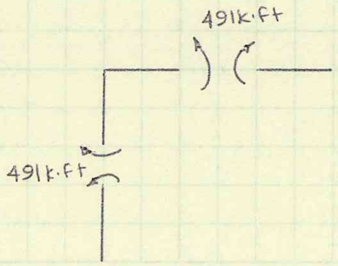


$M_B = EI\left(1/60\,u_2 + 1/30\,u_1\right) = 491$ k·ft

$M_c = EI\left(1/30\,u_2 + 1/60\,u_1\right) = 0$

$C_y = EI\left(+6/120^2\,u_2 + 6/120^2\,u_1\right) = 49.1$ k

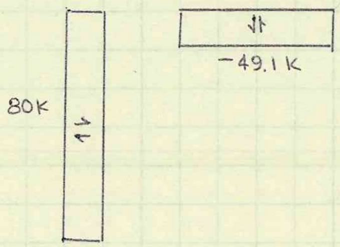
$V_{BB} = -49.1$ k



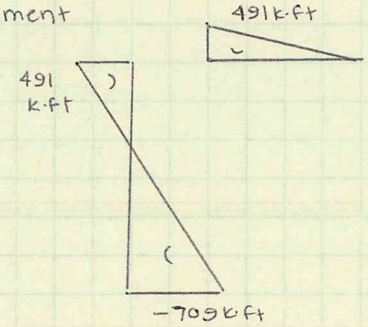
FLEXURE METHODS

Shear and Moment Diagrams

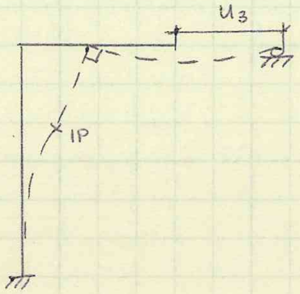
Shear



Moment



Deflected shape



- large displacement indicates that PΔ effect should be considered
- moment at A from load at B (ove 16.0 in!)

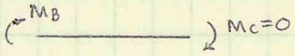
$$\Delta M_A = (49.1k)(16.02in) = 65.6 k\cdot ft$$

about 10% of moment of linear analysis

- can't just subtract; structure would no longer be in equilibrium
- axial force coupled with bending moment

So, how to account for it
- approximate, superimpose
- rederive relationship between axial, bending
- change $\underset{\sim}{K}$ accordingly

Axial force



$$\frac{M_B}{L} = \frac{4EI}{L}\cdot\frac{u_1}{L}$$

$$k_{11}: 4EI\left(1/30 + 1/45\right) + \frac{2P\ell_1}{15}$$

$$k_{33}: +6/5\, P/\ell_1$$

$$k_{31} = -P/10$$

(how, why not derived now)

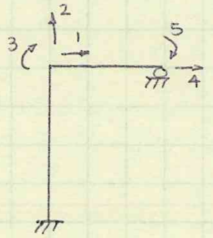
most programs have a P·Δ button

USE IT (when appropriate)

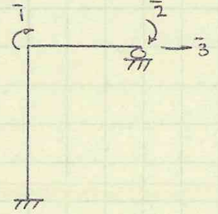
- using modified $\underset{\sim}{K}$, guess P value
- check to calculated P
- alter, re-iterate until estimated P = P (approx.)

STIFFNESS/FLEXIBILITY

considering constraints



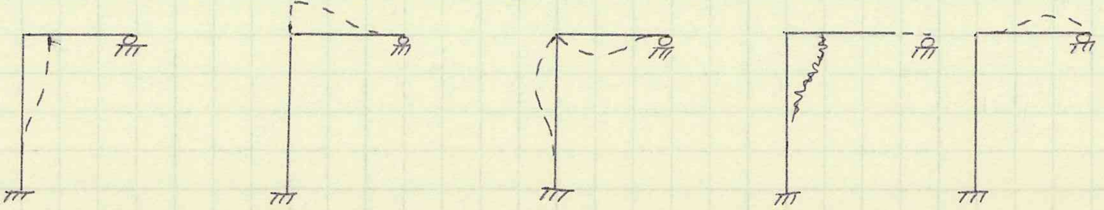
No constraints



constraints

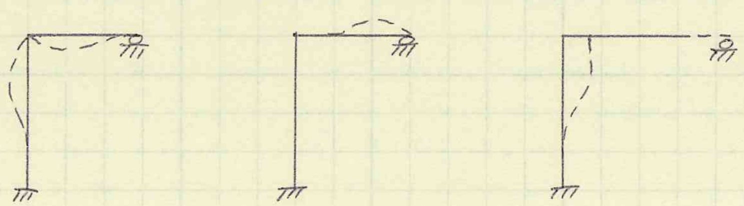
Is there a way to go easily from one to the next?

unconstrained



$u_1 = 1.0$ $u_2 = 1.0$ $u_3 = 1.0$ $u_4 = 1.0$ $u_5 = 1.0$

constrained



$u_{\bar{1}} = 1.0$ $u_{\bar{2}} = 1.0$ $u_{\bar{3}} = 1.0$

coupling

$\underset{\sim}{u}_{uncon.} = \Gamma \, \underset{\sim}{u}_{const}$, $\underset{\sim}{u} = \underset{\sim}{\Gamma} \, \underset{\sim}{\bar{u}}$

capital γ
constraint matrix

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \overset{5\times 3}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \cdot \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix}$$

how do $u, \bar{u}$ relate?

so, if we have $\underset{\sim}{\bar{K}}$

$\underset{\sim}{K} \, \underset{\sim}{u} = \underset{\sim}{F}$ unconstrained

$\underset{\sim}{\Gamma}^T \underset{\sim}{K} (\underset{\sim}{\Gamma} \, \underset{\sim}{\bar{u}}) = \underset{\sim}{\Gamma}^T \underset{\sim}{F}$, $\underline{\underset{\sim}{\Gamma}^T \underset{\sim}{K} \, \underset{\sim}{\Gamma} = \underset{\sim}{\bar{K}}}$ constrained stiffness matrix

$\underline{\underset{\sim}{\bar{K}} \, \underset{\sim}{\bar{u}} = \underset{\sim}{\bar{F}}}$ $\underset{\sim}{\Gamma}^T \underset{\sim}{F} = \underset{\sim}{\bar{F}}$ constrained load vector

work equivalency

W unconstrained = W constrained

$\underset{\sim}{u}^T \underset{\sim}{F} = \underset{\sim}{\bar{u}}^T \underset{\sim}{\bar{F}}$

$(\Gamma \bar{u})^T F = \bar{u} \bar{F}$ $(ab)^T = b^T a^T$

...

$\bar{u}^T (\Gamma^T K \Gamma) \bar{u} = \bar{u}^T \bar{F}$, or $\underset{\sim}{\bar{K}} \, \underset{\sim}{\bar{u}} = \underset{\sim}{\bar{F}}$

so, assumption agrees

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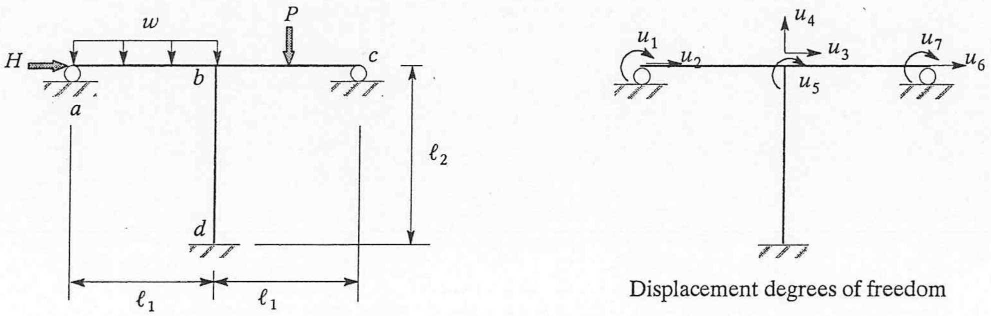
Dept. of Civil, Arch. & Env. Eng.

E. Williamson
CE 363

Example Problem

For small displacements, axial and flexural displacements are uncoupled and therefore can be treated independently. The frame shown in Fig. 1, accounting for both axial and flexural deformations, has seven degrees of freedom, numbered as shown on the sketch. All of the members have the same modulus of elasticity E, moment of inertia I, and cross-sectional area A.

The structure is subjected to a uniformly distributed load that acts on the left-hand beam and a concentrated load acting at the mid-span of the right-hand beam. In addition, the frame is loaded laterally by a load H acting at the left end of the beam. We seek to develop the 7x7 structure stiffness matrix and the 7x1 load vector. Find the joint displacements and rotations, member forces, and support reactions induced by these loads.

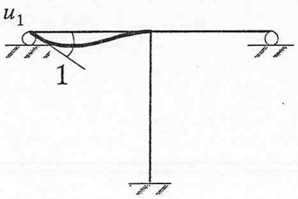


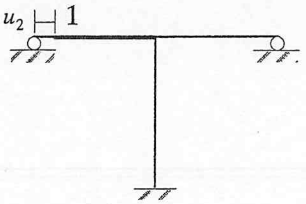
Displacement degrees of freedom

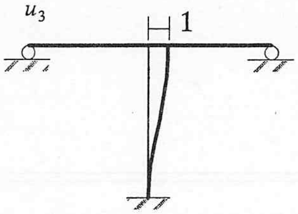
Fig. 1. Description of structure and displacement degrees of freedom

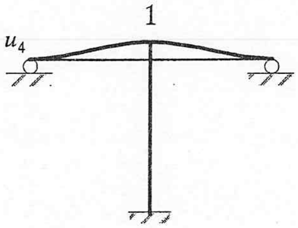
To solve this problem we must first identify the kinematic degrees of freedom. Once the degrees of freedom are identified, we can proceed to determine the relationship between the deformations in each member and the displacements u at these degrees of freedom. In order to accomplish this task, we study the deformed shape of the structure that results when each degree of freedom is moved by a unit amount while holding all other degrees of freedom equal to zero. With these deformations known, the forces that develop in the members can be computed. A solution to the problem is then achieved by enforcing equilibrium at each of the degrees of freedom. For this example, we must solve seven simultaneous equations to determine the displacement quantities at each degree of freedom. With the displacements known, these values can be substituted back into the member force–deformation relationships to determine the shear force, axial force, and bending moment at the member ends. The reaction forces can be determined from equilibrium with the member forces.

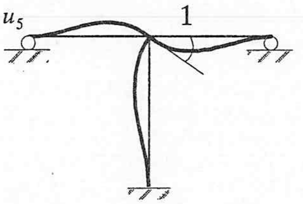
1. Compute member distortions due to nodal displacements

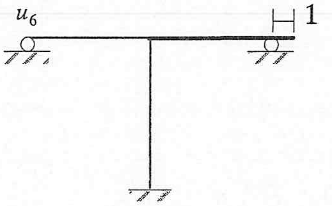


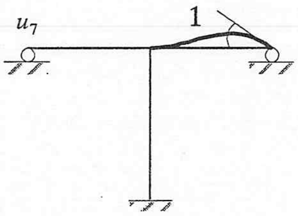










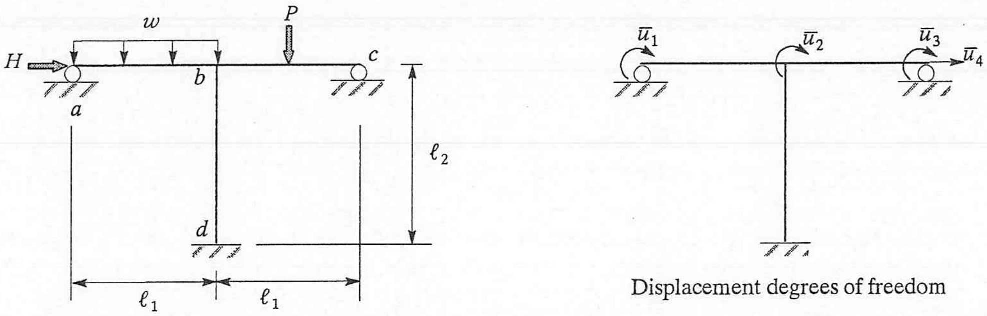


2. Develop the structural stiffness matrix and load vector.

$$\begin{bmatrix} \frac{4EI}{\ell_1} & 0 & 0 & \frac{6EI}{(\ell_1)^2} & \frac{2EI}{\ell_1} & 0 & 0 \\ 0 & \frac{EA}{\ell_1} & -\frac{EA}{\ell_1} & 0 & 0 & 0 & 0 \\ 0 & -\frac{EA}{\ell_1} & 2\frac{EA}{\ell_1}+\frac{12EI}{(\ell_2)^3} & 0 & -\frac{6EI}{(\ell_2)^2} & -\frac{EA}{\ell_1} & 0 \\ \frac{6EI}{(\ell_1)^2} & 0 & 0 & (2)\frac{12EI}{(\ell_1)^3}+\frac{EA}{\ell_2} & 0 & 0 & -\frac{6EI}{(\ell_1)^2} \\ \frac{2EI}{\ell_1} & 0 & -\frac{6EI}{(\ell_2)^2} & 0 & (2)\frac{4EI}{\ell_1}+\frac{4EI}{\ell_2} & 0 & \frac{2EI}{\ell_1} \\ 0 & 0 & -\frac{EA}{\ell_1} & 0 & 0 & \frac{EA}{\ell_1} & 0 \\ 0 & 0 & 0 & -\frac{6EI}{(\ell_1)^2} & \frac{2EI}{\ell_1} & 0 & \frac{4EI}{\ell_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} \frac{w(\ell_1)^2}{12} \\ H \\ 0 \\ -\frac{P}{2}-\frac{w(\ell_1)}{2} \\ \frac{P\ell_1}{8}-\frac{w(\ell_1)^2}{12} \\ 0 \\ -\frac{P\ell_1}{8} \end{bmatrix}$$

Response for Axially Rigid Case

If the above structure is re-analyzed with the assumption that each of the members behaves in an axially rigid manner, the number of degrees of freedom for the structure is reduced from seven to four. The degrees of freedom for the constrained system are shown in the sketch below.



