CE 383T
Plasticity in Structural Concrete

Class Notes
Introduction

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PLASTICITY IN STRUCTURAL CONCRETE

Structural concrete: all concrete used for a structural purpose

total spectrum – plain concrete
- reinforced concrete
  - active reinf. (P/S)
  - passive reinf. (rebar)

  ] or, mixed
- composite
  - tubes
  - W, C shapes

PLASTICITY IN STRUCTURAL CONCRETE

- Definitions & Principles
- Past
- Present
- Future

" The problem to find the state of stress under a specific loading does not make sense! "

- Ernst Melan-

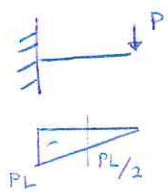
Wien 1938

what does the "state of stress" mean?

- STATE OF STRESS OF A BODY ?

  e.g. Paper Clip

Stress from load is not the same as stress in the member.



$f = \frac{Mc}{I}$ Is this a state of stress?

RESIDUAL STRESSES

- Why do structures stand up ?

- Why do structures collapse ?

I-35W: these questions need to be answered. NTSB is getting there.

Require plastic calculations

-B. Thürlimann-

godfather of plasticity in structural concrete

Strength is essential,
but otherwise unimportant.

" The first requirement for a beautiful bridge is that it must stand up long enough for us to look at it."

- *Hardy Cross*

ASCE Transactions 1932

first published paper on moment-distribution

A structure stands up if it can develop statically admissible stress distributions for all load cases without violating the yield conditions.

→ satisfies the law of equilibrium

-B. Thürlimann-

STRUCTURAL CONCRETE

- Any application of concrete in a structure or structural element

- A continuous spectrum from non-reinforced applications through the most involved combinations of concrete with non-prestressed or prestressed concrete

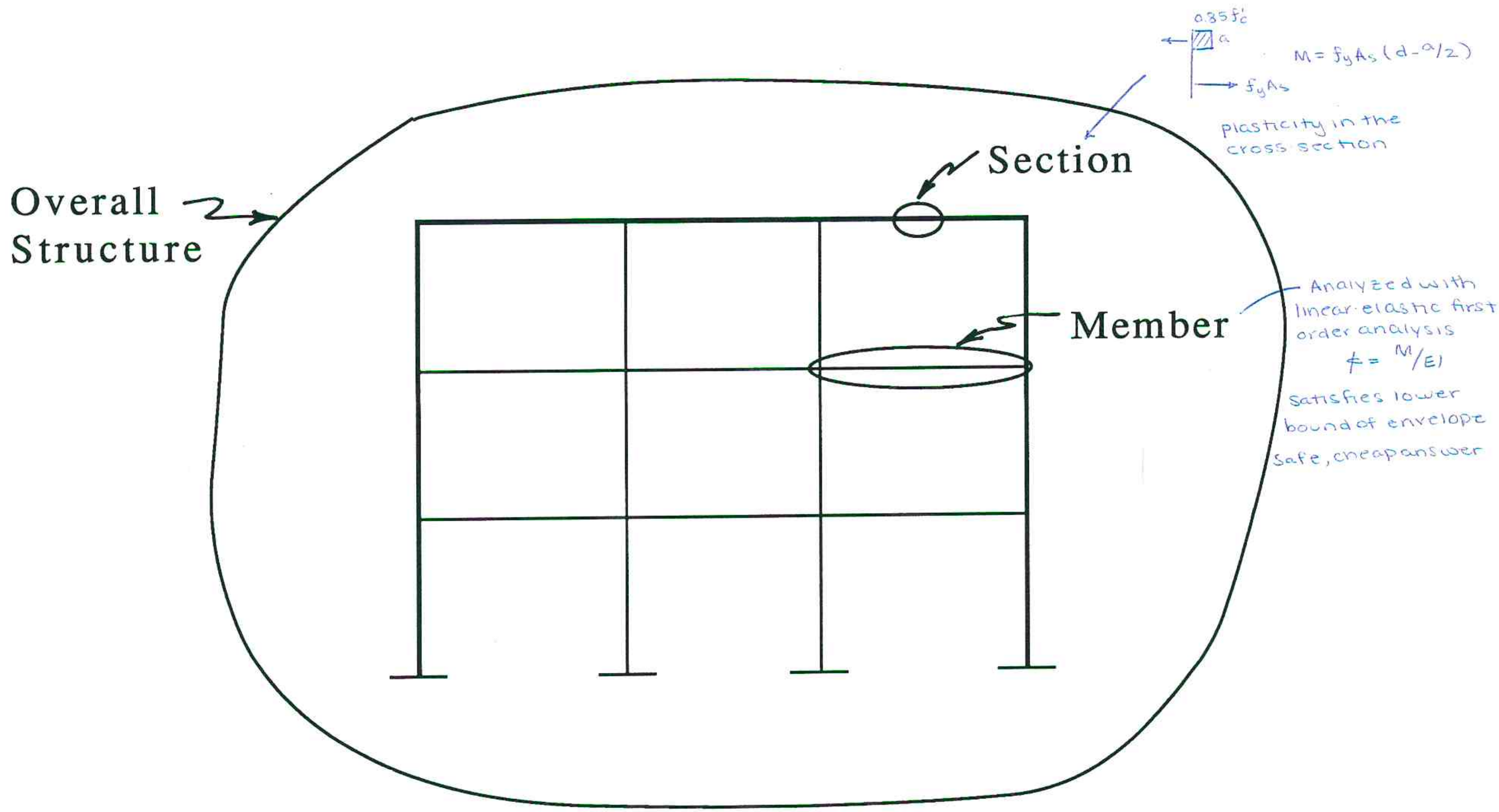
ELASTICITY - The quality or property of a material of returning to an initial form or state following deformation

PLASTICITY - The quality or property of a material which changes shape when pressure is applied and retains its new shape when pressure is removed

PLASTICITY - The quality or property of a material which makes it capable of undergoing continuous deformation without rupture or relaxation

Most important "ϕ factor" is the fee factor — linear analysis is cheap and conservative.

PLASTICITY DOMAINS



STRUCTURAL CONCRETE

	ACTIONS	LINEAR	PLANAR	SPATIAL
N	THRUST	COLUMN PRESTRESSED BEAM	PRESTRESSED SLAB WALL ARCH	SHELL
M	MOMENT	BEAM COLUMN	SLAB WALL ARCH	SHELL
V	SHEAR	BEAM COLUMN	SLAB WALL ARCH	SHELL
T	TORSION	BEAM	SLAB ARCH	SHELL

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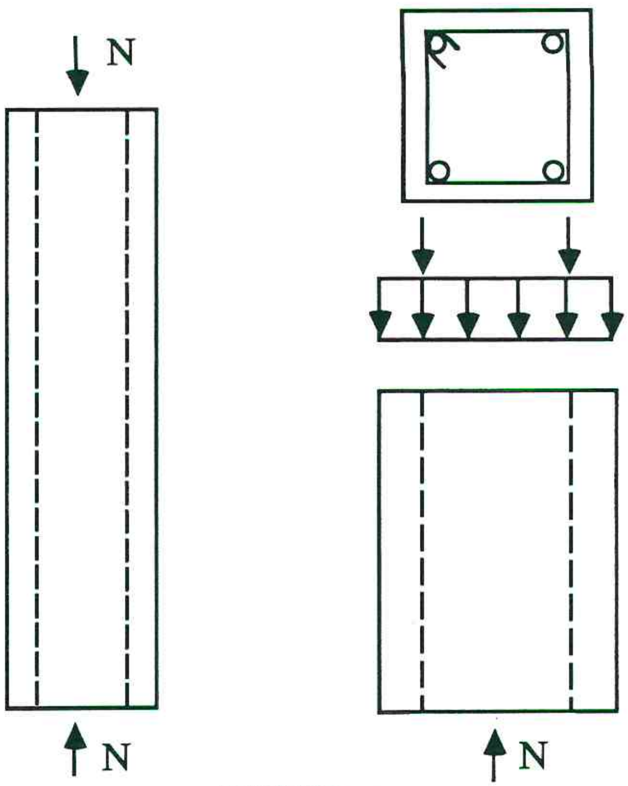
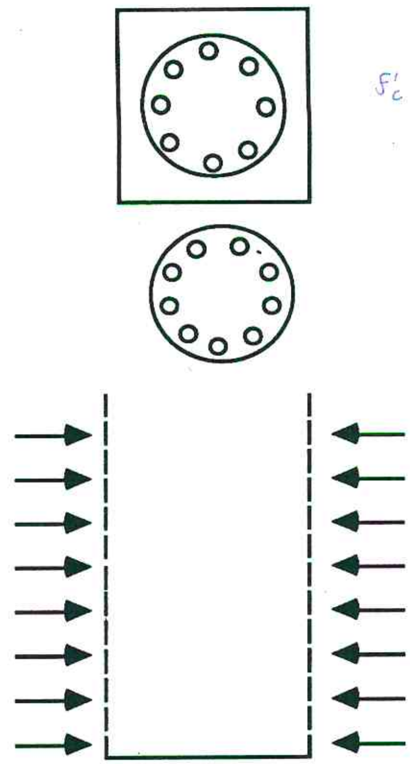
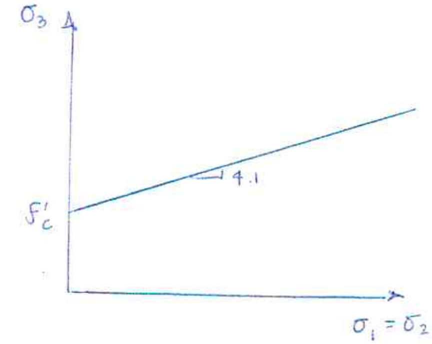
THRUST







$N_s = A_s f_y$

$N_c = 0.85\ f'_c A_g$

Lyse, Richart } ACI Column Program 1930

Considere -1903
Morsch -1920
Richart et al. -1928

TIED

ADDITION LAW - FULLY PLASTIC

$$N = N_c + N_s = 0.85\ f'_c A_g + A_s f_y$$

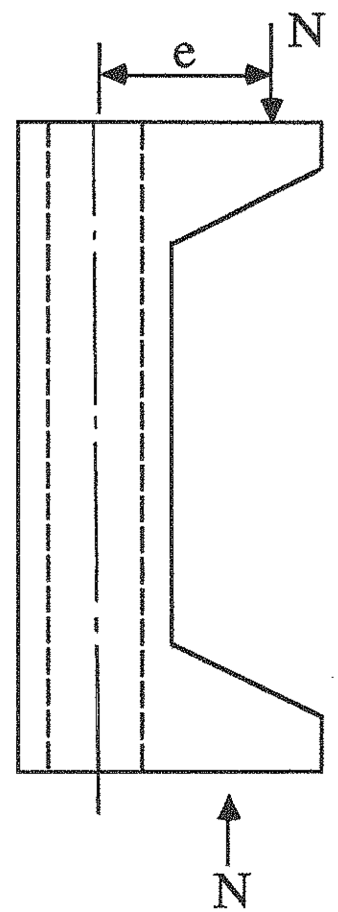
SPIRAL

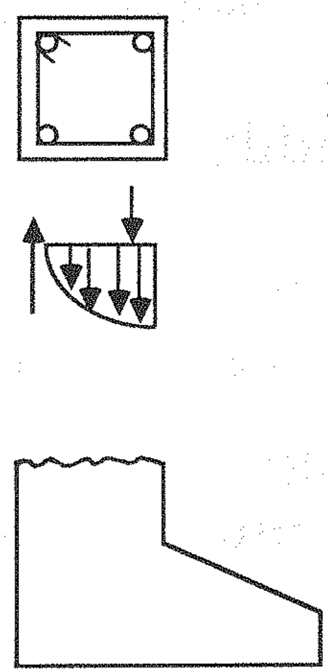
ADDITION LAW - FULLY PLASTIC

Recognizes Lateral Confinement

$$N = N_{c_2} + N_s = 0.85\ f'_c A_c + 4.1\ f_2 A_c + A_s f_y$$

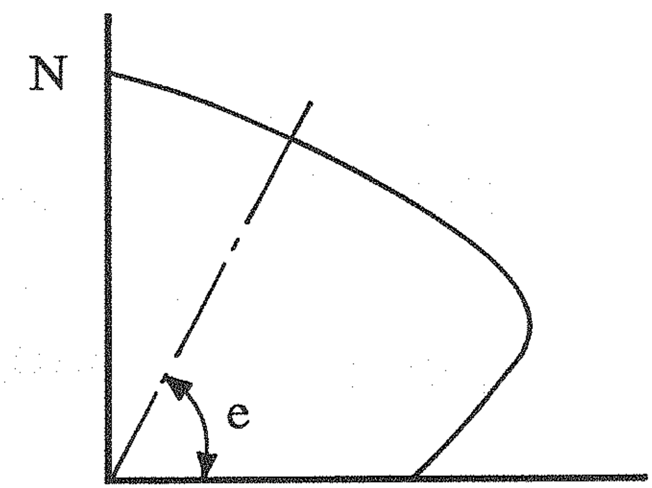
THRUST & MOMENT





Whitney 1942

Hognestad 1951 – went to Illinois after WW2 (Navy)



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RECENT APPROACHES TO PLASTICITY IN STRUCTURAL CONCRETE

- Constitutive Equations and Computer Models
  - Ottosen, Chen, Okamura (Tokyo) – SCC
  - Tassoulas
- Reinforcement Anchorage and Development
  - Ferguson, Goto (Japan), Tepfers, Jirsa, Hess (development length, hook equations)
- Generalized Plasticity Approach for Structural Concrete
  - M.P. Nielsen, W.F. Chen – originators of the study
  - B. Thürlimann (Zurich) et al.