Displacement degrees of freedom

Fig. 2. Description of structure and displacement degrees of freedom assuming all members are axially rigid

Rather than re-deriving the stiffness matrix and load vector for the new degrees of freedom, we can compute it by enforcing the necessary constraints on the system of equations previously developed. Therefore, we simply need to develop a mapping that relates the degrees of freedom in the unconstrained structure to the degrees of freedom for the constrained structure. Thus, let

$$\boldsymbol{u}_{original} = \boldsymbol{\Gamma u}_{constrained} \text{ or } \boldsymbol{u} = \boldsymbol{\Gamma \overline{u}}$$

where, in expanded form, we have the following relationship

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \overline{u}_1 \\ \overline{u}_2 \\ \overline{u}_3 \\ \overline{u}_4 \end{bmatrix}$$

Because of the constraints, the load vector must also be modified, and the constrained load vector is given by

$$\boldsymbol{F}_{constrained} = \boldsymbol{\Gamma}^T \boldsymbol{F}_{original} \text{ or } \boldsymbol{\overline{F}} = \boldsymbol{\Gamma}^T \boldsymbol{F}$$

As a result of these relationships, the equilibrium equations can be modified to give the constrained system of equations

$$\boldsymbol{\overline{K}\overline{u}} = \boldsymbol{\overline{F}} \text{ where } \boldsymbol{\overline{K}} = \boldsymbol{\Gamma}^T \boldsymbol{K \Gamma}$$

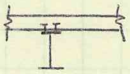
For this example, the resulting stiffness matrix is 4x4 as it should be. The stiffness matrix for the constrained system is given as

$$\begin{bmatrix} \frac{4EI}{\ell_1} & \frac{2EI}{\ell_1} & 0 & 0 \\ \frac{2EI}{\ell_1} & (2)\frac{4EI}{\ell_1}+\frac{4EI}{\ell_2} & \frac{2EI}{\ell_1} & -\frac{6EI}{(\ell_2)^2} \\ 0 & \frac{2EI}{\ell_1} & \frac{4EI}{\ell_1} & 0 \\ 0 & -\frac{6EI}{(\ell_2)^2} & 0 & \frac{12EI}{(\ell_2)^3} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{bmatrix} = \begin{bmatrix} \frac{w(\ell_1)^2}{12} \\ \frac{P\ell_1}{8} - \frac{w(\ell_1)^2}{12} \\ -\frac{P\ell_1}{8} \\ H \end{bmatrix}$$

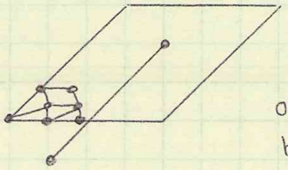
Note that ***none*** of the terms are dependent upon the axial stiffness EA/ℓ of either member. Study the development of the constrained system of equations until you can derive the appropriate expressions for any planar frame.

COUPLED DOFS

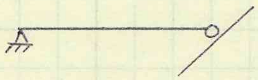
Constraints



inputted into a program as a line with a platesystem on top of it



offset between N.A. of beam, slab — how do you accomplish this?

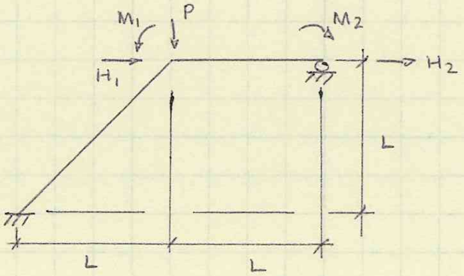


x, y displacements allowed but, x and y movements are related (must stay on the plane).

constraints. assign movement of slab to equal movement of beam

MPC = multi-point constraint

Axial constraints



EI

EA → ∞ (no axial consideration)

How many degrees of freedom?

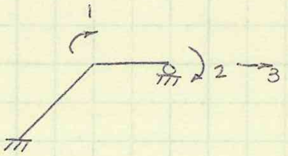
DOF = 3 (joints) − supports (boundaries)

with constrained system, then subtract constraints

- two members axially rigid
- DOF = 3(3) − 4 − 2 = 3

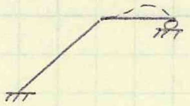
add number of releases (eg, hinge)

define DOFs

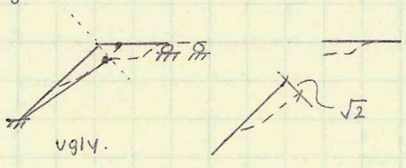


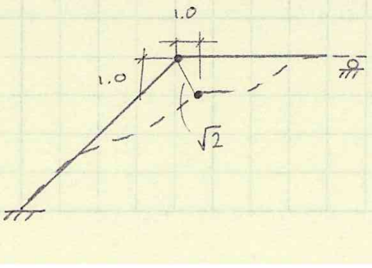
$u_1 = 1.0$

$u_2 = 1.0$



$u_3 = 1.0$

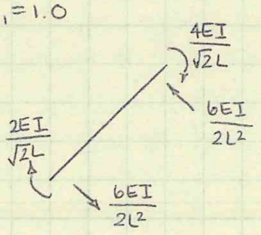


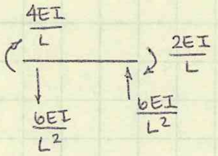


CONSTRAINTS

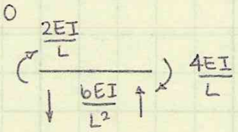
Example – extra steps

$u_1 = 1.0$

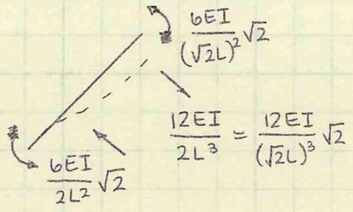


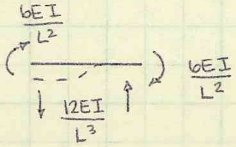


$u_2 = 1.0$



$u_3 = 1.0$





definition for terms in stiffness matrix:

K_{ij}: force at i due to a unit displacement at j

OR: work done by the internal forces that result for a unit displacement of degree of freedom i when moving through displacements associated with DOF j.

eg. $K_{11} = \frac{4EI}{\sqrt{2}L} + \frac{4EI}{L}$

$K_{12} = \frac{2EI}{L}$

each value is equal to F·d or M·θ. Deflection at 3 is caused by force in incline, beam

$K_{13} = \frac{-6EI}{2L^2}\sqrt{2} + \frac{6EI}{L^2}(1)$

in order to move it horizontally, joint actually moves vertically

$K_{22} = \frac{4EI}{L}$

$K_{23} = \frac{6EI}{L^2}$

$K_{33} = \frac{12EI}{2L^3}\sqrt{2} + \frac{12EI}{L^3}$

$$\underset{\sim}{K} = \begin{bmatrix} 4/\sqrt{2}L + 4/L & 2/L & -6\sqrt{2}/2L^2 + 6/L^2 \\ 2/L & 4/L & 6/L^2 \\ \frac{-6\sqrt{2}}{2L^2} + \frac{6}{L^2} & 6/L^2 & 6\sqrt{2}/L^3 + 12/L^3 \end{bmatrix} EI$$

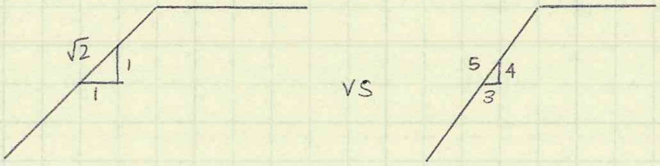
CONSTRAINTS

Example (cont'd)

$$\underset{\sim}{F} = \begin{bmatrix} -M_1 \\ M_2 \\ H_1 + H_2 + P \end{bmatrix}$$

downward load at joint (P) causes horizontal movement at roller

P factors according to geometry

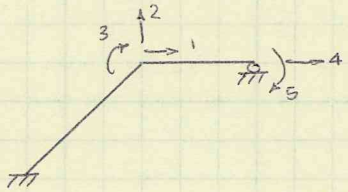


Constraint Matrix

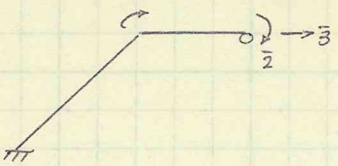
$$\underset{\sim}{u} = \underset{\sim}{\Gamma}\,\underset{\sim}{\bar{u}}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix}$$

unconstrained case: EA → ∞



constrained:



in this case, we know $\underset{\sim}{\bar{u}}$. so, if we develope $\underset{\sim}{\Gamma}$, we can get $\underset{\sim}{u}$.

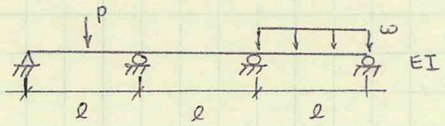
In load case $\underset{\sim}{\bar{u}}_1 = 1.0$, how much do DOFs 1-5 move?

numbers are not always 1 and 0
(2,3) = -1 now, could be $-4/3$

$$\underset{\sim}{\bar{K}} = \underset{\sim}{\Gamma}^T \underset{\sim}{K}\,\underset{\sim}{\Gamma}$$

INTERMEDIATE LOADS

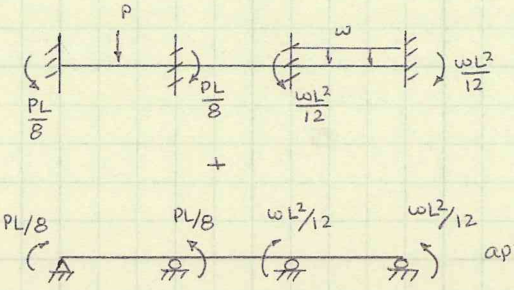
Loads Not at DOFS



Four degrees of freedom

how do loads get put into load vector
SUPERPOSITION

1. Assume all DOFs are fixed

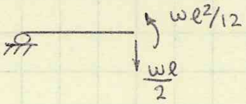


apply negatives of reaction forces

2. solve second structure the way we know how

Mild changes:

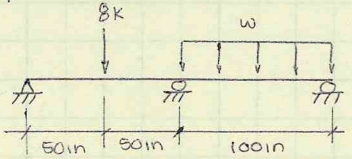
- cantilevered at the right end
  - still fix all points to get moments
  - apply moments, PLUS force at right end



INTERMEDIATE LOADS

Or, loads not at DOFs

example

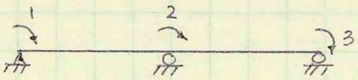


$w = 0.12$ k/in

$E = 10{,}000$ ksi

$I = 100$ in^4

ignore axial effects

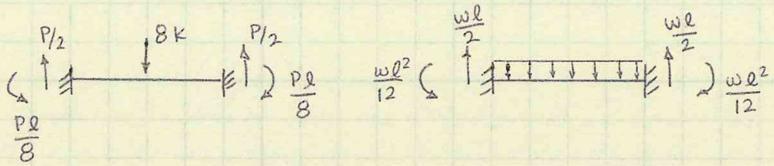


$$\underset{\sim}{K} = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

(done in detail earlier in notes)

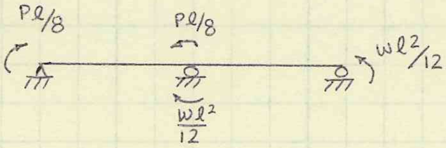
Force vector

- find fixed end forces
- apply opposite to structure



Fixed-end forces

For analysis by stiffness method, loads must be at DOFs

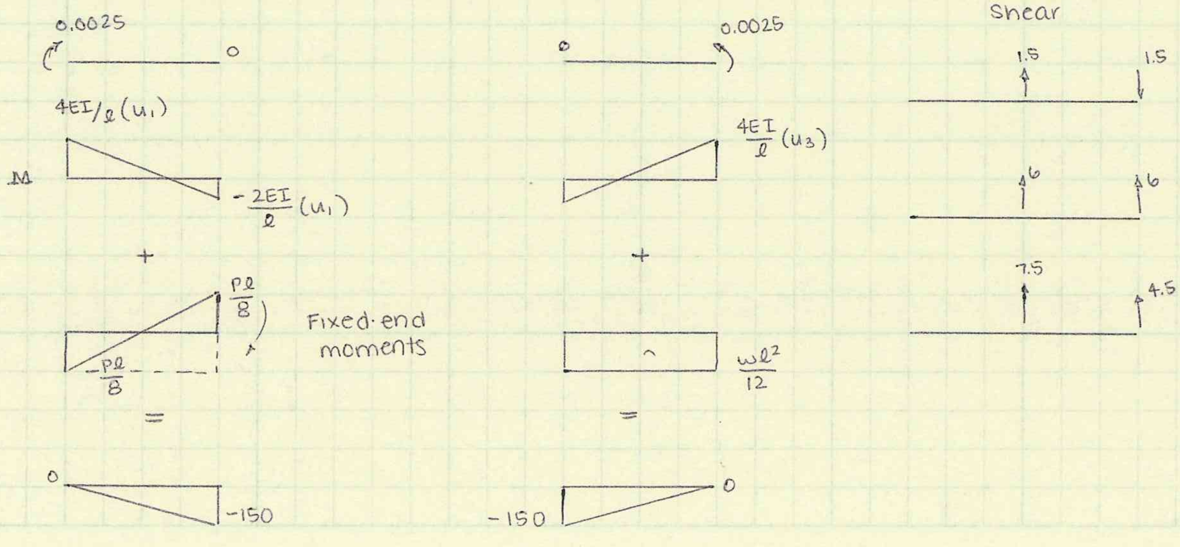


$$\underset{\sim}{F} = \begin{bmatrix} Pl/8 \\ 0 \\ -wl^2/12 \end{bmatrix}$$ or, $\frac{wl^2}{12} - \frac{Pl}{8}$