MORE ADVANCED PLASTICITY CONCEPTS OFTEN IGNORED

- Failure by Progressive Plastic Deformation (Incremental Collapse) - HORNE 1950
- Failure by Alternating Plastic Deformation (Low Cycle Fatigue) - imp. in aircrafts
  - BLEICH 1932
  - MELAN 1936
  - KOITER 1936
- Shakedown Theory Including Elevated Temperature Effects, cyclic temps. - PRAGER 1956
- Bauschinger Effect - Important in Cyclic Loading as for Seismic Actions — Bazant (Northwestern) viscoplasticity
- Concrete Creep and Shrinkage Effects (Viscoplasticity)

plasticity only deals with strength requirements

DESIGN APPROACHES

ANALYSIS METHOD	COMPUTATION OF RESISTANCE	GENERAL NAME
ELASTIC ANALYSIS - LINEAR (Superimpose Load Effects)	Limits on stresses (Elasticity)	Allowable Stress Design
	Best Estimate of strength of critical sections (Plasticity) 1960s	Strength Design
ELASTIC ANALYSIS - 2nd ORDER (No superposition) P·Δ	Best Estimate of strength and stiffness of critical sections (Plasticity)	Second Order Strength Design
INELASTIC RESPONSE TO LIMIT LOAD (No superposition)	Best Estimate of strength of critical sections - arbitrary restrictions to ensure ductility (Plasticity)	Limit Design ACI 428
ELASTIC ANALYSIS AT SERVICE LOAD	Satisfaction of deformation limits and crack control (Elasticity)	Limit States Design
ELASTIC OR INELASTIC ANALYSIS AT LIMIT LOADS	Satisfaction of strength requirements (Plasticity)	

solution of a model – has lots of assumptions (not exact)

Not practical because of no superposition

recognizes that a structure can be unfit for use without falling down (cracks, deflections...)

"Nevertheless a man who wishes to go far in art or science, must, I think, proceed to universal principles and make himself acquainted with them, as far as possible; for sciences, as we have said, deal with universals."

-Aristotle-

Need to learn universal principles
and then generalize from them

".... once we leave the elastic range, all hope disappears of finding both a simple and complete description of the mechanical behavior. When solving problems simplicity comes first. Full generality is not the goal; full generality is complete chaos and contains no information."

- *Drucker* -

Dean of Engineering at UIUC

EM 395: Theory of Plasticity

" Aim of Theory of Plasticity is the detailed study of the stress and strain fields in structures in a state of contained plastic flow."

- *Ch. Massonnet* -

" Part of Theory of Plasticity.... to develop methods for obtaining the value of the plastic collapse load of structures.... is called Limit Analysis."

- *Ch. Massonnet* -

IABSE: International Association for Bridge & Structural Engineers

RECENT APPROACHES TO PLASTICITY IN STRUCTURAL CONCRETE

- Shear in Plain & Reinforced Concrete - Nielsen, Braestrup
- Shear & Torsion in Reinforced & Prestressed Concrete -Collins, Mitchell
  compression Field Theory (CFT)
- Flexure, Shear, & Torsion in Structural Concrete -Thürlimann, Lampert, Marti, Müller, Pralong
  Marti: Now at Toronto (influenced Collins, Mitchell)
  generalized from Nielsen, Braestrup's shear theories
- Transparent Design Models for Structural Concrete - Schlaich & Schaffer
  good for detailing rebar, esp. in trouble areas (e.g. corners)
- Limit Design of Columns in Frames - Ferguson, Breen
- Linear Programming in Limit Design - Cohn, Grierson
  optimization techniques
  Cohn: Univ. of Waterloo, Canada

PLASTICITY VIEWPOINTS

PhD level
- RESEARCHER
- TEACHER
- CODE OFFICIAL
  might not even be a graduate engineer
- DESIGNER (BS / MS)

↓ Increasingly Simple

Need simple models and procedures for the designer to use, while still matching behavior to some level same quote as a few pages back.

".... once we leave the elastic range, all hope disappears of finding both a simple and complete description of the mechanical behavior. When solving problems simplicity comes first. Full generality is not the goal; full generality is complete chaos and contains no information."

- *Drucker* -

PLASTICITY IN STRUCTURAL CONCRETE

KEY QUESTIONS FOR UTILIZATION

- Does it result in better structures ?

  safer, more serviceable ...

  allows us to detail in a sensible way

- Does it result in significant economies ?

  depends on situation

  much more payoff in steel than in concrete

- Does it save design time ?

  Again, maybe

  Takes time to learn how to use plasticity correctly

Plastic theory is great as a design tool.

- congestion in reinforcement (unconstructable)
- failures occur through detailing

cost of steel:

- steel procurement
- fabrication
- placing
- concrete placement (how congested?)

"... it is in relation to design that plastic theory is of the greatest interest."

- M.R. Horne - 1971

MAIN REWARDS OF PLASTICITY IN STRUCTURAL CONCRETE

- Develop clearer understanding of failure mechanisms
- Lessen dependence on empiricism - fewer equations / prescriptive requirements
- Lead to move transparent models (more)
- Enhance intuitive feel of designer
- Leads to improved proportioning and detailing
  more constructable patterns of steel
- Emphasize overall structural system behavior

No necessary big economic benefits

MAIN REWARDS OF PLASTICITY IN STRUCTURAL CONCRETE

- Encourage continuity, redundancy, and ductility thus greatly enhancing General Structural Integrity
- Allows sensible consideration of structural system reserve in design process
- Allow constructability flexibility in design
- Possible materials savings

BENEFITS OF PLASTICITY APPROACH

- Means to study material and system response at advanced levels of loads and deformations

- ☆ Provides a physical basis for fail-safe approach to engineering systems rather than the weakest link approach (elastic limit of one component)

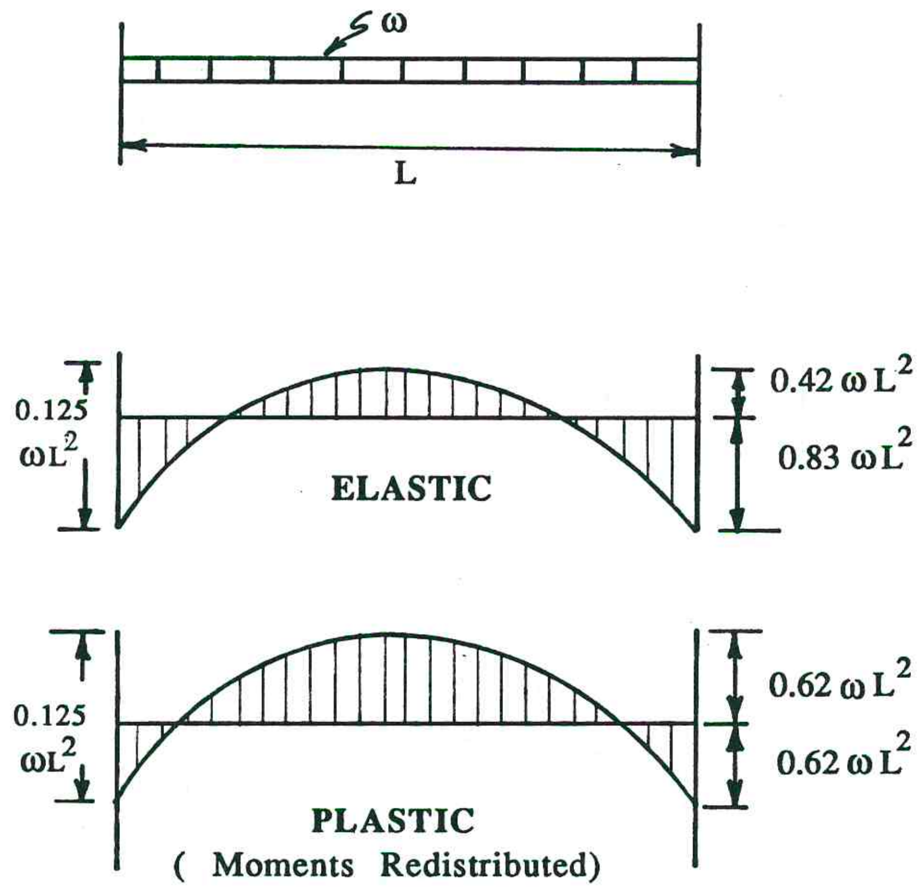
  multiple hinges may form before the structure fails

- Introduces concept of gradual, ductile deterioration of the system rather than sudden, brittle failure

- Allows one to take advantage of capacity and energy reserve of the system, if and as appropriate

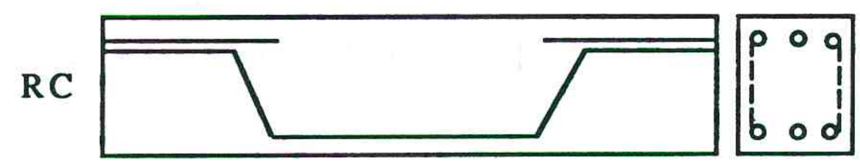
- Cohn 1977

ECONOMIC POTENTIAL



UNIFORM CAPACITY

$M'_{DES} = .083\ \omega L^2 \longrightarrow .062\ \omega L^2$ 25%



TAILORED CAPACITY

Main potential for savings are varied loadings and reducing congestion

"Although concrete is a material of very limited deformability, indications are that the load carrying capacity of reinforced and prestressed structures will, in time, be computed on the basis of the limit theorems of plastic analysis and design."