Equilibrium

$$\underset{\sim}{F} = \underset{\sim}{K}\,\underset{\sim}{u}, \quad \underset{\sim}{u} = \begin{bmatrix} 0.0025 \\ 0 \\ -0.0025 \end{bmatrix} \text{radians}$$

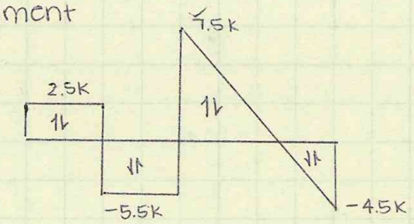
Shear, Moment Diagrams



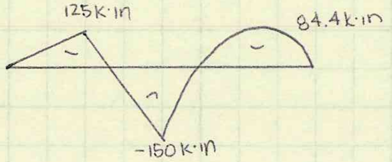
INTERMEDIATE LOADS

Shear and Moment

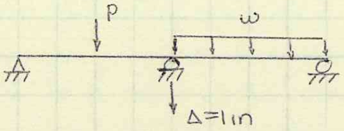
Shear



Moment



Support Settlement



add DOF 4, vertically up at roller B

add row, column to $\underset{\sim}{K} = \left[6EI/\ell^2 \quad 0 \quad -6EI/\ell^2 \quad 24EI/\ell^3 \right]$

add row to $\underset{\sim}{F} = F_4$

add row to $\underset{\sim}{u} = -1.0$ in

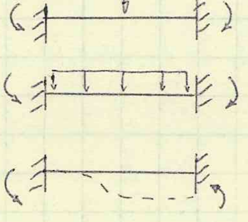
$$\frac{EI}{100}\begin{bmatrix} 4 & 2 & 0 & 6/100 \\ 2 & 8 & 2 & 0 \\ 0 & 2 & 4 & -6/100 \\ 6/100 & 0 & -6/100 & 24/100^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ -1.0 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ -100 \\ B_y \end{bmatrix}$$

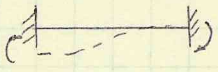
equations to solve are early in notes

$$\underset{\sim}{u}_0 = K_{00}^{-1}(F_0 - K_{01}u_1)$$

$$\underset{\sim}{F}_1 = \left[K_{11} - K_{10}K_{00}^{-1}K_{01}\right]u_1 + K_{10}K_{00}^{-1}F_0$$

OR, include settlement in end moments





add effects of extra moments into load vector, keep $\underset{\sim}{K}$ as ~~is~~ it was

$$\underset{\sim}{F} = \begin{bmatrix} P\ell/8 + 6EI/\ell^2\Delta \\ -P\ell/8 + w\ell^2/12 + 6EI/\ell^2\Delta - \frac{6EI}{\ell^2}\Delta \\ \frac{-w\ell^2}{12} - \frac{6EI}{\ell^2}\Delta \end{bmatrix}$$

COMPLEX CASES

End of support settlement

$$\underset{\sim}{u} = \begin{bmatrix} 0.0175 \\ 0 \\ -0.0175 \end{bmatrix} \text{radians}$$

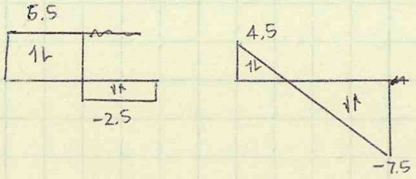
Shear & Moment

$$M_A = \frac{4EI}{\ell} u_1 - \frac{P\ell}{8} - \frac{6EI}{\ell^2}\Delta = 0$$

$$M_{BL} = \frac{-2EI}{\ell} u_1 - \frac{P\ell}{8} + \frac{6EI}{\ell^2}\Delta = \text{~~1200 k·in~~ } 150 \text{ k·in}$$

$$M_{BR} = \frac{-2EI}{\ell} u_3 - \frac{w\ell^2}{12} + \frac{6EI}{\ell^2}\Delta = 150 \text{ k·in}$$

$$M_C = -\frac{4EI}{\ell} u_3 - \frac{w\ell^2}{12} - \frac{6EI}{\ell^2}\Delta = 0$$

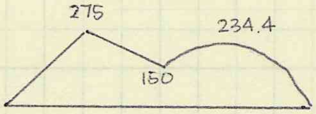


$$V_A = \frac{-6EI}{\ell^2}\Delta_1 + P/2 \cancel{+\frac{w\ell}{2}} + \frac{12EI}{\ell^3}\Delta_1 = 5.5\text{K}$$

$$V_{BL} = \frac{6EI}{\ell^2}\Delta_1 + P/2 - \frac{12EI}{\ell^3}\Delta = 2.5\text{K}$$

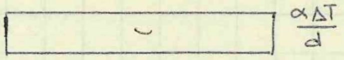
$$V_{BR} = \frac{6EI}{\ell^2}\Delta_3 + \frac{w\ell}{2} - \frac{12EI}{\ell^3}\Delta =$$

$$V_C = -\frac{6EI}{\ell^2}\Delta_3 + \frac{w\ell}{2} + \frac{12EI}{\ell^3}\Delta =$$

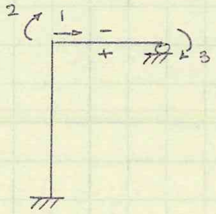


MISCELLANEOUS TOPICS

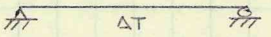
Thermal loads by the stiffness method



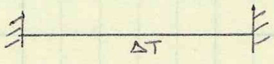
uniform temperature applied to beam, not on bottom

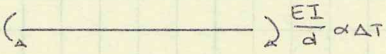


consider reaction with fixed ends



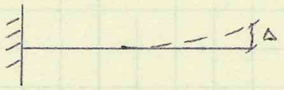
no moments at ends





to solve:

- pick primary structure



- pick two secondary structures

(ignore axial)

- solve

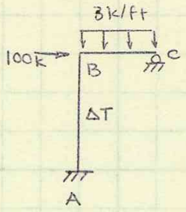
$$M_B = \frac{-EI}{d}\alpha T = M_A$$

$$B_y = 0$$

SWITCH SIGNS IN APPLICATION TO STRUCTURE!

$$F = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \frac{EI}{d}\alpha\Delta T$$

Example



$l_1 = 12$ ft

$l_2 = 8$ ft

ΔT varies linearly

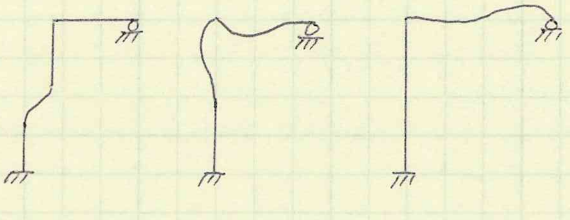
$\alpha = 6.5\times10^{-6}/°F$

$d = 12$ in

$E = 29000$ ksi

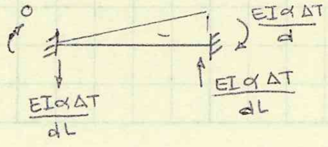
$b = 2$ in

$$\underset{\sim}{K} = \begin{bmatrix} 12/144^3 & -6/144^2 & 0 \\ -6/144^2 & \frac{4}{144}+\frac{4}{96} & 2/96 \\ 0 & 2/96 & 4/96 \end{bmatrix}$$

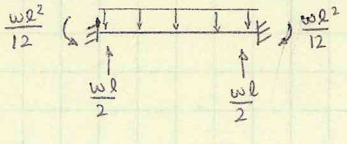


Equivalent forces

temperature – derive at home (woo!)



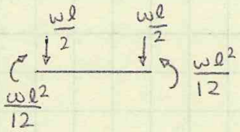
distributed load

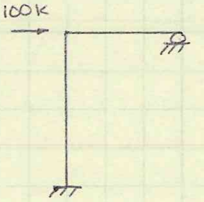


TEMPERATURE, ROLLERS

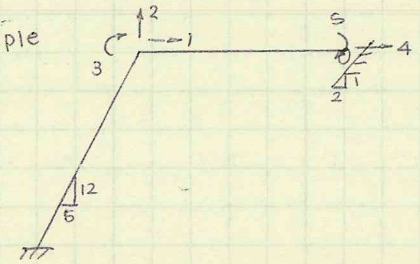
Example, cont'd

$$\underset{\sim}{F} = \begin{bmatrix} 100 + \frac{EI}{dL}\alpha\Delta T \\ \frac{wl^2}{12} - \frac{EI}{d}\alpha\Delta T \\ -wl^2/12 \end{bmatrix}$$

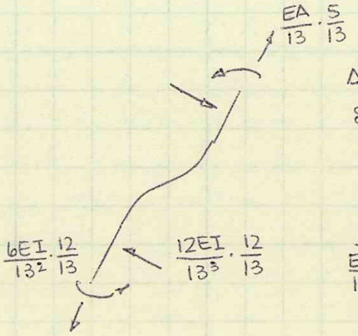




Example

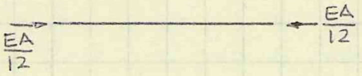


$u_1 = 1.0$



$\Delta = 12/13$

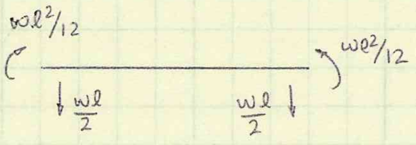
$\delta = 5/13$



$$K_{11} = \frac{EA}{12} + \frac{12EI}{13^3}\cdot\frac{12}{13}\cdot\frac{12}{13} + \frac{EA}{13}\cdot\frac{5}{13}\cdot\frac{5}{13}$$

$$K_{12} = \frac{EA}{13}\cdot\frac{5}{13}\cdot\frac{12}{13} + \frac{-12EI}{13^3}\cdot\frac{12}{13}\cdot\frac{5}{13}$$

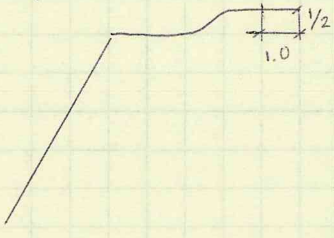
distributed load on beam



$$\underset{\sim}{F} = \begin{bmatrix} 0 \\ \cdots \\ wl^2/12 \\ -wl/2(1/2) \\ -wl^2/12 \end{bmatrix}$$

If roller was not inclined, value would = 0

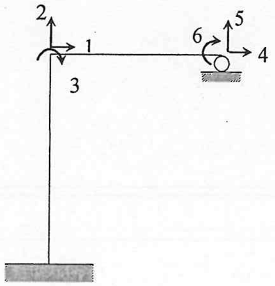
$u_4 = 1.0$



$F_2 = -\frac{wl}{2}$

Oblique Supports & Generalized Constraints

$$K(E,I,A,L) := \begin{pmatrix} \frac{12 \cdot E \cdot I}{L^3} + \frac{E \cdot A}{L} & 0 & \frac{6 \cdot E \cdot I}{L^2} & \frac{-E \cdot A}{L} & 0 & 0 \\ 0 & \frac{12 \cdot E \cdot I}{L^3} + \frac{E \cdot A}{L} & \frac{6 \cdot E \cdot I}{L^2} & 0 & \frac{-12 \cdot E \cdot I}{L^3} & \frac{6 \cdot E \cdot I}{L^2} \\ \frac{6 \cdot E \cdot I}{L^2} & \frac{6 \cdot E \cdot I}{L^2} & \frac{8 \cdot E \cdot I}{L} & 0 & \frac{-6 \cdot E \cdot I}{L^2} & \frac{2 \cdot E \cdot I}{L} \\ \frac{-E \cdot A}{L} & 0 & 0 & \frac{E \cdot A}{L} & 0 & 0 \\ 0 & \frac{-12 \cdot E \cdot I}{L^3} & \frac{-6 \cdot E \cdot I}{L^2} & 0 & \frac{12 \cdot E \cdot I}{L^3} & \frac{-6 \cdot E \cdot I}{L^2} \\ 0 & \frac{6 \cdot E \cdot I}{L^2} & \frac{2 \cdot E \cdot I}{L} & 0 & \frac{-6 \cdot E \cdot I}{L^2} & \frac{4 \cdot E \cdot I}{L} \end{pmatrix}$$