- *Drucker, IABSE,-1961*

(stability failures are also brittle)

REQUIRED ROTATION CAPACITY

Less steel:
- less moment capacity
- more ductility

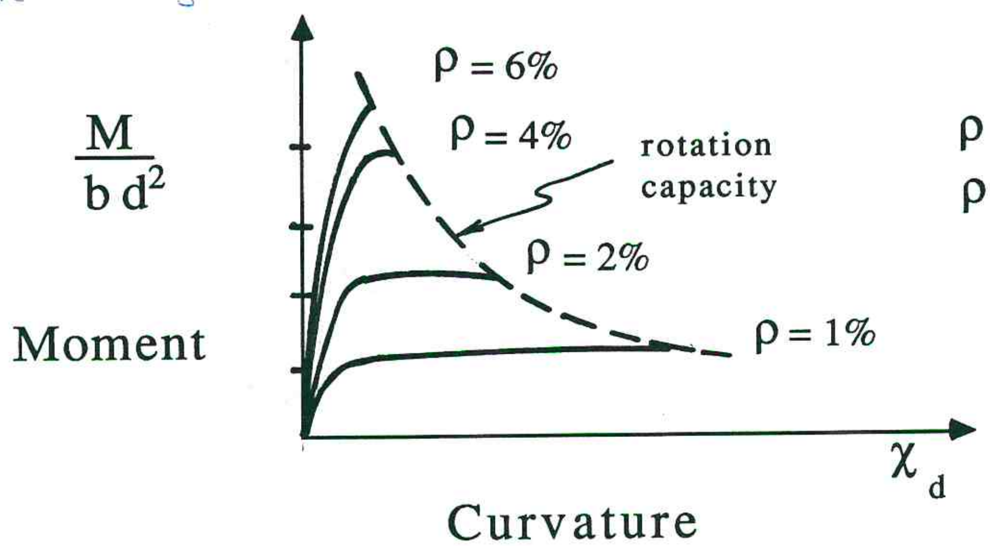
Grade 30 Concrete (4.2 ksi)

Grade 250 Reinforcement (36 ksi (G400 = 50 ksi))

$\rho_b = 5.5\%$

$\rho_{max} = 3/4\ \rho_b = 4.1\%$

$\rho_{redistr.} < \frac{1}{2}\ \rho_b = 2.75\%$ (redistribute)



FUTURE OF PLASTICITY IN STRUCTURAL CONCRETE

- **MORE FREEDOM TO "ASSIGN" OR "REDISTRIBUTE" REINFORCEMENT**

  Thürlimann et al. provided scientific base for use of plasticity in combined actions. Codes and designers can concentrate on applications to reduce congestion and provide improved structural integrity.

- **FOCUS ATTENTION ON OVERALL, INTERACTIVE STRUCTURAL ACTION**

  Aided by computers for detailed calculation, designer can use plasticity to determine weak link in design and weigh action required

28 January 2008

FUTURE OF PLASTICITY IN STRUCTURAL CONCRETE

- GROWTH IN USE IN DETAILING — greatest benefit, (J. Breen)

  Transparent Models for Discontinuity

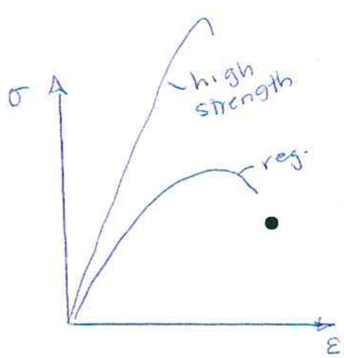
  Safe Lower Bound In Many Cases

Sleipner — maybe the largest structural failure of the last 25 years... but no one was killed. (oil platform in Norway)

- Research must identify where lack of ductility imposes limits

  This is a concern with <u>more brittle</u> high strength concretes

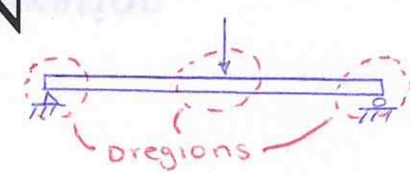
  Limits on brittleness: wide early cracking, stability



- Polymer concrete and fiber reinforced concrete may be <u>more ductile</u> — FRC included in ACI 318-08 for the first time

PLASTICITY IN DETAILING

- MAJOR PAYOFF FOR DESIGNER
- RATIONAL BASIS FOR STRUT-AND TIE MODELS
- SUBSTANTIAL SIMPLIFICATION IN DISCONTINUITY REGIONS



- DEVELOPMENT OF THEORY COMPLEX
  - application becomes very complex
  - how about simpler method?
- USE OF MODELS EASIER

MAJOR PROBLEMS OF PLASTICITY IN STRUCTURAL CONCRETE

- INHERENTLY LOW EXCESS CAPACITY (not a lot of excess in design)

  Tailoring of Reinforcement
  High Forming Costs
  Substantial Economics in Standardization

- ACCOMODATING NEW MATERIALS

  Higher Strength- Lower Ductility
  Less Bulky - More Prone to Instability slimmer members – less stability
  Larger Bars and Chemical Coatings Reduce Bond
  ACI 318-08 – introduces headed bars

- INCORPORATION OF PROBABILISTIC CONCEPTS

  To date largely deterministic

MAJOR PROBLEMS OF PLASTICITY IN STRUCTURAL CONCRETE

- DEFORMATION BOUNDS
  - Acceptable Displacements (not serviceability here)
  - Limited Ductility
  - Discontinuities Due to Cracking
  - Effective Crack Control at Service States
  - Confinement Effects on Strain Capacity
    adds ductility (used a lot in earthquake)
    increases strain capacity

- TIME EFFECTS

  Maturing Structure
  Creep & Shrinkage – not considered (limit state / strength)
  Cyclic Loading - Seismic and Thermal

concrete failures most common during construction
– low concrete strength
– high construction loads

"All around us we see the results of neglect of long-term problems, and yet we persist in concentrating our attention on short-term design. Least first cost is a short-term objective. Long useful life is a long-term objective. We need them both."

-Thomas R. Kuesel

"There are several further reasons for bringing the notions of bounding techniques to the attention of engineers. They appeal to and make use of the physical intuition developed by designers; the methods, once appreciated, require little mathematical expertise."

- Johnson and Mellor
Engineering Plasticity-1983

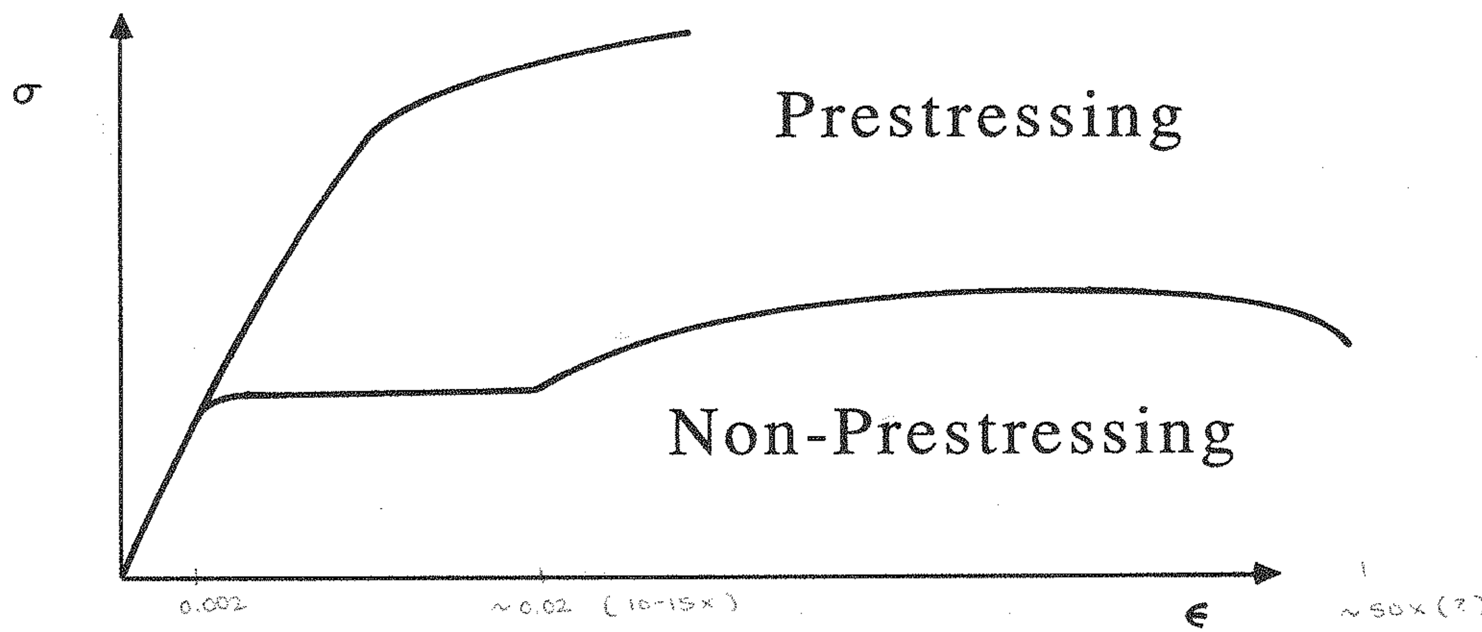
"The extension of a piece of metal is in a sense more complicated than the working of a pocket watch and to hope to derive information about its mechanism from two or three data derived from measurement during the tensile test is perhaps as optimistic as would be an attempt to learn about the working of a pocket watch by determining its compressive strength."

- *E. Orowan, F.R.S., 1944*

Fellow to the Royal Society (v. high in UK)

MATERIAL PROPERTIES

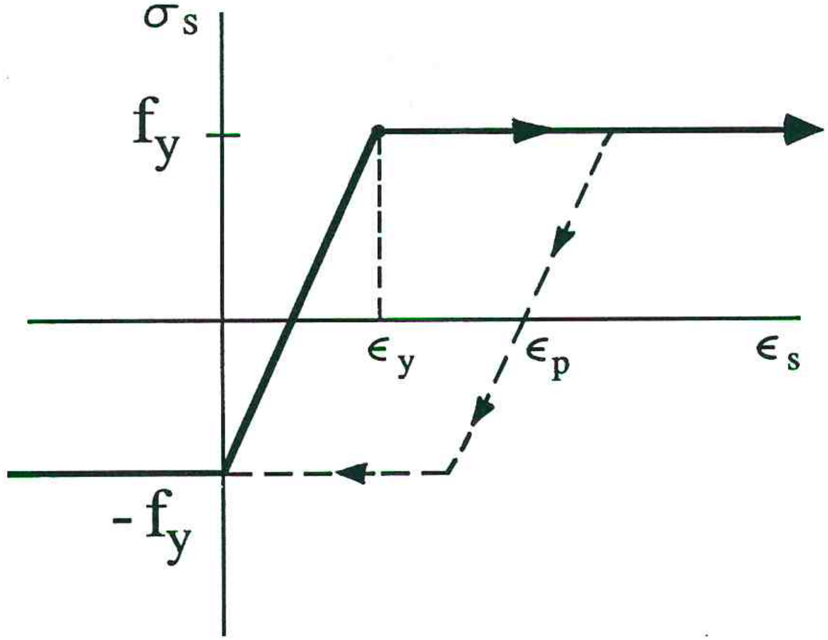
- STEEL - Almost always linear elements loaded parallel to axis



Mainly tension
Compression similar if stability provided

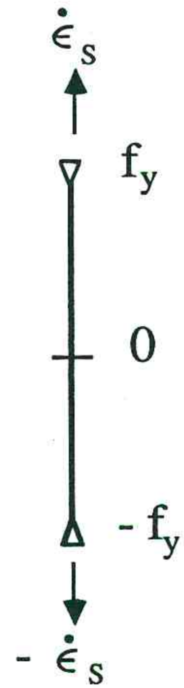
ASSUMED MATERIAL PROPERTIES

- STEEL - <u>UNAXIAL</u>



Elastic - Plastic

Ductile $\epsilon_p >> \epsilon_y$



$\sigma_s = f_y \rightarrow \dot{\epsilon}_s > 0$

$-f_y < \sigma_s < f_y \rightarrow \dot{\epsilon}_s > 0$

$\sigma_s = -f_y \rightarrow \dot{\epsilon}_s > 0$

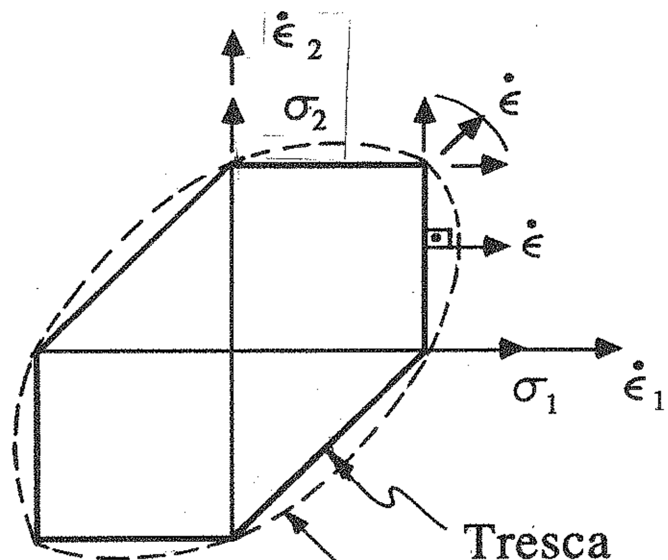
Rigid-Plastic (uniaxail yield criterion)