If we have a horizontal roller at the far end of the beam -> r5 = 0

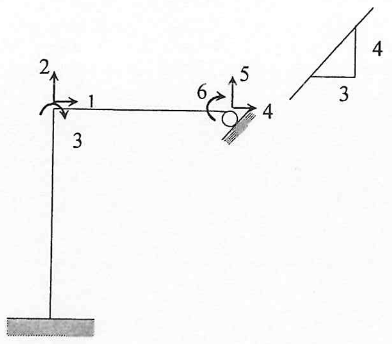
$$\Gamma_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \left.\begin{matrix} \\ \\ \end{matrix}\right]$$

only ones that change
due to roller alone

$$K_1(E,I,A,L) := \Gamma_1^{T}\cdot K(E,I,A,L)\cdot\Gamma_1 \rightarrow \begin{pmatrix} 12\cdot E\cdot\frac{I}{L^3}+E\cdot\frac{A}{L} & 0 & 6\cdot E\cdot\frac{I}{L^2} & -E\cdot\frac{A}{L} & 0 \\ 0 & 12\cdot E\cdot\frac{I}{L^3}+E\cdot\frac{A}{L} & 6\cdot E\cdot\frac{I}{L^2} & 0 & 6\cdot E\cdot\frac{I}{L^2} \\ 6\cdot E\cdot\frac{I}{L^2} & 6\cdot E\cdot\frac{I}{L^2} & 8\cdot E\cdot\frac{I}{L} & 0 & 2\cdot E\cdot\frac{I}{L} \\ -E\cdot\frac{A}{L} & 0 & 0 & E\cdot\frac{A}{L} & 0 \\ 0 & 6\cdot E\cdot\frac{I}{L^2} & 2\cdot E\cdot\frac{I}{L} & 0 & 4\cdot E\cdot\frac{I}{L} \end{pmatrix}$$

If we have an inclined roller at the far end of the beam -> r5 = 4/3 r4

$$\Gamma_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



$$K_1(E,I,A,L) := \Gamma_1^{T}\cdot K(E,I,A,L)\cdot\Gamma_1 \rightarrow \begin{pmatrix} 12\cdot E\cdot\frac{I}{L^3}+E\cdot\frac{A}{L} & 0 & 6\cdot E\cdot\frac{I}{L^2} & -E\cdot\frac{A}{L} & 0 \\ 0 & 12\cdot E\cdot\frac{I}{L^3}+E\cdot\frac{A}{L} & 6\cdot E\cdot\frac{I}{L^2} & -16\cdot E\cdot\frac{I}{L^3} & 6\cdot E\cdot\frac{I}{L^2} \\ 6\cdot E\cdot\frac{I}{L^2} & 6\cdot E\cdot\frac{I}{L^2} & 8\cdot E\cdot\frac{I}{L} & -8\cdot E\cdot\frac{I}{L^2} & 2\cdot E\cdot\frac{I}{L} \\ -E\cdot\frac{A}{L} & -16\cdot E\cdot\frac{I}{L^3} & -8\cdot E\cdot\frac{I}{L^2} & E\cdot\frac{A}{L}+\frac{64}{3}\cdot E\cdot\frac{I}{L^3} & -8\cdot E\cdot\frac{I}{L^2} \\ 0 & 6\cdot E\cdot\frac{I}{L^2} & 2\cdot E\cdot\frac{I}{L} & -8\cdot E\cdot\frac{I}{L^2} & 4\cdot E\cdot\frac{I}{L} \end{pmatrix}$$

If we have a horizontal roller and both members are axially rigid

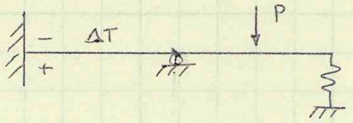
$$\Gamma_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad K_2(E,I,A,L) := \Gamma_2^{T} \cdot K(E,I,A,L) \cdot \Gamma_2 \rightarrow \begin{pmatrix} 12 \cdot E \cdot \frac{I}{L^3} & 6 \cdot E \cdot \frac{I}{L^2} & 0 \\ 6 \cdot E \cdot \frac{I}{L^2} & 8 \cdot E \cdot \frac{I}{L} & 2 \cdot E \cdot \frac{I}{L} \\ 0 & 2 \cdot E \cdot \frac{I}{L} & 4 \cdot E \cdot \frac{I}{L} \end{pmatrix}$$

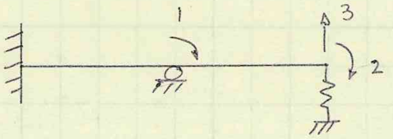
If we have an inclined roller and both members are axially rigid

$$\Gamma_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{4}{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad K_3(E,I,A,L) := \Gamma_3^{T} \cdot K(E,I,A,L) \cdot \Gamma_3 \rightarrow \begin{pmatrix} \frac{100}{3} \cdot E \cdot \frac{I}{L^3} & -2 \cdot E \cdot \frac{I}{L^2} & -8 \cdot E \cdot \frac{I}{L^2} \\ -2 \cdot E \cdot \frac{I}{L^2} & 8 \cdot E \cdot \frac{I}{L} & 2 \cdot E \cdot \frac{I}{L} \\ -8 \cdot E \cdot \frac{I}{L^2} & 2 \cdot E \cdot \frac{I}{L} & 4 \cdot E \cdot \frac{I}{L} \end{pmatrix}$$

EXTRA PROBLEMS

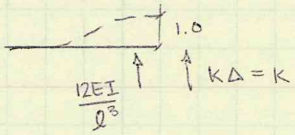
Spring support





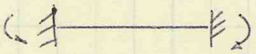
$$K = \begin{bmatrix} 8/\ell & 2/\ell & 6/\ell^2 \\ 2/\ell & 4/\ell & 6/\ell^2 \\ 6/\ell^2 & 6/\ell^2 & 12/\ell^3 + k \end{bmatrix} EI$$

always positive
(diagonals are (+))



maybe have to consider special circumstances with concrete, buckle-able steel springs...

to consider temperature change
- ~~uniform~~ linear through depth
- uniform along length

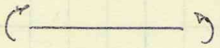


Fixed end forces

$\frac{EI}{d}\alpha\Delta T = M$

$F = 0$

ALWAYS TAKE OPPOSITE VALUES!
(negative)



Applied forces

Force vector

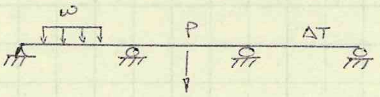
$$F = \begin{bmatrix} -\frac{EI}{d}\alpha\Delta T + \frac{P\ell}{8} \\ -P\ell/8 \\ -P\ell/2 \end{bmatrix}$$

Temperature change in truss

use negatives again, include in load vector

MORE REVIEW

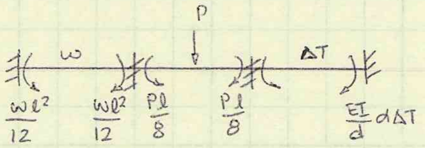
Stepping through superposition



how do you get loads on nodes
to produce expected rotations?

make kinematically determinate system(s)

(1)

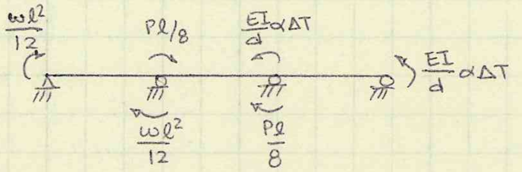


however, this says $\theta = 0$ at all pins,
and $M \neq 0$, neither of which are true

when solved, using negatives, moments exist
eg, at left, $M = \frac{\omega l^2}{12}$

so, subtract it back out!

(2)



(1) + (2) = 0 at the right places
intermediate rotations, deflections work out, too

$\theta_{A(1)} = 0$, $\theta_{A(2)} =$ right value

$M_{A(1)} = \frac{\omega l^2}{12}$, $M_{A(2)} = \frac{\omega l^2}{2}$ $\quad \Sigma = 0$

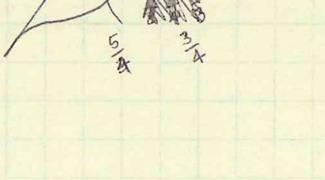
displacements (rotations) are easy, because (1) = 0 for all ends

$$M_A: \frac{4EI}{l_1}\theta_1 + \frac{2EI}{l_1}\theta_2 - \frac{\omega l^2}{12} = 0$$ if not, there are problems.

Stiffness method indeterminacy
- adding members ~~and~~ internally doesn't change DOFs
- adding supports removes DOFs

Roller on incline

$u_3 = 1.0$

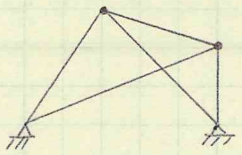


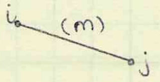
COMPUTER BASED WORK

Direct stiffness Method

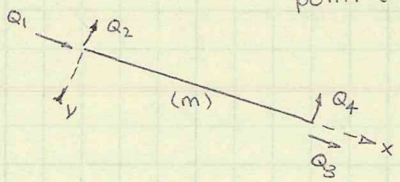
1. Look at individual member responses
2. Assemble individual responses to get structure response

Axial force (truss) members



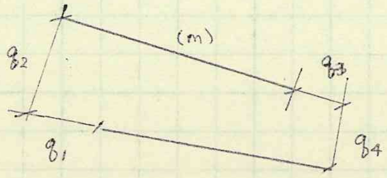


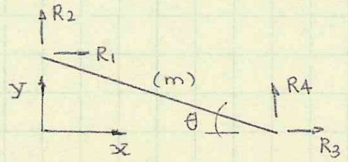
member m, from point i to j



local coordinates

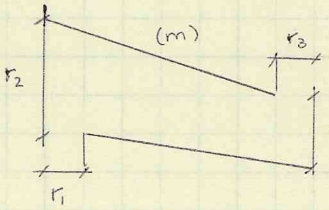
under load, (m) moves





global coordinates

θ = angle between x and X



displacement values locally and globally

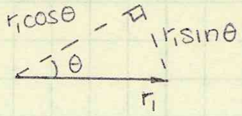
solve for local forces (displacements) (in terms of global response)

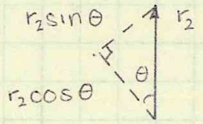
$$q_1 = r_1 \cos\theta + r_2 \sin\theta$$

$$q_2 = -r_1 \sin\theta + r_2 \cos\theta$$

$$q_3 = r_3 \cos\theta + r_4 \sin\theta$$

$$q_4 = -r_3 \sin\theta + r_4 \cos\theta$$





$$\underset{\sim}{q} = \underset{\sim}{T}\,\underset{\sim}{r}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

$s = \sin\theta$
$c = \cos\theta$

q, r can be substituted as Q, R

$$\underset{\sim}{Q} = \underset{\sim}{T} \cdot \underset{\sim}{R}$$

DIRECT STIFFNESS

Local to Global

if written out,

$r_1 = q_1 \cos\theta + (-q_2)\sin\theta$

$r_2 = q_1 \sin\theta + q_2 \cos\theta$ etc.

$\underset{\sim}{r} = \underset{\sim}{T}^T \underset{\sim}{q}$ or, $\underset{\sim}{r} = \underset{\sim}{T}^{-1} \underset{\sim}{q}$

[T] is an orthogonal matrix

$T^T = T^{-1}$

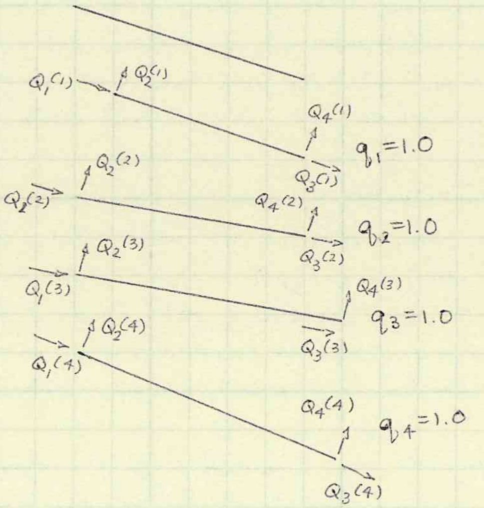
so much nicer, easier to "calculate"

- all coordinate-changing matricies are orthogonal
- maintains length (vectors do)
  - T just changes orientation

Relate forces and displacements: Stiffness matrix

$\underset{\sim}{Q} = \underset{\sim}{K}_{local} \cdot \underset{\sim}{q}$

↳ local stiffness matrix



$Q_1 = Q_1^{(1)} + Q_1^{(2)} + Q_1^{(3)} + Q_1^{(4)}$

$Q_1 = \frac{EA}{L} q_1 + 0 + (-1)\frac{EA}{L} q_3 + 0$

$Q_2 = 0 + 0 + 0 + 0$ (axial force member)

$Q_3 = \frac{-EA}{L} q_1 + 0 + \frac{EA}{L} q_3 + 0$

$Q_4 = 0$

$$K_{local} = \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

about stiffness matrix

- diagonals are all non-negative
- singular: no inverse
  - ↳ determinant = 0
    - ∘ unstable
    - ∘ no equilibrium
    - ∘ no constraints

  } singular matrix

- need to be able to shift it to go from local to global
  - ▸ if in global coordinates, K values can just be added to the K values from other members that influence deflections and forces
  - ▸ global displacements — think Baber's class

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CE 363

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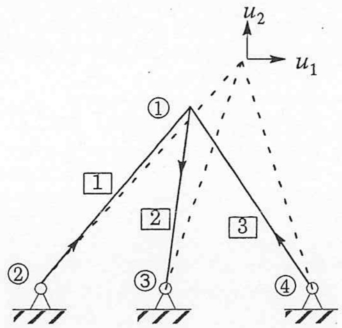
Assembly of the Structural Stiffness Matrix

Assembly of the structural stiffness matrix can be thought of as a mapping process in which terms from each of the member stiffness matrices are added to the structural stiffness matrix according to the way in which local member displacement quantities "map" to the structural degrees of freedom.