$\dot{\epsilon}_s \longrightarrow$ Strain rate

ASSUMED MATERIAL PROPERTIES

- STEEL - BIAXIAL YIELD CRITERION

  FLOW RULE



Tresca (1864))
Maximum shearing stress

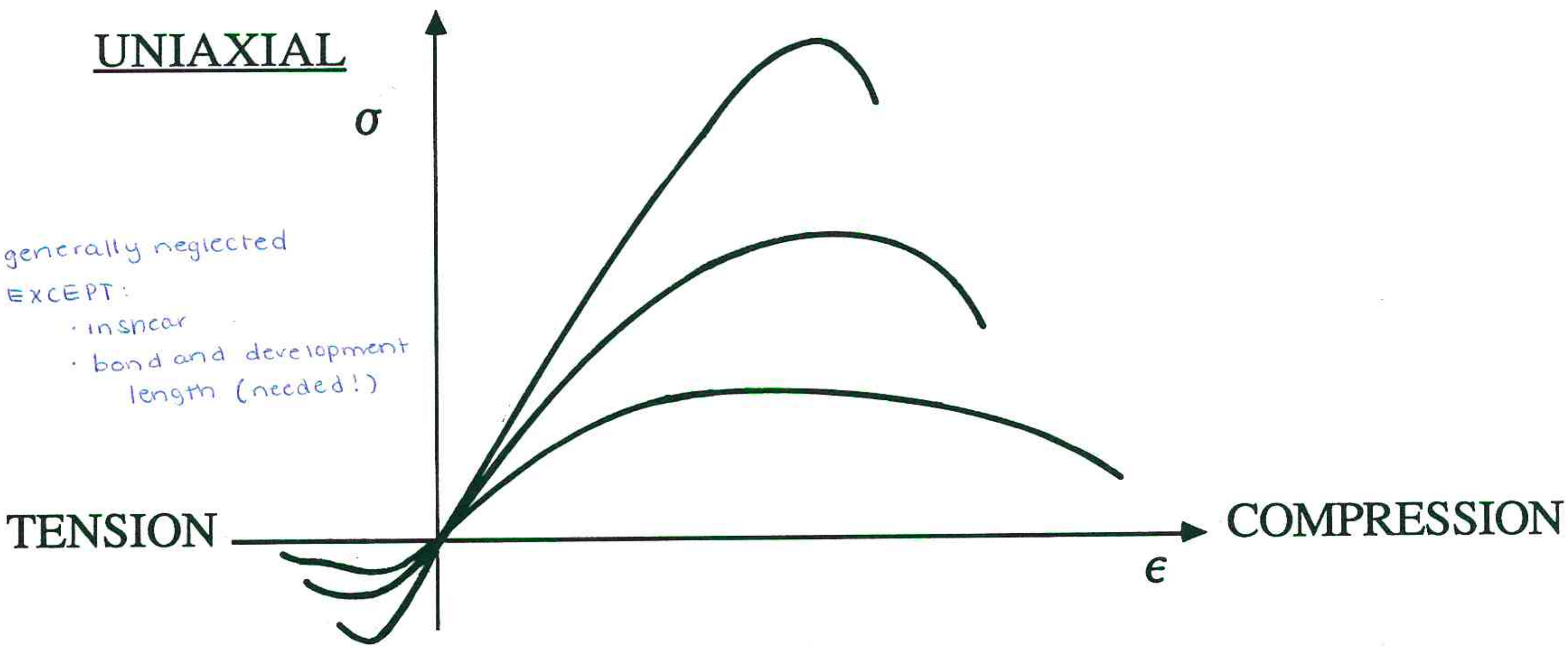
Von Mises (1913)
Octahedral shearing stress
Constant energy of distortion

MATERIAL PROPERTIES

CONCRETE - May be uniaxial, biaxial, triaxial depending on application

UNIAXIAL

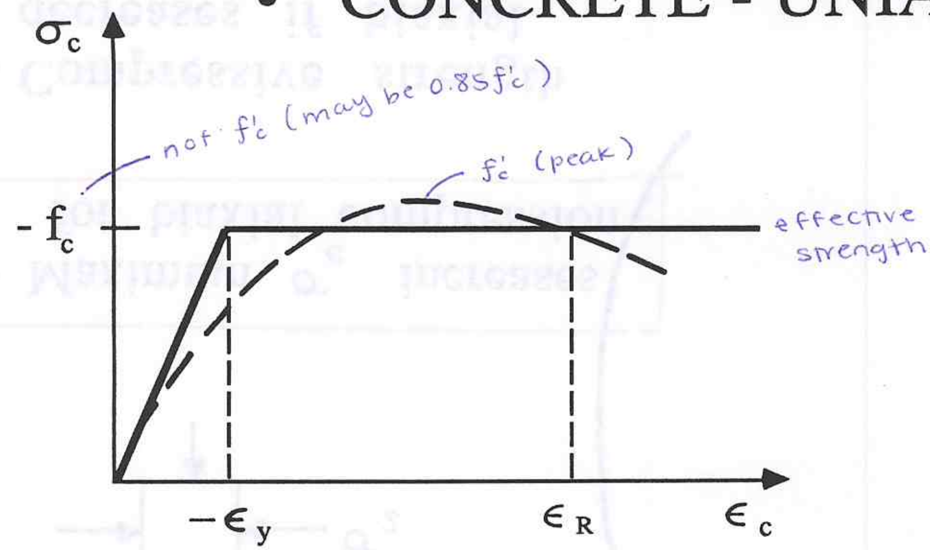
- generally neglected
- EXCEPT:
  - in shear
  - bond and development length (needed!)



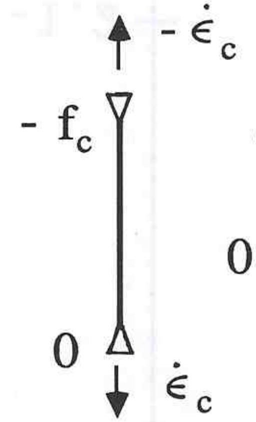
- INCREASINGLY BRITTLE WITH STRENGTH INCREASE
- VERY LOW TENSILE STRENGTH

ASSUMED MATERIAL PROPERTIES

• CONCRETE - UNIAXIAL



Elastic-Plastic



$\sigma_c = -f_c \rightarrow \dot{\epsilon}_c < 0$

$0 > \sigma_c > -f_c \rightarrow \dot{\epsilon}_c = 0$

$\epsilon_c = 0 \rightarrow \dot{\epsilon}_c > 0$

Rigid-Plastic

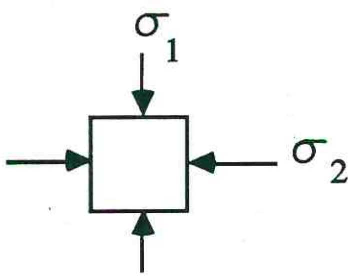
PROBLEMS:

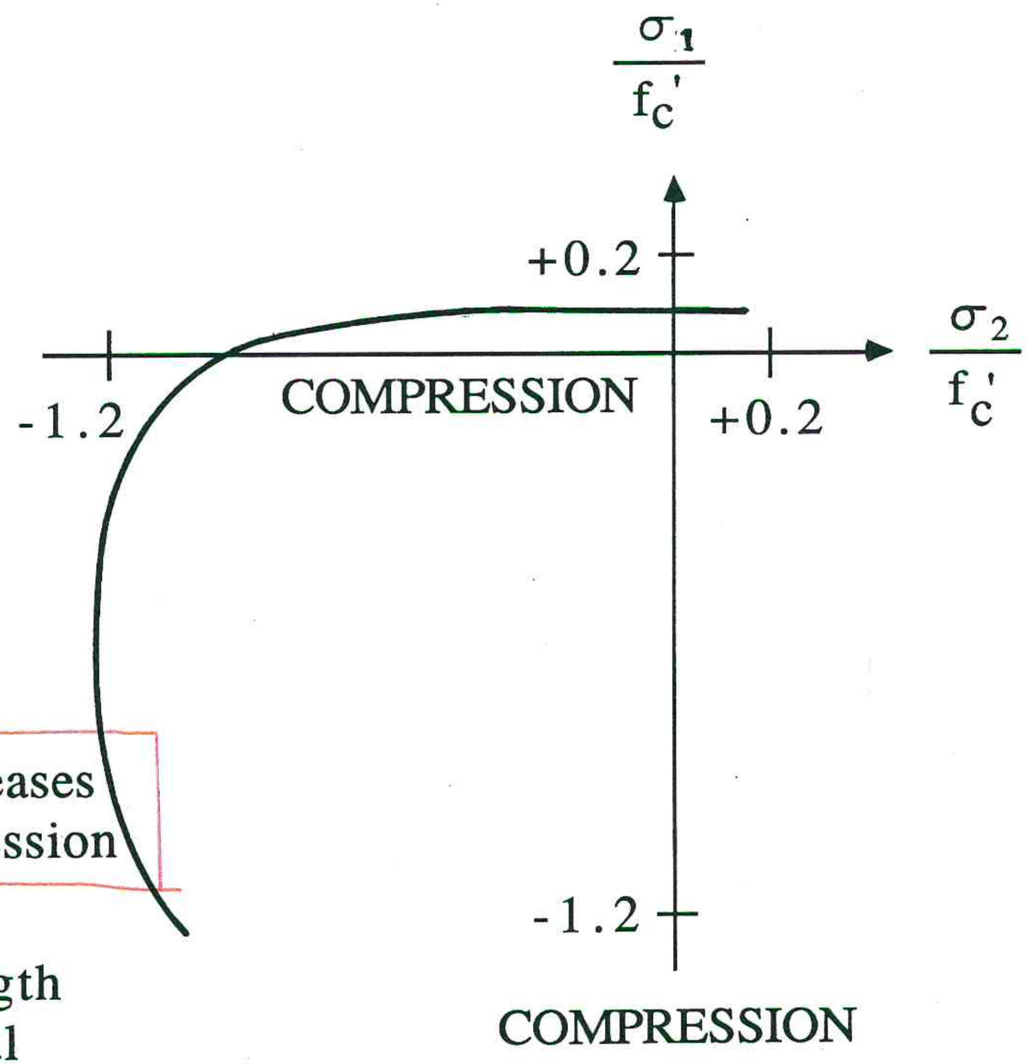
Brittleness of ultra high strength concrete
Varies with time
Essential neglect of tensile strength
Highly simplified
$f_c = \nu_e f'_c$ (Effective strength depends on cracking at failure)
Positive $\dot{\epsilon}_c$ - no tensile strength means discrete cracks