To accomplish this mapping, let us create an array of indices called $\boldsymbol{cn}$ (short for code number). The columns of $\boldsymbol{cn}$ will be in one-to-one correspondence with the members of the structure. The rows of $\boldsymbol{cn}$ will be in one-to-one correspondence with member degrees-of-freedom in global coordinates. The element $\boldsymbol{cn}_{ij}$ will be the global degree-of-freedom number which corresponds to member degree-of-freedom i in member j. For the example structure $\boldsymbol{cn}$ takes the form:



$$\boldsymbol{cn} = \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \textbf{Member number} \\ 0 & 1 & 0 & \mathbf{1} \\ 0 & 2 & 0 & \mathbf{2} \\ 1 & 0 & 1 & \mathbf{3} \\ 2 & 0 & 2 & \mathbf{4} \end{array} \quad \textbf{Local degree-of-freedom number}$$

Fig. 1. Example Structure

The contribution of member m to the global stiffness matrix is constructed by assigning its elements in the following way

$$\boldsymbol{S}[cn(i,m), cn(j,m)] \;\leftarrow\; \boldsymbol{K}^m(i,j) \tag{1}$$

where the symbol ← means "assemble into" or added to the existing value. In other words, the stiffness coefficient $\boldsymbol{K}_{ij}^m$ is assembled into the $\boldsymbol{cn}(i,m)$ row and $\boldsymbol{cn}(j,m)$ column of the global stiffness matrix $\boldsymbol{S}$, as shown in Fig. 2. It is important to realize that more than one member might contribute to a particular position in $\boldsymbol{S}$. To construct the entire stiffness matrix we loop over elements as shown in the following FORTRAN algorithm:

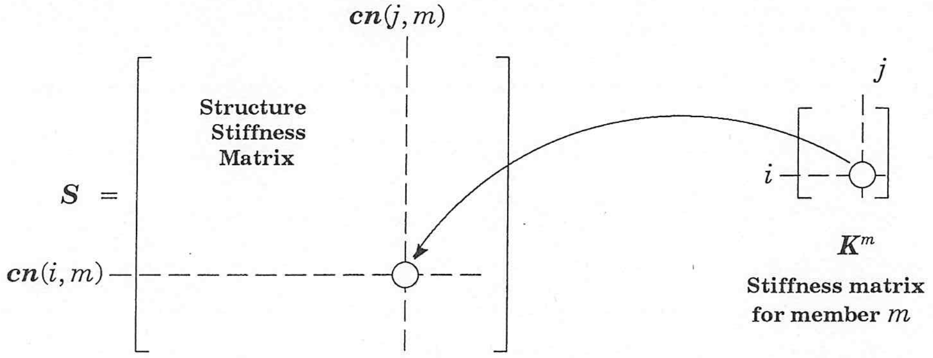


Fig. 2. Assembly Procedure

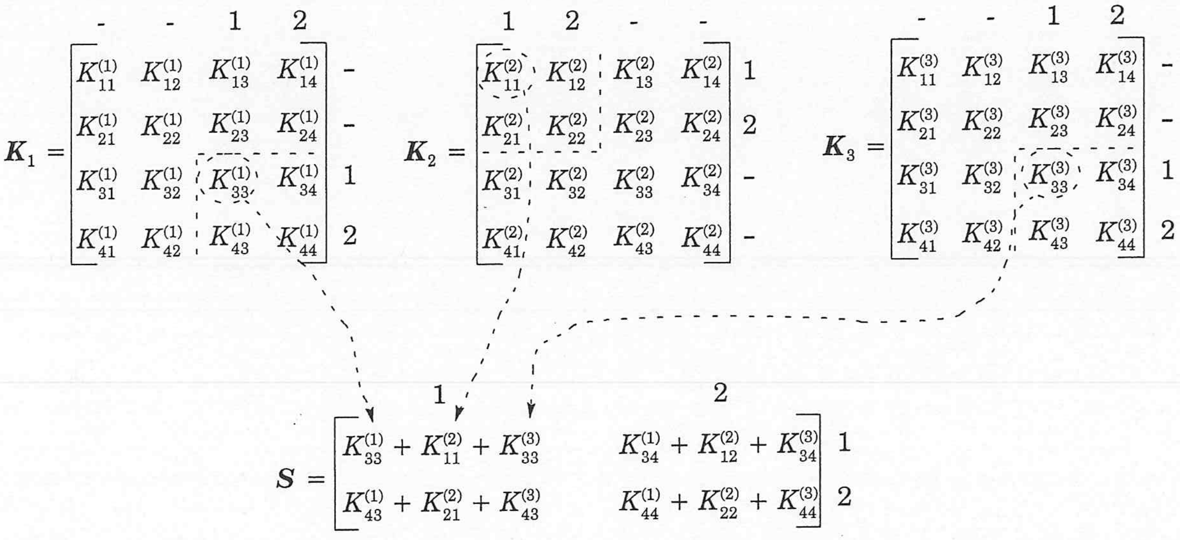
```
    subroutine assemble_global_stiffness (ndof, nmem, S, cn)
    integer ndof, nmem, i, j, k, l, m, n, cn(4,nmem)
c....     ndof = number of degrees-of-freedom in structure
c....     nmem = number of members is structure
    double precision K(4,4), S(ndof,ndof)
c....     K = member stiffness matrix in global coordinates
c....     S = global stiffness matrix

    do k=1,ndof
        do l=1,ndof
            S(k,l)=0    !initialize global stiffness matrix to zero
        end do
    end do

    do m= 1,nmem
        call Get_Local_Stiffness (K,m) !subroutine to get stiffness matrix for member m
        do i=1,4
            do j=1,4
                S(cn(i,m), cn(j,m))= S(cn(i,m), cn(j,m)) +  K(i,j)
            end do
        end do
    end do
```

In the above FORTRAN subroutine, a call is made to the subroutine *Get_Local_Stiffness* which retrieves the array ***K*** for a specified member. Note that the above routine has been developed specifically for the analysis of trusses. In order to employ this same assembly technique for beams and frames, the dimensions of the *cn* matrix and local member stiffness matrix must be increased to account for the additional degrees of freedom at the ends of these types of members. Be sure that you can make the required modifications to the procedure above in order to analyze structures with beam and frame members.

Note that the member end displacements that correspond to restrained degrees-of-freedom (corresponding to the support conditions) do not get assembled into the structural stiffness matrix. The structural stiffness matrix and corresponding equilibrium equations are expressed in terms of the unknown displacement quantities corresponding to the structural degrees-of-freedom. We have accounted for the boundary conditions through manipulation of the *cn* matrix. For each completely restrained boundary degree-of-freedom we assign the global degree-of-freedom number zero. When assembly is carried out, each time a zero is encountered simply *do nothing*. The result is a matrix without boundary degrees-of-freedom. For this particular example, the resulting structural stiffness matrix will be 2 × 2. A schematic of the resulting assembly process is shown in Fig. 3.



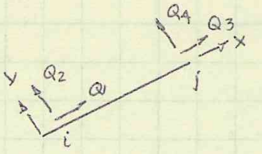
$$\boldsymbol{K}_1 = \begin{array}{cccc|c} - & - & 1 & 2 & \\ \hline K^{(1)}_{11} & K^{(1)}_{12} & K^{(1)}_{13} & K^{(1)}_{14} & - \\ K^{(1)}_{21} & K^{(1)}_{22} & K^{(1)}_{23} & K^{(1)}_{24} & - \\ K^{(1)}_{31} & K^{(1)}_{32} & K^{(1)}_{33} & K^{(1)}_{34} & 1 \\ K^{(1)}_{41} & K^{(1)}_{42} & K^{(1)}_{43} & K^{(1)}_{44} & 2 \end{array} \qquad \boldsymbol{K}_2 = \begin{array}{cccc|c} 1 & 2 & - & - & \\ \hline K^{(2)}_{11} & K^{(2)}_{12} & K^{(2)}_{13} & K^{(2)}_{14} & 1 \\ K^{(2)}_{21} & K^{(2)}_{22} & K^{(2)}_{23} & K^{(2)}_{24} & 2 \\ K^{(2)}_{31} & K^{(2)}_{32} & K^{(2)}_{33} & K^{(2)}_{34} & - \\ K^{(2)}_{41} & K^{(2)}_{42} & K^{(2)}_{43} & K^{(2)}_{44} & - \end{array} \qquad \boldsymbol{K}_3 = \begin{array}{cccc|c} - & - & 1 & 2 & \\ \hline K^{(3)}_{11} & K^{(3)}_{12} & K^{(3)}_{13} & K^{(3)}_{14} & - \\ K^{(3)}_{21} & K^{(3)}_{22} & K^{(3)}_{23} & K^{(3)}_{24} & - \\ K^{(3)}_{31} & K^{(3)}_{32} & K^{(3)}_{33} & K^{(3)}_{34} & 1 \\ K^{(3)}_{41} & K^{(3)}_{42} & K^{(3)}_{43} & K^{(3)}_{44} & 2 \end{array}$$

$$\boldsymbol{S} = \begin{array}{cc|c} 1 & 2 & \\ \hline K^{(1)}_{33} + K^{(2)}_{11} + K^{(3)}_{33} & K^{(1)}_{34} + K^{(2)}_{12} + K^{(3)}_{34} & 1 \\ K^{(1)}_{43} + K^{(2)}_{21} + K^{(3)}_{43} & K^{(1)}_{44} + K^{(2)}_{22} + K^{(3)}_{44} & 2 \end{array}$$

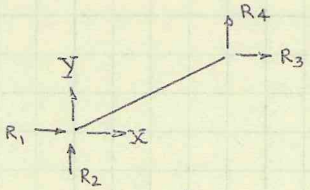
Fig. 3. Schematic showing assembly process for example structure

DIRECT STIFFNESS

Recap of last class



local



global

displacements – same numbers, lowercase letters

$Q = k_{local} \cdot q$

Relate k_{local} to global coordinates

$Q = k_{local} \cdot q$, $Q = T \cdot R$, $q = T \cdot r$

So, $T \cdot R = k_{local} T \cdot r$

$T^{-1} \cdot T \cdot R = T^{-1} k_{local} \cdot T \cdot r$

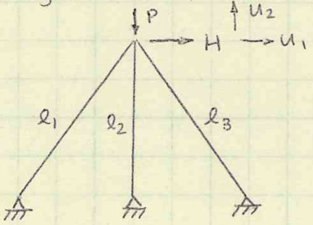
$$\underset{\sim}{R} = \underset{\sim}{T}^{T} \cdot \underset{\sim}{k}_{local} \cdot \underset{\sim}{T} \cdot \underset{\sim}{r} = \underline{(T^{T} \cdot k_{local} \cdot T)\, r}$$

↳ stiffness matrix in global coordinates

$k_{global} = T^{T} k_{local} T$

Assembly

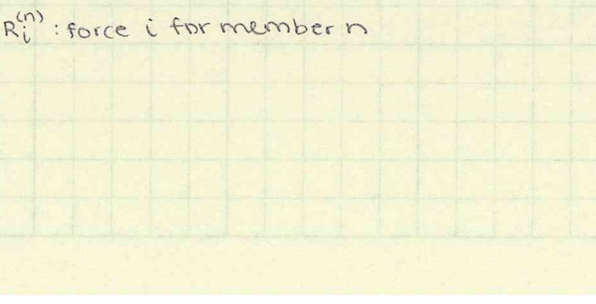
using an example



EA constant

1. Identify displacement degrees of freedom
2. Define geometry
3. Label forces
4. Establish equilibrium
5. consider compatibility
6. compute forces

geometry definition



Equilibrium

$\Sigma F_x = 0 : \; R_3^{(1)} + R_1^{(2)} + R_3^{(3)} = -H$

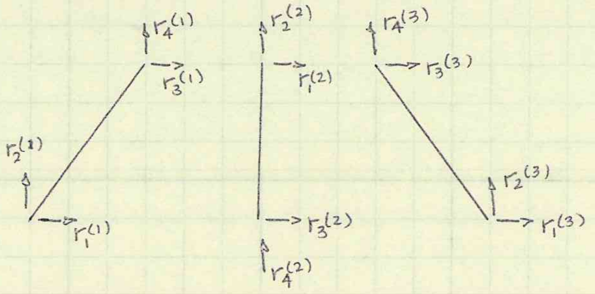
$\Sigma F_y = 0 : \; R_4^{(1)} + R_2^{(2)} + R_4^{(3)} = P$

$R_i^{(n)}$: force i for member n

DIRECT STIFFNESS

Assembly example

compatibility



$r_3^{(1)} = r_1^{(2)} = r_3^{(3)}$ else, top will ~~nos~~ not stay connected

also $= u_1$

$r_4^{(1)} \pm r_2^{(2)} \pm r_4^{(3)} = u_2$

compute forces

$$R_3^{(1)} = K_{31}^{(1)} \cdot r_1^{(1)} + K_{32}^{(1)} \cdot r_2^{(1)} + K_{33}^{(1)} \cdot r_3^{(1)} + K_{34}^{(1)} \cdot r_4^{(1)}$$

$$\begin{bmatrix} R_1^{(1)} \\ R_2^{(1)} \\ R_3^{(1)} \\ R_4^{(1)} \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ & & & \end{bmatrix} \cdot \begin{bmatrix} r_1^{(1)} \\ r_2^{(1)} \\ r_3^{(1)} \\ r_4^{(1)} \end{bmatrix}$$

global K matrix

$r_1^{(1)} = r_2^{(1)} = 0$
(pinned connection)

$$R_3^{(1)} = K_{33}^{(1)} r_3^{(1)} + K_{34}^{(1)} r_4^{(1)}$$

$r_3 = u_1$, $r_4 = u_2$

$$R_3^{(1)} = K_{33}^{(1)} u_1 + K_{34}^{(1)} u_2 \qquad R_4^{(1)} = K_{43}^{(1)} u_1 + K_{44}^{(1)} u_2$$

$$R_1^{(2)} = K_{11}^{(2)} u_1 + K_{12}^{(2)} u_2 \qquad R_2^{(2)} = K_{21}^{(2)} u_1 + K_{22}^{(2)} u_2$$

$$R_3^{(3)} = K_{33}^{(3)} u_1 + K_{34}^{(3)} u_2 \qquad R_4^{(3)} = K_{43}^{(3)} u_1 + K_{44}^{(3)} u_2$$

Add it all together

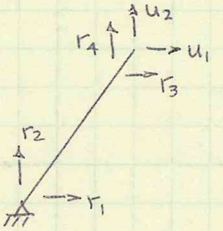
$$\begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -H \\ P \end{bmatrix}$$

Structure stiffness matrix

Further explanation

$$K_{\cancel{3}global}^{(1)} = \frac{EA}{l_1} \begin{array}{c} \left[\begin{array}{cccc} - & - & 1 & 2 \\ k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{array} \right] \end{array} \begin{array}{l} \\ \text{– all for member (1)} \\ - \\ 1 \\ 2 \end{array}$$

indicating where or how local numbers correspond to global ones

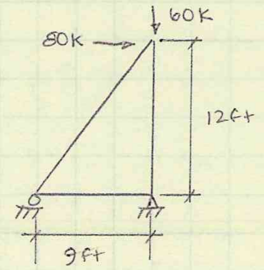


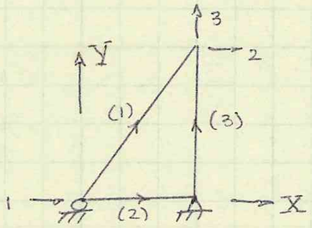
so, $K_{33}^{(1)}$ goes in 1,1 spot
$K_{43}^{(1)}$ goes in 2, ~~2~~1 etc.
(backwards of obvious)

row index along rows, column ~~tt~~ index along columns

DIRECT STIFFNESS

Full example Numero Uno





arrows indicate i to j

Member one

$$\underset{\sim}{K}_{local} = \frac{EA}{15}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow$$

always the local stiffness matrix

$$\frac{EA}{l} = \frac{(15000\,ksi)(24in^2)}{180in} = 2000\ ^k/in$$

$$\cos\theta = \frac{x_j - x_i}{L} = {}^9/_{15} = 0.6$$

$$\sin\theta = \frac{y_j - y_i}{L} = {}^{12}/_{15} = 0.8$$

$$\underset{\sim}{T} = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix}$$

$$K^{(1)}_{global} = T^T K_{local} T = \begin{array}{c} \left[\begin{array}{cccc} 720 & 960 & -720 & -960 \\ 960 & 1280 & -960 & -1280 \\ -720 & -960 & 720 & 960 \\ -960 & -1280 & 960 & 1280 \end{array}\right] \begin{array}{c} 1 \\ - \\ 2 \\ 3 \end{array} \\ \begin{array}{cccc} 1 & - & 2 & 3 \end{array} \end{array}$$

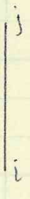
Member Two

$$\left.\begin{array}{l} \cos\theta = 1.0 \\ \sin\theta = 0 \end{array}\right\} \begin{array}{l} T \text{ is an identity matrix} \\ \quad T^T K_{local} T = K_{local} \end{array}$$

$$\frac{EA}{L_2} = 3333.3\ K/in$$

$$K^{(2)}_{global} = \begin{array}{c} \begin{array}{cccc} 1 & - & - & - \end{array} \\ \left[\begin{array}{cccc} 3333.3 & 0 & -3333.3 & 0 \\ 0 & 0 & 0 & 0 \\ -3333.3 & 0 & 3333.3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} 1 \\ - \\ - \\ - \end{array} \end{array}$$

Member Three



$$\frac{EA}{L_3} = 2500K/in$$

$$K^{(3)}_{\cancel{global}\ local} = \begin{bmatrix} 2500 & 0 & -2500 & 0 \\ 0 & 0 & 0 & 0 \\ -2500 & 0 & 2500 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

ASSEMBLY

Example (cont'd)

Member Three

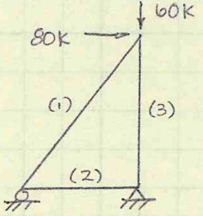
$$K_{global} = \begin{array}{cccc|c} - & - & 2 & 3 & \\ 0 & 0 & 0 & 0 & - \\ 0 & 2500 & 0 & -2500 & - \\ 0 & 0 & 0 & 0 & 2 \\ 0 & -2500 & 0 & 2500 & 3 \end{array}$$

Global stiffness matrix (for entire structure)

$$\underset{\sim}{K} = \begin{bmatrix} 3333.3+720 & -720 & -960 \\ -720+0 & 720+0 & 960 \\ -960 & 960+0 & 1280+2500 \end{bmatrix}$$

DIRECT STIFFNESS EXAMPLES

continuing last problem



solving for $\underset{\sim}{u} = \begin{bmatrix} 0.024 \\ 0.224 \\ -0.0667 \end{bmatrix}$ in

computing member forces (disassembly)

$R = k_{glob} \cdot r \ , \ k_{glob} = T^T k_{loc} T \ , \ Q = T \cdot R$

subscript 1 always refers to horizontal

$\underset{\sim}{r}^{(1)} = \begin{bmatrix} 0.024 \\ 0 \\ 0.224 \\ -0.0667 \end{bmatrix}$ in

$\underset{\sim}{R}^{(1)} = \begin{bmatrix} -80 \\ -106.67 \\ 80 \\ 106.67 \end{bmatrix}$ k

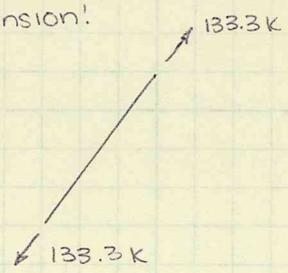
Forces should indicate member is in tension or compression

$Q^{(2)} = 80$ k comp

$Q^{(3)} = 166.67$ k comp

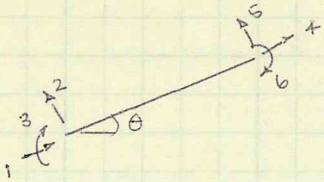
$\underset{\sim}{Q}^{(1)} = \begin{bmatrix} -133.3 \\ 0 \\ 133.3 \\ 0 \end{bmatrix}$ k

in tension!



Frame modifications

local stiffness matrix changes



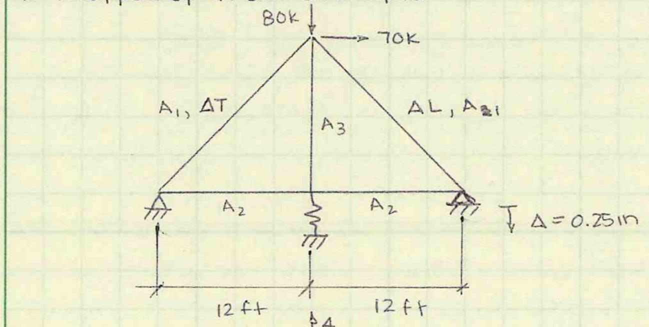
$$\underset{\sim}{k}_{local} = \begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ & 12EI/L^3 & -6EI/L^2 & 0 & -12EI/L^3 & -6EI/L^2 \\ & & 4EI/L & 0 & 6EI/L^2 & 2EI/L \\ & & & EA/L & 0 & 0 \\ & & & & 12EI/L^3 & 6EI/L^2 \\ & & & & & 4EI/L \end{bmatrix}$$

$\underset{\sim}{K}_{glob} = T^T k_{loc} T$

$$T = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

INTERMEDIATE EFFECTS

All wrapped up in one example



$E = 20000$ ksi

$A_1 = 8$ in^2

$A_2 = 6$ in^2

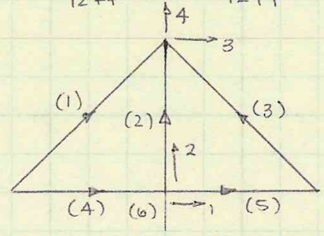
$A_3 = 10$ in^2

$\Delta T = 60°F$

$\alpha = 6.5 \times 10^{-6}/°F$

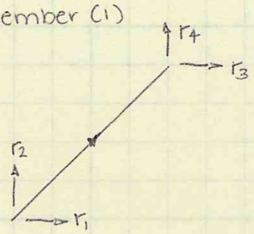
$\Delta L = 0.125$ in

$k_s = 1250$ k/in



degrees of freedom and member designation

Member (1)



$$\underset{\sim}{k}^{(1)}_{loc} = \frac{(20000\,ksi)(8\,in^2)}{(144\,in)\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{T} = \begin{bmatrix} 1 & +1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \frac{\sqrt{2}}{2}$$

$$k^{(1)}_{glob} = \begin{array}{c} \begin{array}{cccc} - & - & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{array}\right] \begin{array}{c} - \\ - \\ 3 \\ 4 \end{array} \end{array} (392.84)$$

$$\underset{\sim}{k}^{(6)}_{loc} = k \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

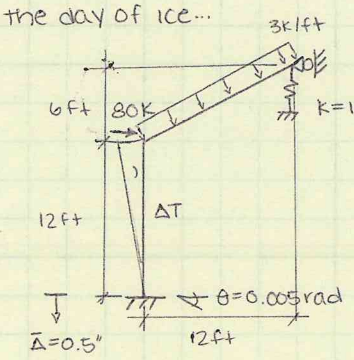
truss members in the orthogonal direction as degrees of freedom (eg. (2), 3), need not consider contribution (should be zero).

Answers

$$K = \begin{bmatrix} 2(833.3) & 0 & 0 & 0 \\ & 1389 + k & 0 & -1389 \\ & & 2(392.8) & 0 \\ & & & 2(392.8) + 1389 \end{bmatrix} \text{k/in}$$

FRAME EXAMPLE

on, the day of ice...



EI, EA

$\Delta T = 100° F$

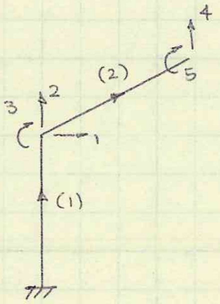
$\alpha = 6.5 \times 10^{-6} /°F$

$E = 30000$ ksi

$I = 1000 \text{ in}^4$

$A = 20 \text{ in}^2$

$d = 14 \text{ in}$



Stiffness matricies

(1)

$x_i = 0$, $x_j = 0$ sin = 1

$y_i = 0$, $y_j = 144$ cos = 0

$$K^{(1)}_{local} = \begin{bmatrix} EA/144 & 0 \\ 0 & 12EI/144^3 \\ 0 & -6EI/144^2 \\ -EA/144 & 0 \\ 0 & -12EI/144^3 \\ 0 & -6EI/144^2 \end{bmatrix} \quad \text{etc...}$$

$$K^{(1)}_{glob} = \begin{array}{c|cccccc|c} & - & - & - & 1 & 2 & 3 & \\ \hline & 120.56 & & & & & & - \\ & 0 & 4166.7 & & & & & - \\ & 8680.6 & 0 & 833333 & & & & - \\ & -120.6 & 0 & -8680.6 & 120.6 & & & 1 \\ & 0 & -4166.7 & 0 & 0 & 4167 & & 2 \\ & 8680.6 & 0 & 416,667 & -8680.6 & 0 & 833333 & 3 \end{array}$$

(2)

$$K^{(2)}_{glob} = \begin{array}{c|cccccc|c} & 1 & 2 & 3 & - & 4 & 5 & \\ \hline & 2999 & & & & & & 1 \\ & 1456 & 814 & & & & & 2 \\ & 3106 & -6211 & 745356 & & & & 3 \\ & -2999 & -1456 & -3106 & 2999 & & & - \\ & -1456 & -814 & 6211 & 1456 & 814 & & 4 \\ & 3106 & -6211 & 372678 & -3106 & 6211 & 745356 & 5 \end{array}$$

Assembly:

$$\begin{bmatrix} 3119.6 & & & & \\ 1456 & 4980.7 & & & \\ -5575 & -6211 & 1578689 & & \\ -1456 & -814 & 6211 & 814 & \\ 3106 & -6211 & 372678 & 6211 & 745356 \end{bmatrix}$$

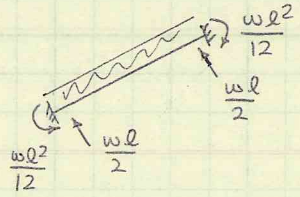
FRAME EXAMPLE

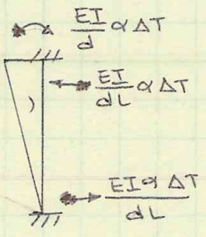
Example, cont'd

K_{44} : must add in spring force

$814+100=914$

Force matrix / vector





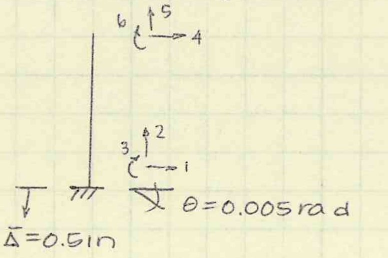
solve using flexibility method - remove redundants, etc.

$$Q_{FE} = \begin{bmatrix} 0 \\ 20.1 \\ -540 \\ 0 \\ 20.1 \\ 540 \end{bmatrix}$$

$$R_{FE} = T^T Q_{FE}^{(2)} = \begin{array}{cc} \begin{bmatrix} -9 \\ 18 \\ -540 \\ -9 \\ 18 \\ 540 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ - \\ 4 \\ 5 \end{matrix} \end{array}$$

need to add the negatives into load vector

support settlement / rotation



$$R_{FE}^{(1)} = \begin{array}{cc} \begin{bmatrix} 9.67 \\ 0 \\ 0 \\ -9.67 \\ 0 \\ 1393 \end{bmatrix} & \begin{matrix} - \\ - \\ - \\ 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

(from temp.)

$$\underset{\sim}{r}_{FE}^{(1)} = \begin{bmatrix} 0 \\ -0.5 \\ 0.005 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

global coordinate displacement values

goal: get $R_{FE}^{(1)}$

$q = T \cdot r$

$Q_{FE} = K_{local} \cdot q_{FE}$

$R_{FE} = T^T Q_{FE}$

so.

$$R_{FE} = \underbrace{T^T \cdot K_{local} \cdot T}_{\to K_{global}} \cdot r_{FE} = \begin{array}{cc} \begin{bmatrix} 43.4 \\ -2083 \\ 4167 \\ -43.4 \\ 2083 \\ 2083 \end{bmatrix} & \begin{matrix} - \\ - \\ - \\ 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

$F = F_o - F_{FE}$

F_o = load vector due to loads at DOF

F_{EE} = fixed end forces / moments associated with all members

$$\underset{\sim}{F} = \begin{bmatrix} 142.1 \\ -2101 \\ -2936 \\ -18 \\ -540 \end{bmatrix}$$

FRAME EXAMPLE

Example, con'd (or cont'd).