- direction of cracking
- direction of strut

- CONCRETE

BIAXIAL





- Maximum σ_c increases for biaxial compression

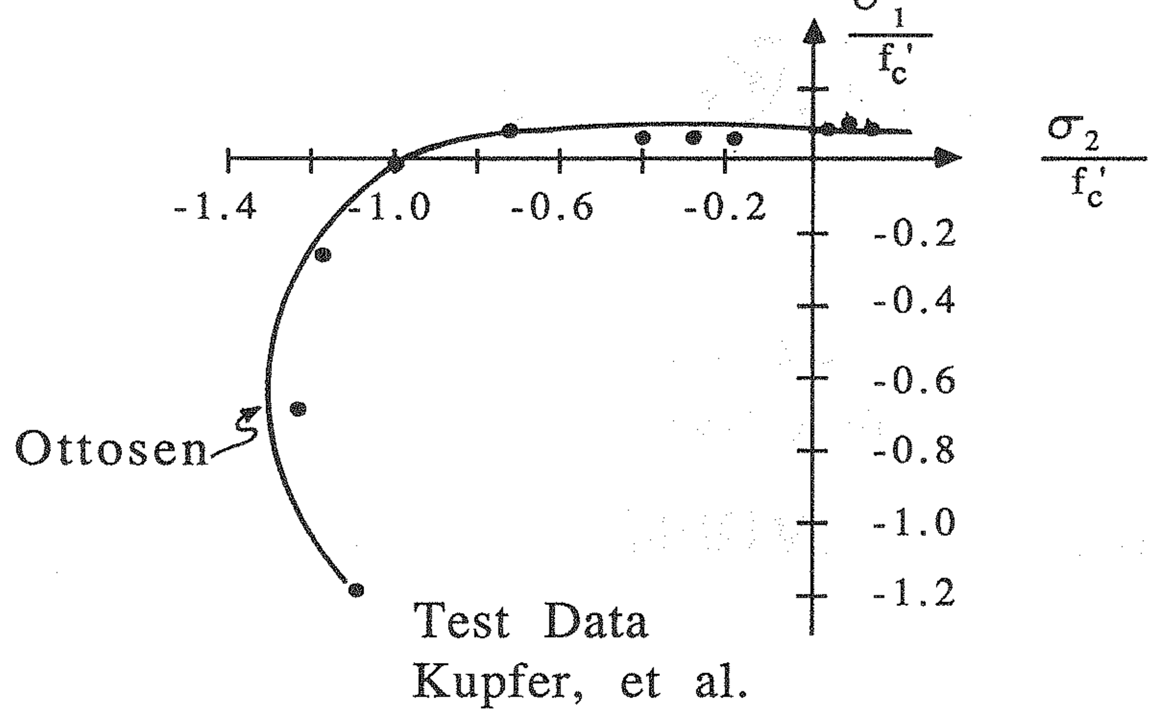
- Compressive strength decreases if biaxial tension-compression present

-Kupfer, Hilsdorf, Rusch, 1969-

ASSUMED MATERIAL PROPERTIES

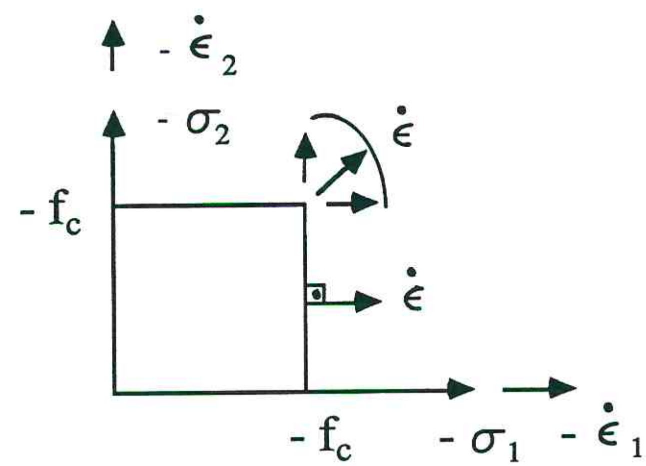
Numerical Modelling - Ottosen Criterion for Concrete

$$f(I_1, J_2, \cos 3\theta) = a\frac{J_2}{f_c'^2} + \lambda\frac{\sqrt{J_2}}{f_c'} + b\frac{I_1}{f_c'} - 1 = 0$$



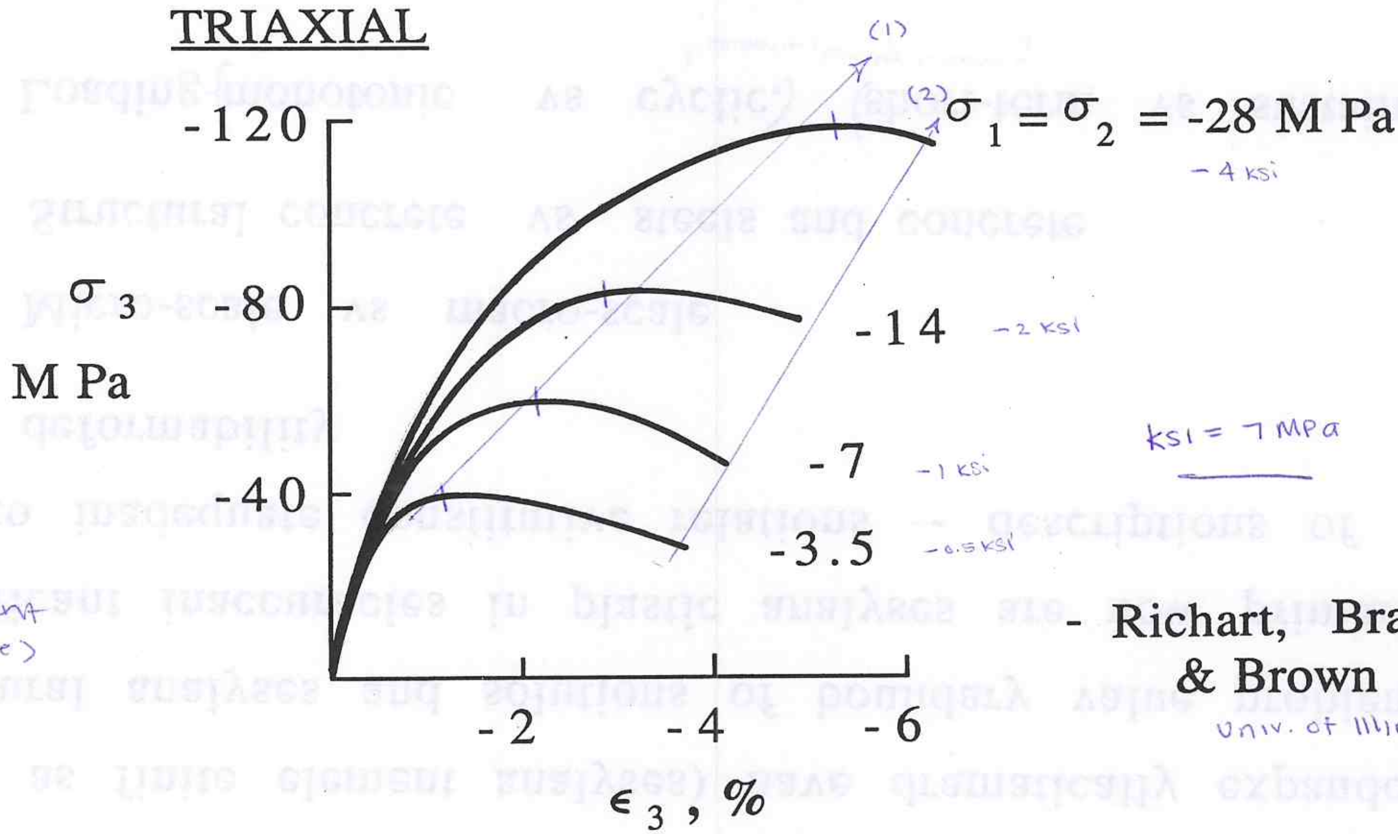
ASSUMED MATERIAL PROPERTIES

- CONCRETE - BIAXIAL YIELD CRITERION
  FLOW RULE



- **CONCRETE**

use confinement for:
- increased comp. stress (1)
- increased ductility (2)



- Richart, Brandtzaeg, & Brown - *1928*

Univ. of Illinois

ksi = 7 MPa

lateral confinement (even just a little) increases strength significantly.

- Confinement increases compressive strength
- Confinement produces substantial ductility

28 January 08

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Developments of Computer Techniques and Numerical Methods (such as finite element analyses) have dramatically expanded structural analyses and solutions of boundary value problems. Significant inaccuracies in plastic analyses are now primarily due to inadequate constitutive relations -- descriptions of local deformability

- Micro-scale vs macro-scale
- Structural concrete vs steels and concrete
- Loading-(monotonic vs cyclic,) (short-term vs sustained)
  (seismic, bridge loading)
- Rigid Plastic vs elastic-work hardening

RESEARCH CHALLENGES

- DEFINE WHEN AND HOW MUCH RELIANCE CAN BE PLACED ON CONCRETE TENSILE CAPACITY can't overrely!

- RELATE CONVINCINGLY TO REINFORCEMENT DEVELOPMENT
  strut and tie vs. development theory

- TIE IN HIGH COMPRESSION MEMBERS
  - LOW ECCENTRICITY COLUMNS
  - POST - TENSIONED ANCHORAGES
  - FULLY PRESTRESSED MEMBERS

  } insufficient testing

Andenheggen – ETHZ (zurich)
M. Schlaich / P. Steffen } two famous students
strut-and-tie models, computer programs, etc.

RESEARCH CHALLENGES

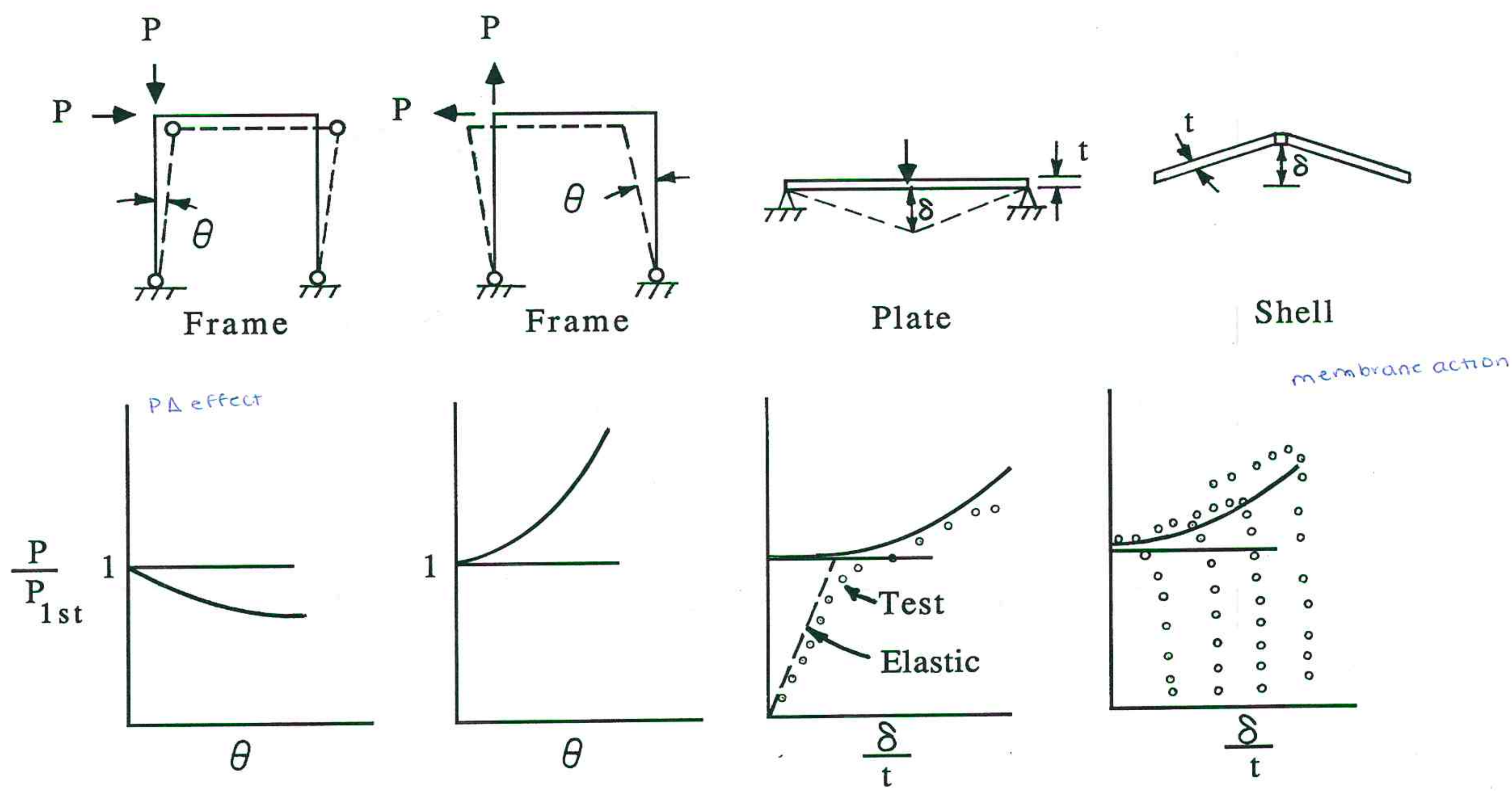
- DEVELOP USER - FRIENDLY COMPUTER MODELS WITH GOOD GRAPHICS
    - ILLUSTRATE LOAD PATHS IN STRUCTURES
    - IDENTIFY "D" REGIONS
    - REFINED NON-LINEAR FEA OF "D" REGIONS
    - INCLUDE REALISTIC CONCRETE FAILURE MODEL
- DEVELOP A FRAMEWORK TO INCLUDE LOW CYCLE (SEISMIC) AND HIGH CYCLE (FATIGUE) LOADINGS