Kglobal, we have.

$$\underset{\sim}{F} = \begin{bmatrix} 142.1 \\ -~~20~~\,2101 \\ -2936 \\ -18 \\ -540 \end{bmatrix} \qquad \text{So,} \quad \underset{\sim}{u} = \begin{bmatrix} 0.3688 \\ -0.5127 \\ -0.0016 \\ 0.1705 \\ -0.0072 \end{bmatrix} \begin{matrix} \text{in} \\ \text{in} \\ \text{rad} \\ \text{in} \\ \text{rad} \end{matrix}$$

what now?

- compare to allowable values for building
  - out-of-~~plub~~ plumbness
  - linear calculations OK?
- calculate member forces

Member Forces

$$Q = K_{local} \cdot q + Q_{FE} \quad , \text{ where } q = T \cdot r$$

r is the global displacement matrix

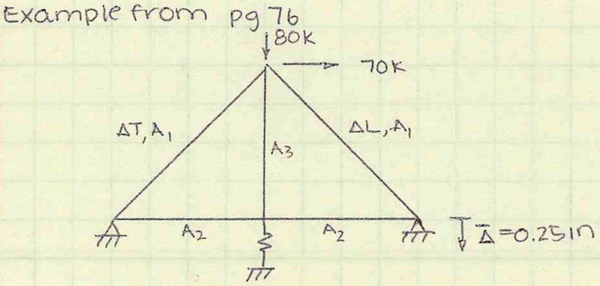
- easy to get

$$r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.3688 \\ -0.5127 \\ -0.0016 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.5127 \\ -0.3688 \\ -0.0016 \end{bmatrix}$$

$$Q_{FE}^{(1)} = \begin{bmatrix} 0 - 2083 \\ -9.67 - 43.4 \\ 4167 \\ 2083.3 \\ 9.67 + 43.4 \\ 1393 + 2083 \end{bmatrix}, \quad Q = \begin{bmatrix} 53.25 \\ 5.28 \\ 299 \\ -53.0 \\ -5.28 \\ -1059 \end{bmatrix}$$

TRUSS EXAMPLE

Example from pg 76



E = 20000 ksi

$A_1 = 8\ in^2$

$A_2 = 6\ in^2$

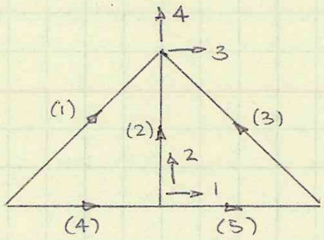
$A_3 = 10\ in^2$

ΔT = 60°F

$\alpha = 6.5 \times 10^{-6}\ °F^{-1}$

ΔL = 0.125 in

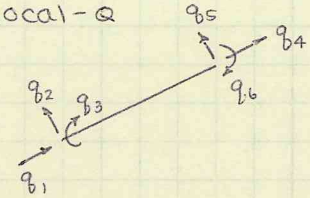
$k_s = 1250\ K/in$



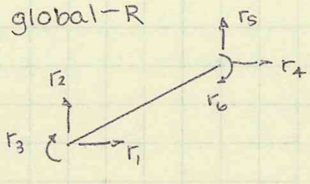
$$K = \begin{bmatrix} 2(833.3) & 0 & 0 & 0 \\ 0 & 1389 + k & 0 & -1389 \\ 0 & 0 & 2(392.8) & 0 \\ 0 & -1389 & 0 & 2(392.8)+1389 \end{bmatrix}$$

To Review:

local – Q



global – R

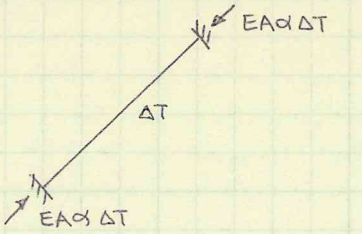


$$\underset{\sim}{K}_{global} = T^T k_{local}\, T$$

Form force vector

loads at DOF + member effects (distributed load, misfits, temp.)

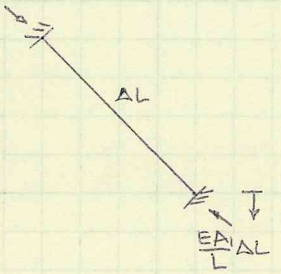
Member (1) : temperature change

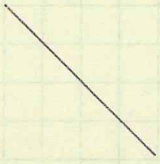


$$Q_{FE}^{(1)} = \begin{bmatrix} EA_1\alpha\Delta T \\ 0 \\ -EA_1\alpha\Delta T \\ 0 \end{bmatrix} \qquad \underline{R_{FE}^{(1)} = T^T Q_{FE}^{(1)}} = 44.12 \begin{array}{c} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \end{array} \begin{array}{l} - \\ - \ k \\ 3 \\ 4 \end{array}$$

STILL FIXED END FORCES
– later, use negatives

Member (3)



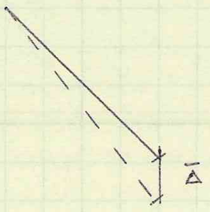


$$Q_{FE}^{(3)} = \begin{bmatrix} EA_1/L\,\Delta L \\ 0 \\ -EA_1/L\,\Delta L \\ 0 \end{bmatrix}, \quad R_{FE}^{(3)} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}(69.4) \begin{array}{l} - \\ - \\ 3 \\ 4 \end{array}$$

TRUSS EXAMPLE

Force vector determination

Member (3) – settlement



$$\underline{R_{FE} = k_{global} \cdot r_{FE}}$$

$$r_{FE} = \begin{bmatrix} 0 \\ -0.25 \\ 0 \\ 0 \end{bmatrix}, \quad R_{FE} = \begin{array}{[c]l} 98.2 & - \\ -98.2 & - \\ -98.2 & 3 \\ 98.2 & 4 \end{array}$$

Member ($\overset{5}{\cancel{4}}$) settlement

$$r_{FE}^{(5)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.25 \end{bmatrix}, \quad R_{FE}^{(5)} = \begin{array}{[c]l} 0 & 1 \\ 0 & 2 \\ 0 & \cancel{3} - \\ 0 & \cancel{4} - \end{array}$$

Summation of forces

$$F = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 70 \\ -80 \end{bmatrix}}_{F_0} - \underbrace{\left[\begin{bmatrix} \overset{0}{\cancel{44.12}} \\ \overset{0}{\cancel{44.12}} \\ -44.12 \\ -44.12 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 69.4 \\ -69.4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -98.2 \\ 98.2 \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 142.92 \\ -64.68 \end{bmatrix}}_{F_{FE}}$$

Solve for displacements

$\cancel{KU}$ $\underline{K u = F}$, $K^{-1} F = u$

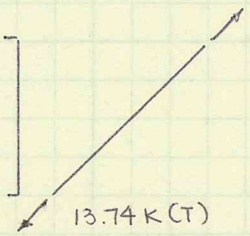
$$u = \begin{bmatrix} 0 \\ -0.0234 \\ 0.18192 \\ -0.04447 \end{bmatrix} \text{ in}$$

very small values.
- trusses are pretty stiff
- EA/L is large

member forces

$\underline{Q = k_{local} \cdot q + Q_{FE}}$, $\underline{q = T \cdot r}$

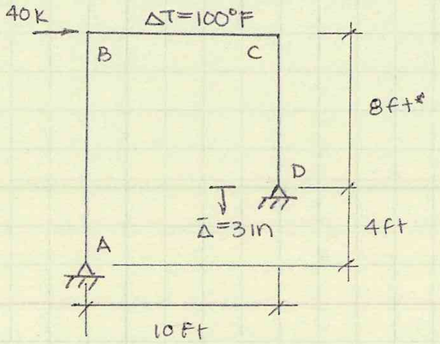
$$r_1 = \begin{bmatrix} 0 \\ 0 \\ 0.1819 \\ -0.044 \end{bmatrix}, \quad Q_{FE}^{(1)} = \begin{bmatrix} EA_1 \alpha \Delta T \\ 0 \\ -EA_1 \alpha \Delta T \\ 0 \end{bmatrix}, \quad Q^{(1)} = \begin{bmatrix} -13.74 \\ 0 \\ 13.74 \\ 0 \end{bmatrix}$$



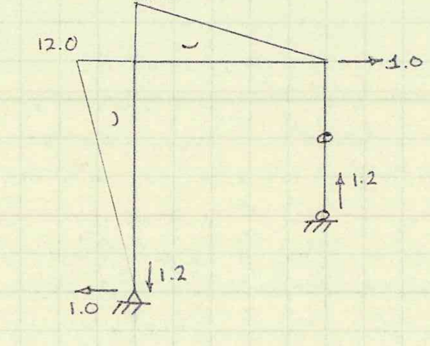
- compare to member capacity
  $\sigma = F/A$
- change A, reiterate
- YAY computers.

EXAMPLES / REVIEW

From Exam 1



moment diagram given

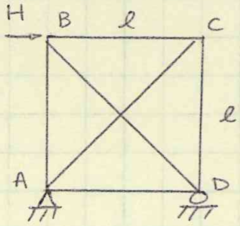


$$\Delta = \int m \frac{M}{EI} dx + \int \frac{\alpha}{d} \Delta T m\, dx + \text{RBM}$$

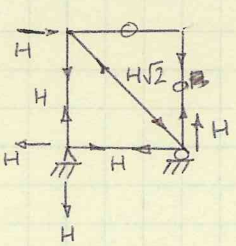
$$\Delta = \tfrac{1}{2}(12)\frac{83.11}{EI}(2/3)(8) + \tfrac{1}{2}(2.39)\frac{83.11}{EI}\overset{11.04}{(\cancel{\text{[illegible]}})} - \tfrac{1}{2}(7.61)\frac{264.59}{EI}(1/3)(7.61)(12/10)$$

$$+ \frac{\alpha \Delta T}{d}(10)(6) + 3.6 \text{ in } = \underline{4.48 \text{ in right}}$$

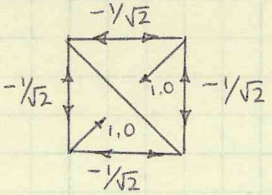
Statically indeterminate truss



Primary



secondary



	L	$\hat{N}$	n	$\frac{\hat{N}nL}{EA}$	$\frac{n^2L}{EA}$
ab	ℓ	H	$-1/\sqrt{2}$		
bc	ℓ	0	$-1/\sqrt{2}$		
cd	ℓ	0	$-1/\sqrt{2}$		
ad	ℓ	H	$-1/\sqrt{2}$		
ac	$\sqrt{2}\ell$	0	1.0		
bd	$\sqrt{2}\ell$	$-\sqrt{2}H$	1.0		

$$\Delta = \frac{\hat{N}nL}{EA} \text{ (summation)} \quad , \quad f = \Sigma \frac{n^2L}{EA}$$

negative value means
assumed force is backwards;
points a and c move apart

$\Sigma n \Delta L$ mis~~s~~fit
$\Sigma n \alpha \Delta T L$ temp

included in Δ
calculation, not
right side of compat.

compatibility:

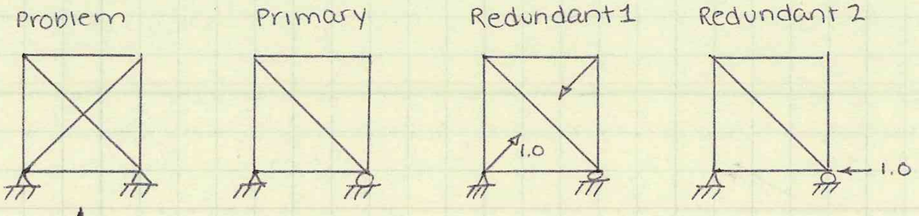
$$\Delta + F_{AC} f = 0$$

REVIEW FOR FINAL

Support settlement

- no effect if internally redundant
- yes effect if externally redundant

Two degrees of redundancy



$$\Delta_1 = \Sigma \frac{\hat{N} n_1 L}{EA}, \quad \Delta_2 = \Sigma \frac{\hat{N} n_2 L}{EA}$$

$$f_{11} = \Sigma \frac{n_1^2 L}{EA}, \quad f_{12} = \Sigma \frac{n_1 n_2 L}{EA}, \quad f_{22} = \Sigma \frac{n_2^2 L}{EA}$$

compatibility

$$\Delta_1 + F_{AC} f_{11} + D_x f_{12} = 0$$

$$\Delta_2 + F_{AC} f_{21} + D_x f_{22} = 0$$

f_{ij} ~~k_{ij}~~: displacement at i due to a unit load at j.

k_{ij}: force at i due to a unit load at j

all applied loads (forces, temp, disp.) ~~ar~~ put on this one structure

$$\Delta = \Sigma \left(\frac{nNL}{EA} + n\Delta L + n\alpha\Delta T L \right)$$, where N are the final structure member forces, and n are the member forces from new primary structure: the load in the place & direction of desired deflection.

Springs

always add.

similar to diagonals in K matrix — all parts add

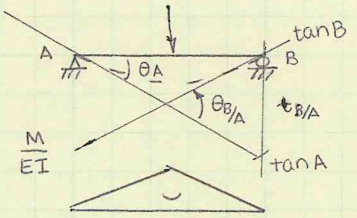
$$\Delta + f_{11} F_1 + f_{12} F_2 = -F_1/K$$

(-) sign means when added to f_{11}, all combine positively.

CE 363: STRUCT. ANALYSIS

C. HOVELL
26 October 05

MOMENT · AREA



$$|\theta_B| \pm |\theta_{B/A}| = |\theta_A|$$

Sign conventions:

- if $\theta_{B/A}$ is positive, from tan A to tan B is counter-clockwise ; moment diagram is positive
- if $t_{B/A}$ is positive, tan B is above tan A.

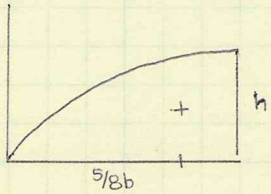
$\theta_{B/A}$ = moment / EI curve area

$t_{B/A} = \theta_{B/A}$ (distance to centroid from pt. B)

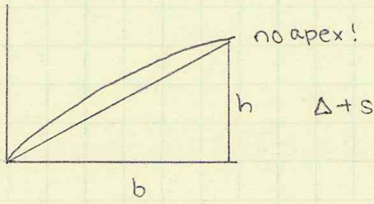
- can be used for general equations / points
- cannot be used (easily) for shear, axial deflections
- requires accurate deflected shape diagram
  - check answers to shapes.

stepwise process

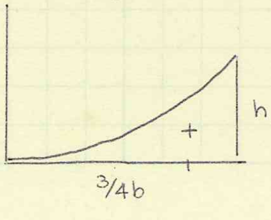
1. calculate reactions, draw M/EI diagram
2. Draw deflected shape as accurately as possible
3. use knowns ($\theta = 0$, $\Delta = 0$) to calculate relationships to unknowns
   - work with small distances, if possible
4. back out to interested value



A = 2/3bh



Δ + sliver
↳ $A = \frac{wx^3}{12}$, $\bar{x} = 1/2\,b$



A = 1/3bh

VIRTUAL WORK

With indeterminacy, step by step:

1. Establish degree of indeterminacy
2. Choose redundant(s)
    - must leave stable structure
3. Compute response of primary structure
    - break down influences & superimpose
        - loads
        - temperature
        - rigid body motion
    - draw curvature diagram (M/EI)
4. draw secondary ~~s~~ structure and curvature
5. Solve for Δ_0, θ_0

    $$\Delta = \int \frac{M}{EI} \cdot m\,dx \qquad \text{Area} \cdot m_{\bar{x}}$$

6. ~~Enforce capat~~ Calculate $f_{11}, f_{12} \ldots$ using secondary structures
7. Enforce capatibility

    $$\Delta_0 + f_{11} R_1 = \text{actual } \Delta$$

    $$\theta_0 + f_{11} M_1 = \text{actual } \theta$$

8. Reapply to original structure, enforce equilibrium
9. Draw shear, moment diagrams
10. Use actual structure, modified primary to calculate other Δ values.

Important Side Notes:

- always split at discontinuity
- indeterminacy: $2j = m + r$
- f_{14} = displacement at 1 due to a load at 4
- $k \equiv$ form factor $= A \int_A \left(\frac{Q}{It}\right)^2 dA$

    $= 1.2$ rectangle
    $= 10/9$ circle
    $= 1.0$ W-shape
- WATCH SIGNS, UNITS!

Frames and Beams

$\Delta = \int m \frac{M}{EI} dx + \Sigma \frac{nNL}{EA} +$ ~~$\Sigma \frac{nN}{GA} k$~~ $+ \int \frac{vV}{GA} \underline{\underline{k}} dx + \Sigma \frac{\alpha \Delta T}{d} L m$ + rigid body motion

flexure   axial   shear   thermal

$\llcorner \int K_T m\,dx$ (area under therm. curve)

Trusses - MAKE TABLE!

$$\Delta = \Sigma \frac{n\hat{N}L}{EA} + \Sigma n \alpha \Delta T L + \Sigma n \Delta L + \text{rigid body motion}$$

axial   thermal   misfit

$$f_{11} = \frac{n_1^2 L}{EA}, \quad f_{12} = \frac{n_1 n_2 L}{EA}$$

$N_{final} = \hat{N}_{primary} + Fn$ for each member

CE363: STRUCT. ANALYSIS

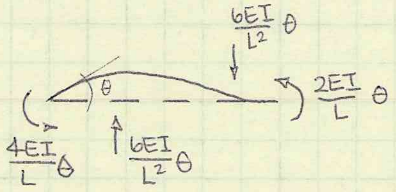
C. HOVELL

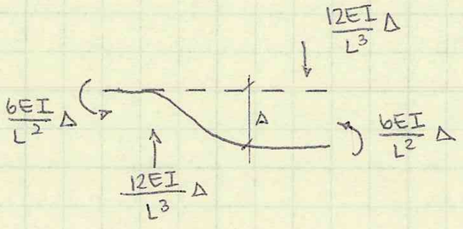
30 November 05

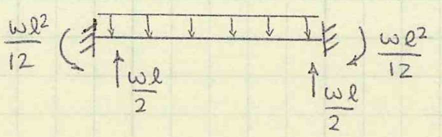
REVIEW SHEET

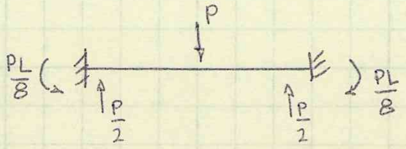
- DRAW EVERYTHING!
- SWITCH SIGNS of fixed-end forces, in application
- DRAW ALL DEFLECTED SHAPES

Standard Moment Values

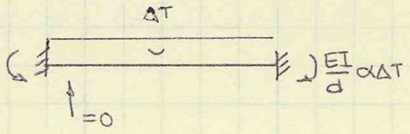


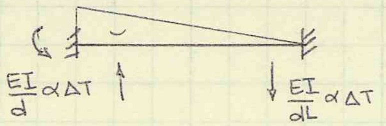






Fixed-end reactions





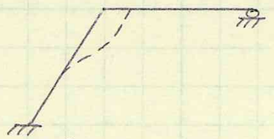
Temperature gradients (bottom hot)

General Rules

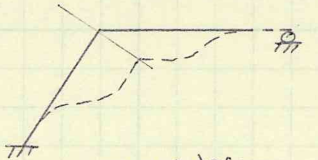
- $\underset{\sim}{F} = \underset{\sim}{K}\,\underset{\sim}{u}$
- $F = \frac{EA}{L}\Delta$
- k_{ij} = force induced at i due to a unit displacement at j

  = work done by forces in system $u_i = 1.0$ to cause displacements of system $u_j = 1.0$

- if no A value, assume axially rigid



deforms axially



does not deform

CE363: STRUCT. ANALYSIS

C.HOVELL
30 November 05

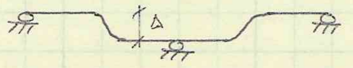
REVIEW SHEET

Intermediate loads
- fix both ends of member
- find reactions
- apply negatives to Force vector
- use directly in M/V calcs

Thermal loads
- like an intermediate load
- fixed-end method

For trusses:
- $F = EA\alpha\Delta T$ temp
- $F = \frac{EA}{L}\Delta L$ misfit

Support Settlement
- partitioned matrix
- include as external load



$\Delta \neq 1.0$

- USE NEGATIVES OF FORCES in vector $\underset{\sim}{F}$

Inclined supports
- mean 2 DOFs are coupled
- draw separately, use Γ to couple
  draw one, consider other
  label DOF up incline
- DRAW ALL DEFLECTED SHAPES

Transformations – Γ Matrix

considering each deformed, constrained shape, how much does each unconstrained DOF move?

$\bar{K}\bar{u} = \bar{F}$

$u = \Gamma\bar{u}$

$\Gamma^T K \Gamma = \bar{K}$

$\Gamma^T F = \bar{F}$ bar = constrained

Partitioned Matrix

$$\left[\begin{array}{cc|c} a^2 & ab & ac \\ ba & b^2 & bc \\ \hline \cancel{k}ca & cb & c^2 \end{array}\right]\left[\begin{array}{c} ua \\ ub \\ \hline uc \end{array}\right] = \left[\begin{array}{c} Fa \\ Fb \\ \hline Fc \end{array}\right]$$

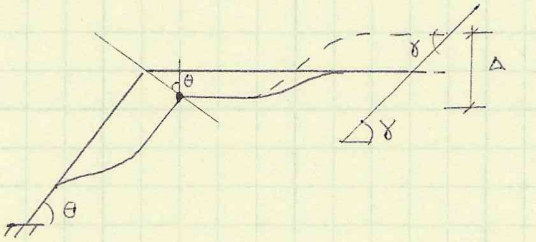
where ua, ub, Fc are unknown, uc, Fa, Fb are known

$$\begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}$$

$u_0 = K_{00}^{-1}(F_0 - K_{01}u_1)$

$F_1 = [K_{11} - K_{10}K_{00}^{-1}K_{01}]u_1 + K_{10}K_{00}^{-1}F_0$

Ah, geometry...



REVIEW SHEET

Global / Direct Stiffness Method

$Q \equiv$ LOCAL
$R \equiv$ GLOBAL

TRUSSES (no moment at joints):

$$T(c,s) := \begin{pmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{pmatrix} \qquad k_{local}(A,E,L) := \frac{E\cdot A}{L}\cdot\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

FRAMES (non-zero moment capacity):

$$k_{local}(A,E,I,L) := \begin{pmatrix} \frac{E\cdot A}{L} & 0 & 0 & \frac{-E\cdot A}{L} & 0 & 0 \\ 0 & \frac{12\cdot E\cdot I}{L^3} & \frac{-6\cdot E\cdot I}{L^2} & 0 & \frac{-12\cdot E\cdot I}{L^3} & \frac{-6\cdot E\cdot I}{L^2} \\ 0 & \frac{-6\cdot E\cdot I}{L^2} & \frac{4E\cdot I}{L} & 0 & \frac{6\cdot E\cdot I}{L^2} & \frac{2E\cdot I}{L} \\ \frac{-E\cdot A}{L} & 0 & 0 & \frac{E\cdot A}{L} & 0 & 0 \\ 0 & \frac{-12\cdot E\cdot I}{L^3} & \frac{6\cdot E\cdot I}{L^2} & 0 & \frac{12\cdot E\cdot I}{L^3} & \frac{6E\cdot I}{L^2} \\ 0 & \frac{-6E\cdot I}{L^2} & \frac{2\cdot E\cdot I}{L} & 0 & \frac{6E\cdot I}{L^2} & \frac{4E\cdot I}{L} \end{pmatrix}$$

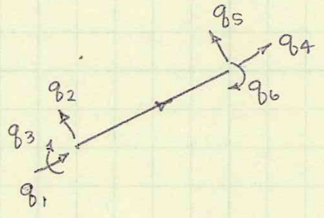
$$T(c,s) := \begin{pmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\cos = \frac{x_j - x_i}{L}$$

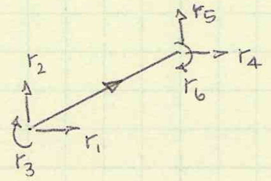
$$\sin = \frac{y_j - y_i}{L}$$

Matrix Manipulations

- $K\,u = F = F_0 - F_{FE}$
- $Q = k_{local} \cdot q$
  $R = k_{global} \cdot r$
- $Q = TR$
  $q = Tr$
- $k_{global} = T^T k_{local}\, T$
- $R_{FE} = T^T Q_{FE}$
- $Q = k_{local} \cdot q + Q_{FE}$



local



global

CE363: STRUCT. ANALYSIS

C. HOVELL

15 December 05

REVIEW SHEET

Direct stiffness – complications

- springs
  - add to diagonal K value in global matrix
- fixed-end forces
  - (loads, ΔT, ΔL)
  - ∘ determine Q_{FE} : local fixed-end forces
  - ∘ calculate $R_{FE} = T^T Q_{FE}$
  - ∘ assemble to global $\underset{\sim}{F}$ vector
- support settlement
  - ∘ write local r_{FE} vector; known
  - ∘ calculate $R_{FE} = k_{global} \cdot r_{FE}$ $\quad q = T r_{FE}$, $Q = k_{local} \cdot q_{FE}$

    $R_{FE} = T^T Q_{FE}$, $R = k_{global} \cdot r$
  - ∘ assemble to global $\underset{\sim}{F}$ vector