STABILITY - Changes in Geometry Significant

- Overall Stability - structure as a whole
- Member Stability - within the length of a member
- Local Stability - within the member and affecting its cross-sectional shape

think of P/S beams with optimized shapes

PLASTICITY AND STABILITY



STABILITY CHECK GUIDANCE

$$\frac{\lambda_c}{\lambda_o} = \frac{\text{Elastic Stability Limit Multiplier}}{\text{Plastic Limit Load Multiplier}}$$ (big is good)

Example : 1975 EC (Eurocode) Steel

$\frac{\lambda_c}{\lambda_o} > 10$ First order theory

$4 < \frac{\lambda_c}{\lambda_o} \lessapprox 10$ Particular consideration to stability checks

$\frac{\lambda_c}{\lambda_o} < 4$ Second-order elastic plastic analysis

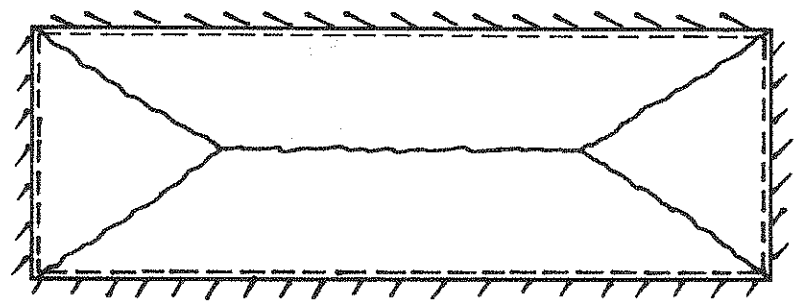
Similar rules in ACI 318-08

"... an alternative definition of collapse is when the stiffness of the structure becomes zero!"

- *S.S.J. Moy*

SLAB PLASTICITY - Yield Line Analysis

$\rho << \rho_{bal}$ High ductility



1. Assume yield line pattern – upper bound, used for design of slabs (high ductility)
2. Check Mechanism - Virtual work - Easiest- Upper Bound
3. Check Equilibrium - Complicated by nodal correction forces

   - also Upper Bound
4. Revise yield line pattern to find lowest load

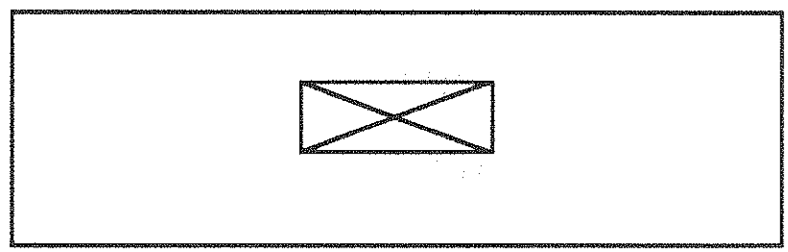
SLAB PLASTICITY - Yield Line Analysis

- Ignores strain hardening and membrane action
- Conservative compared to tests
- Inclusion of deformation effects requires step by step non linear procedure

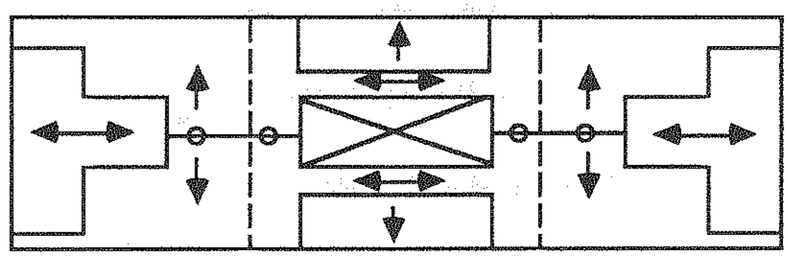
- *Ingerslev - 1921, 1923*
- *Johansen - 1931, 1943*
- *Ockleston - 1955*
- *Wood - 1961*
- *Neilsen - 1963*
- *Park - 1964*

SLAB PLASTICITY - STRIP METHOD (lower bound)

$\rho \ll \rho_{bal}$ High Ductility



SLAB WITH OPENING



cant.

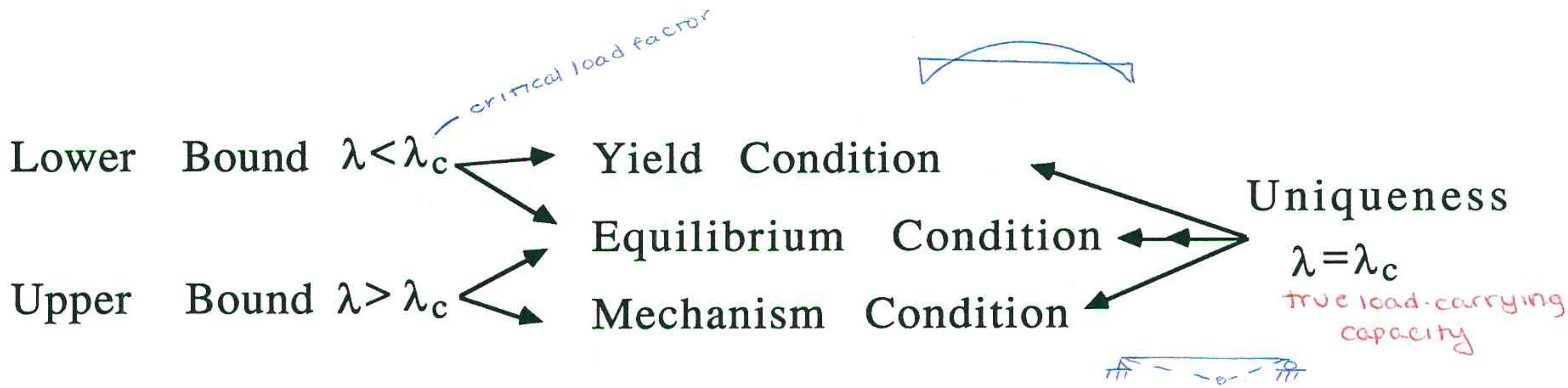
FLOW OF FORCES

LOWER BOUND - EQUILIBRIUM AND YIELDING

- HILLERBORG 1956, 1959

just wrote a new book 200_ ?

THEOREMS OF LIMIT ANALYSIS



- *Gvozdev -1938* USSR
- *Prager -1949* (also developed it; no communication)
- *Hill -1951*
- *Drucker, Prager & Greenberg -1952*

31 January 08

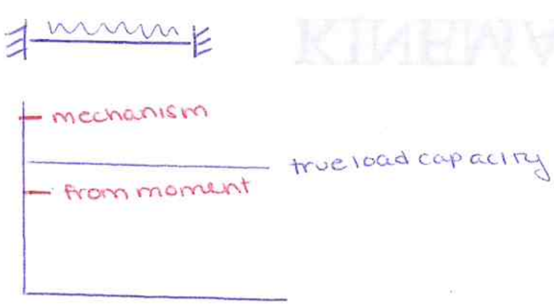
STATIC OR LOWER BOUND THEOREM

"If, at any load factor λ , it is possible to find a bending moment distribution in equilibrium with the applied loads and everywhere satisfying the yield condition, then λ is either equal to or less than the load factor at ultimate."

– $M \leq M_y$

COROLLARY

"The collapse load of a structure cannot be decreased by increasing the strength of any part."



Try various methods
– what load causes a moment that induces failure?
will be ≤ true capacity
– mechanism, plastic load
upper bound

KINEMATIC OR UPPER BOUND THEOREM

"If for any assumed plastic mechanism, (as many hinges as possible) the external work done by the loads at a positive load factor λ is equal to the internal work at the plastic hinges, then λ is either equal to or greater than the load factor at failure."

COROLLARY

"The collapse load of a structure cannot be increased by decreasing the strength of any part."

- *M.R. Horne*

UNIQUENESS THEOREM

"If, at any load factor λ, a bending moment distribution can be found which satisfies the three conditions of equilibrium, mechanism, and yield, then that load factor is the collapse load factor λ_p."

COROLLARY 1

"The initial internal state of stress has no effect on the collapse load."

- *M.R. Horne*

COROLLARY 2

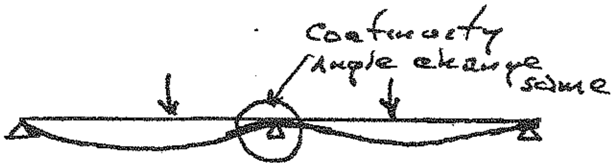
"If a structure is subjected to any programme of proportional or non-proportional loading, collapse will occur at the first combination of loads for which a bending moment distribution satisfying the conditions of equilibrium, mechanism, and yield can be found."

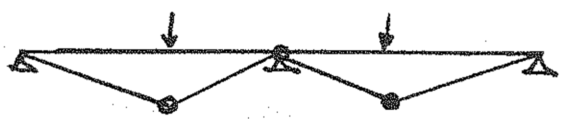
- *M.R. Horne*

(Assume change of geometry, instability and alternating plasticity are unimportant)

Essentials

Elastic Solutions	Plastic Solutions
1. EQUILIBRIUM	1. EQUILIBRIUM ← Most Important
2. Elastic Conditions (Hooke's Law)	2. Yield Criteria, Flow Rules (strain rate perpendicular to yield surface)
3. Compatibility	3. Mechanism (Rotation concentrates at hinges)





Limit Theorems of Plasticity

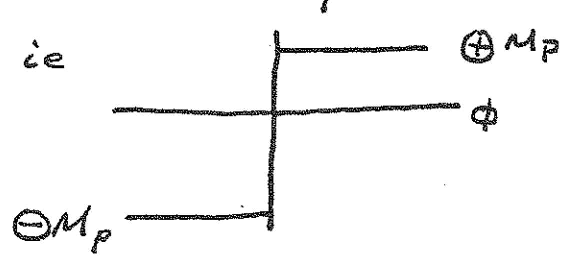
1) Lower Bound (Static Theorem)

a) Must fulfill Equilibrium

statically admissable stress field
statical boundary conditions

b) Must fulfill Yield criteria

ie



$\ominus M_p \leq M \leq \oplus M_p$

In steel $\oplus/\ominus$ M_p often same

In concrete $\oplus/\ominus$ M_p often differ

c) May not fulfill mechanism

Example -- Elastic structure satisfies (a) and (b) but not a mechanism

Corresponding load $P_L \leq P_{COLLAPSE}$

2) Upper Bound (Kinematic Theorem)

a) Must fulfill Equilibrium

-- use principle of virtual work

b) May not fulfill yield criteria

(Assumed mechanism may require $M > |M_p|$)

c) Must fulfill Mechanism

Corresponding Load $P_U \geq P_{COLLAPSE}$

3) Uniqueness Theorem -- satisfies both the static Theorem and the Kinematic Theorem simultaneously --

static theorem | Kinematic Theorem

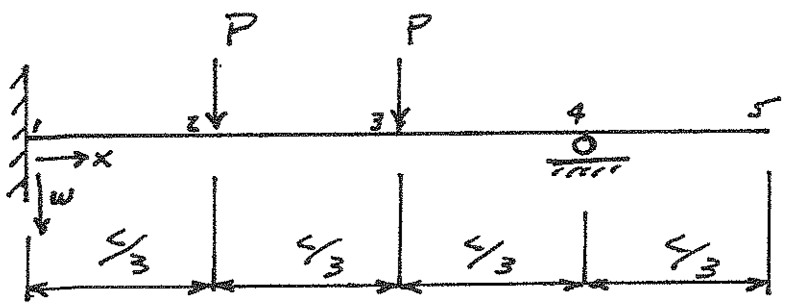
$$P_L \leq P_{collapse} \leq P_U$$

Thus $P_L = P_U = P_{collapse}$

VIP If you can demonstrate that you have satisfied all 3 parts of either P_L or P_U, you need not go to the other

One solution suffices to prove uniqueness

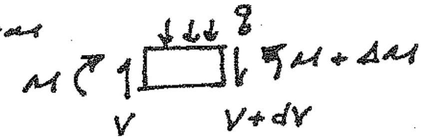
EXAMPLE TO ILLUSTRATE PROCEDURES



Uniform section

$\oplus M_p = \ominus M_p = M_p$

EQUILIBRIUM

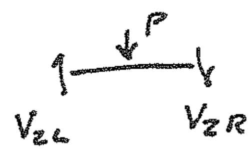


$$\frac{dM}{dx} = V \qquad \frac{d^2M}{dx^2} = -q$$

$$\frac{dV}{dx} = -q$$

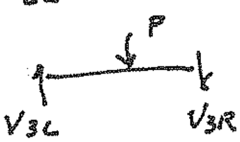
Static Boundary Conditions $M_5 = 0 \quad V_5 = 0$

Transfer Conditions ②



$P + V_{2R} - V_{2L} = 0$

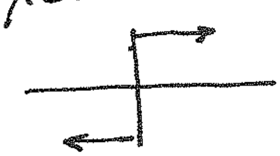
③



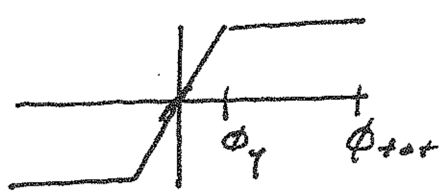
$P + V_{3R} - V_{3L} = 0$

Plasticity Assumptions

$\oplus M_p = \ominus M_p = M_p \quad \therefore$ Yield Criterion $\ominus M_p \le M_p \ge \oplus M_p$



$\phi' = \frac{d\phi}{dt}$ strain rate



$$\frac{\phi_y}{\phi_{tot}} \longrightarrow 0$$

Ductility required to disregard elastic part

We know that 4 unknowns - M_1, R_1, H_1, R_4

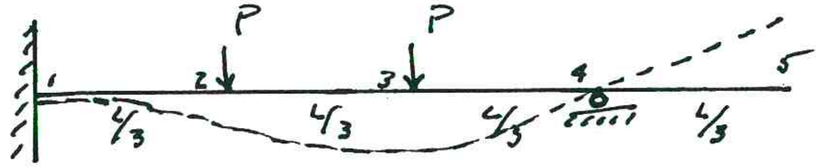
∴ 1 Time indeterminate

1 Hinge reduces to static, determinate

2 Hinges produce mechanism

Temporarily ignore just to see various capacities

Ⓐ Elastic Solution -- Using Moment Distribution



FEM: $\frac{Pab^2}{L^2}$ $\frac{Pa^2b}{L^2}$

$\frac{P\frac{L}{3}\left(\frac{2L}{3}\right)^2}{L^2} = \frac{4}{27}PL$

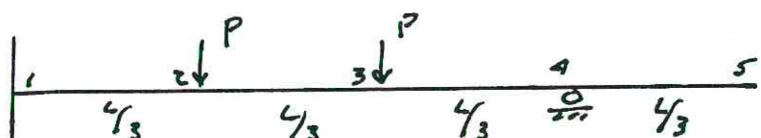
$\frac{P\left(\frac{L}{3}\right)^2\left(\frac{2L}{3}\right)}{L^2} = \frac{2}{27}PL$

FEM	+.222 PL	-.222 PL
1 Dist	0	+.222PL
1 CO	+.111 PL	0
Final M	+.333 PL	0

$FEM_1 = \frac{4}{27} + \frac{2}{27} = \frac{6}{27} = \frac{2}{9}$ PL

$FEM_4 = \frac{2}{27} + \frac{4}{27} = \frac{6}{27} = \frac{2}{9}$ PL

$M_1 =$ ↺ ⊖.333 PL



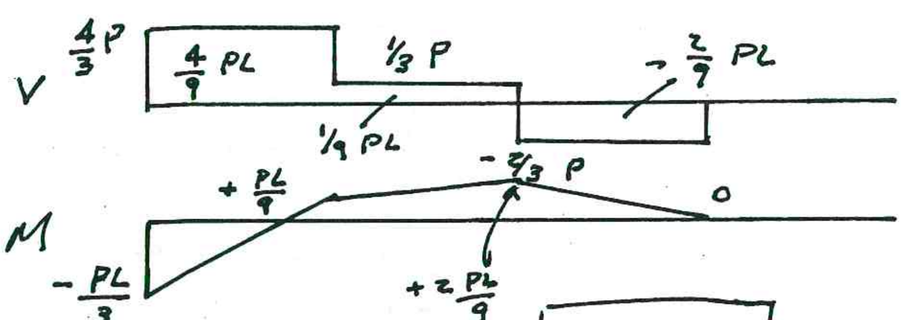
$\uparrow R_1 = \frac{4}{3}P$ $\uparrow R_4 = \frac{2}{3}P$

$\Sigma M_1 = 0$ +↻

$-.333\,PL + P\frac{L}{3} + P\frac{2L}{3} - R_4 L = 0$

$R_4 = \frac{2\,PL}{3\,L} = \frac{2}{3}P$ $\Sigma V = 0 \uparrow +$ $R_1 - P - P + \frac{2}{3}P = 0$

$R_1 = 1\frac{1}{3}P$



① Satisfies Equilibrium

② Consistent Deformation

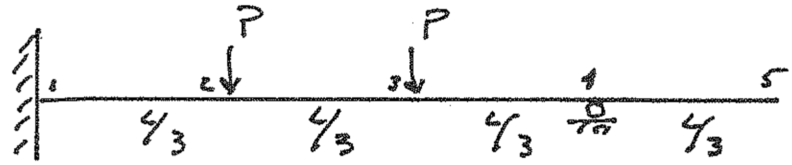
③ Only 1 Section at M_p

All other sections Less than M_p (Yie Crit)

Is a Lower Bound Solution

$\therefore -M_p = -\frac{PL}{3}$

$$P = \frac{3M_p}{L}$$

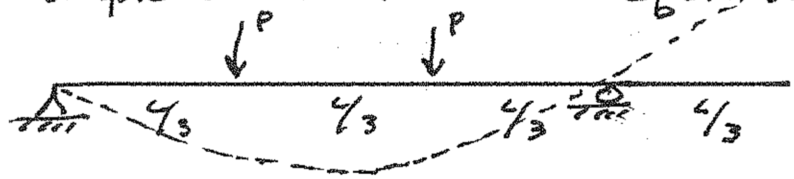


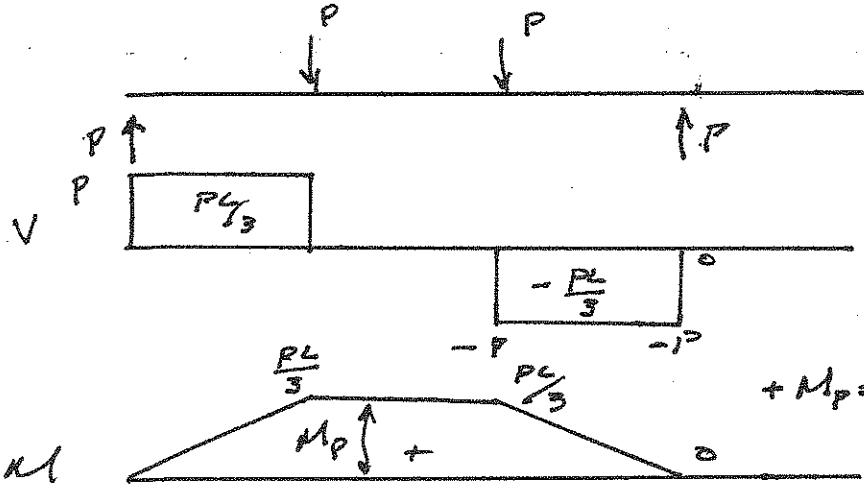
Elastic Solution

$P = 3.00 \frac{M_p}{L}$

(B) Weaken beam by burning flanges at (1) --

Now - Simple beam -- fulfills all equilibrium conditions





by inspection

(1) Equilibrium

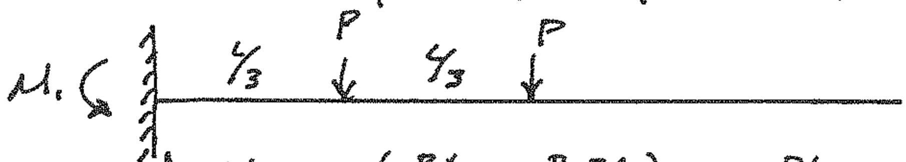
(2) Moment does not exceed M_p yield crit. met

$+M_p = \frac{PL}{3}$

$P = \frac{3M_p}{L}$

Lower Bound

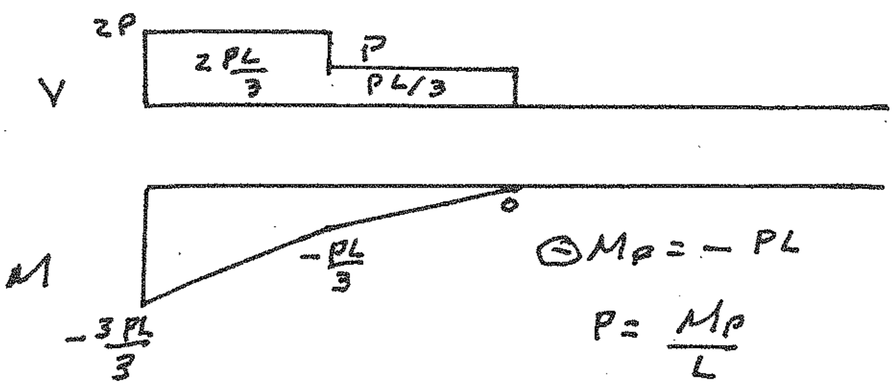
(C) Weaken beam by completely removing support at (4)



Cantilever - still fulfills equilibrium

$M_1 = -\left(\frac{PL}{3} + P\frac{2L}{3}\right) = -PL$

$R_1 = P + P = 2P$



$P = 1.0 \frac{M_p}{L}$

Lower bound

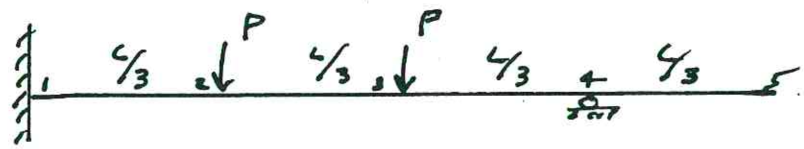
(1) Equilibrium

(2) Moment does not exceed M_p yield criterion met

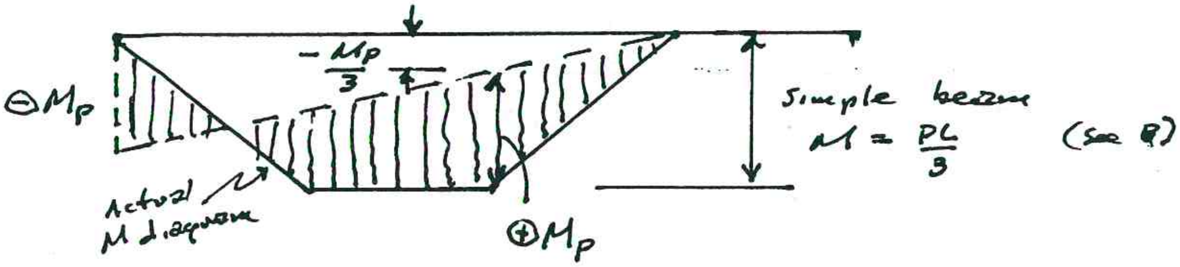
$(-)M_p = -PL$

$P = \frac{M_p}{L}$

(D) Now let load P increase till plastic hinges forms at support 1 and under load at 3



Moment Diagram consists of simple beam (inverted) and support moment ⊖Mp



$$M_3 = \frac{PL}{3} - \frac{M_p}{3} = +M_p \quad \therefore \quad \frac{PL}{3} = \frac{4}{3} M_p$$

$$\boxed{P = \frac{4 M_p}{L}}$$

Again, this is a lower bound

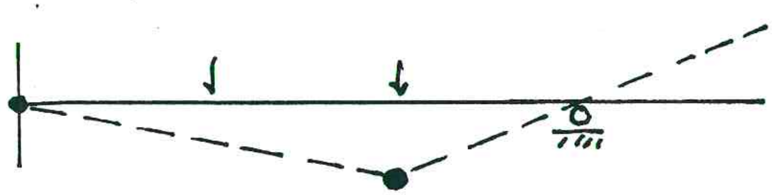
(1) Equilibrium is satisfied

(2) Moment does not exceed Mp at any point

For example $M_2 = \frac{PL}{3} - \frac{2}{3} M_p$

For $P = 4\frac{M_p}{L}$ $\quad M_2 = 4\frac{M_p}{L} \cdot \frac{L}{3} - \frac{2}{3} M_p = \frac{2}{3} M_p < M_p$

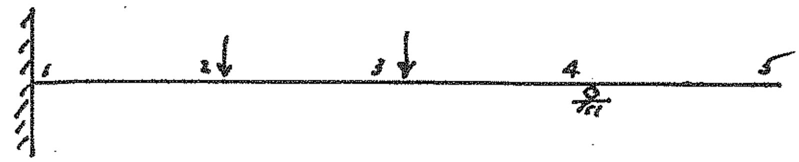
Actually, a mechanism forms



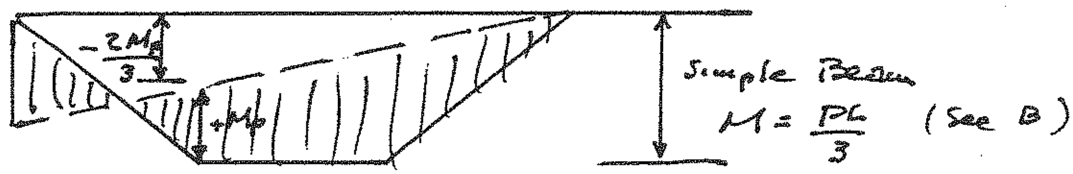
So this also satisfies upper bound criteria and is the unique solution -- but we will continue to examine lower bound cases

Note - This load is 33% higher than elastic solution

(E) Now let load P increase till plastic hinges form at support 1 and under load at 2



Moment Diagram as in (D)



$$M_2 = \frac{PL}{3} - \frac{2}{3}M_p = +M_p \quad \therefore \quad \frac{PL}{3} = \frac{5}{3}M_p$$

$$P = 5\frac{M_p}{L} \leftarrow \text{Higher than (D)}$$

But $M_3 = \frac{PL}{3} - \frac{1}{3}M_p$ If $P = 5\frac{M_p}{L}$

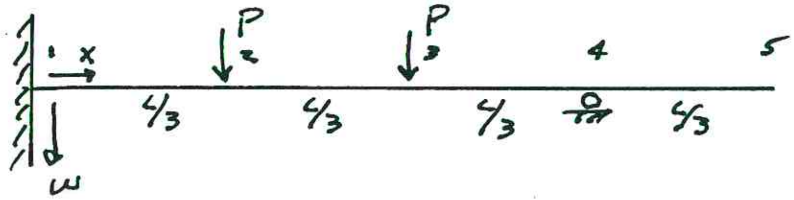
$$M_3 = 5\frac{M_p}{L} \cdot \frac{L}{3} - \frac{1}{3}M_p = \frac{4}{3}M_p \leftarrow \text{Violates the yield criteria}$$

Thus this is not a valid solution from a lower bound perspective since M_3 cannot be higher than M_p and satisfy yield criteria

---

Highest of the lower bound solutions is D and $P = 4\frac{M_p}{L}$

Now we will look at Kinematic solutions to determine upper bounds -- must be mechanisms



B.C. $w = 0$ $\quad \dot{w} = 0$ $\qquad\qquad w = 0$ $\quad \dot{w}_L = \dot{w}_R$ if no hinge

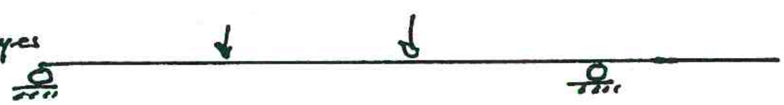
[Since this structure is 1 degree indeterminate it needs 2 hinges to develop a mechanism]

Possible configurations

Ⓐ Elastic

Not a mechanism

Ⓑ cut flanges at ①

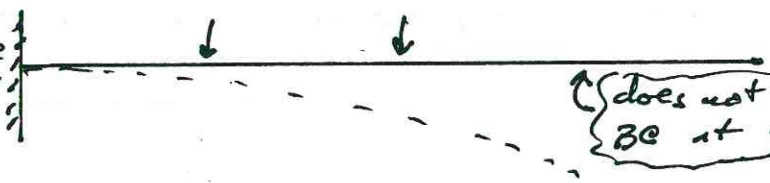


Not a mechanism

Ⓒ let plastic hinge develop at 1

Not a mechanism

Ⓓ Remove support at 4

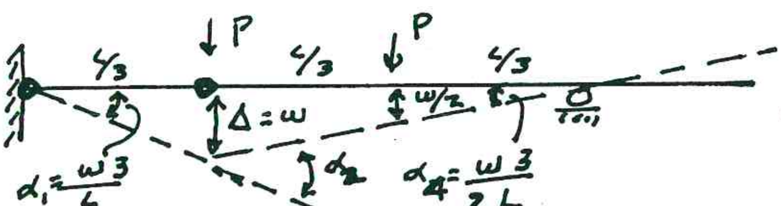


Not a mechanism

(does not satisfy BC at 4)

If yields at support is a mechanism but does not satisfy BC at 4

Ⓔ let Plastic hinges develop at 1 and 2



Equilibrium – Principle of Virtual Work

$W_e = W_i$ — Dissipation

(external) (internal)

$\alpha_2 = \alpha_1 + \alpha_4 = \frac{9}{2} wL$

$W_e = P \cdot w + P \cdot \frac{w}{2} = \frac{3}{2} Pw$

This is a mechanism

$W_i = M_p \alpha_1 + M_p \alpha_2 = M_p \frac{3w}{L} + M_p \frac{9}{2}\frac{w}{L} = \frac{15}{2} M_p \frac{w}{L}$

$W_e = W_i = \frac{3}{2} P\, w = \frac{15}{2} M_p \frac{w}{L}$

$P\theta + \frac{3}{2} M_p \theta$

$P\Delta + \frac{1}{2} P\Delta$

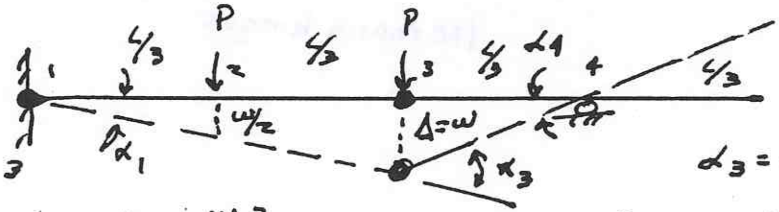
$= \frac{L}{3}\theta$

$M_p = \frac{3}{2} P \cdot \frac{L}{3}$

$P = \frac{5M_p}{L}$

(E) – we know from lower bound case (E) that at 3 the yield criteria was violated $M > M_p$ -- we would have to add cover plates in a steel structure or extra reinforcement in a concrete structure to beef up strength in region around 3 to make this mechanism work

(F) Let plastic hinges develop at 1 and 3. This is a mechanism



$\alpha_3 = \alpha_1 + \alpha_4$

$\alpha_1 = \frac{w\,3}{2L}$ $\qquad \alpha_4 = \frac{w\,3}{L}$ $\qquad \alpha_3 = \frac{3}{2}\frac{w}{L} + \frac{3w}{L} = \frac{9}{2}\frac{w}{L}$

$$W_e = P\cdot\frac{w}{2} + P\cdot w = \frac{3}{2}Pw$$

$$W_i = M_p\alpha_1 + M_p\alpha_3 = M_p\frac{3}{2}\frac{w}{L} + M_p\frac{9}{2}\frac{w}{L} = \frac{12}{2}\frac{w}{L}M_p$$

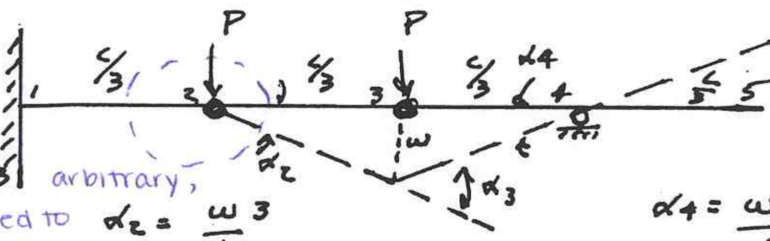
$$W_e = W_i = \frac{3}{2}Pw = \frac{12}{2}\frac{w}{L}M_p$$

$$\boxed{P = 4\frac{M_p}{L}}$$

Upper Bound but same as Lower Bound

Unique Solution

(G) Let Plastic Hinges Develop at 2 and 3



arbitrary, don't need to bend cantilever

$\alpha_2 = \frac{w\,3}{L}$ $\qquad \alpha_4 = \frac{w\,3}{L}$ $\qquad \alpha_3 = \alpha_2 + \alpha_4$

$W_e = P\cdot w$ $\qquad W_i = M_p\alpha_2 + M_p\alpha_3$

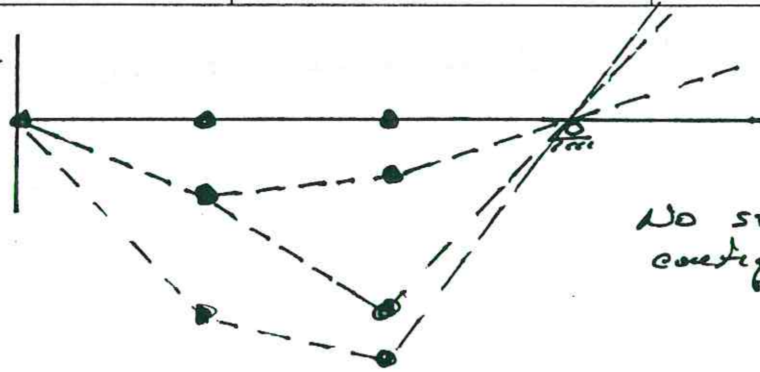
$$= M_p\left(\frac{w3}{L} + \frac{w3}{L} + \frac{w3}{L}\right) = M_p\frac{9w}{L}$$

$W_e = W_i$

$$\boxed{P = 9\frac{M_p}{L}}$$

Upper Bound – Much higher. Satisfies Equilibrium and mechanism

(H) Let Plastic Hinges form at 1, 2, 3



No stable configuration

too many hinges