CE 383T
Plasticity in Structural Concrete

Class Notes
Shear & Bending

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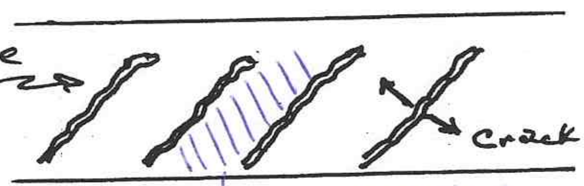
Structural Concrete Beams

Diagonal Compression Field

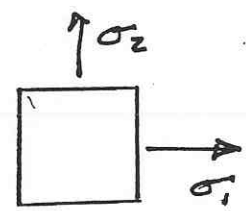
Individual Diagonals in a Truss



Along cracks the compression diagonals are interlocked by aggregate interlock → crack movement ⊥ to crack

↳ load is transfered through ties, friction

Combined Shear & Bending

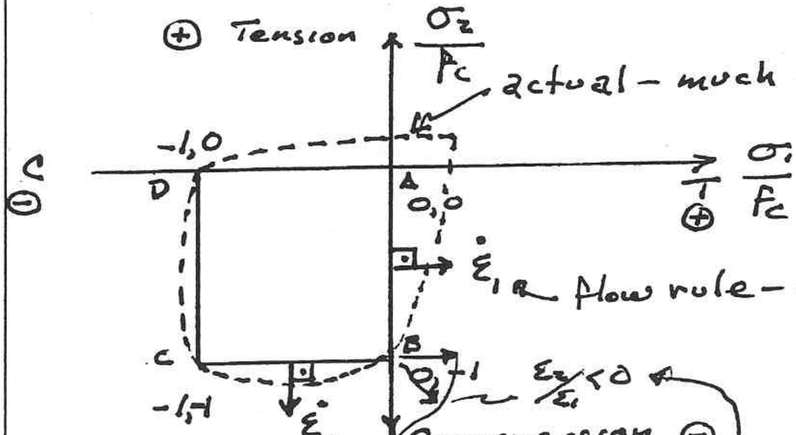


Assumptions:

1) Concrete -- No tensile strength

f_c = <u>effective</u> concrete strength $= \nu f_c'$

(not necessarily cylinder strength)

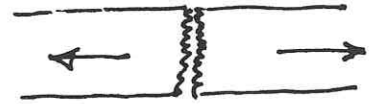


actual - much more complicated possible i.e. see W.F. Chen

flow rule - means if tension is applied crack opens perpendicular to applied stress

crack opening also ⊥ to applied compression on struts

Singularity

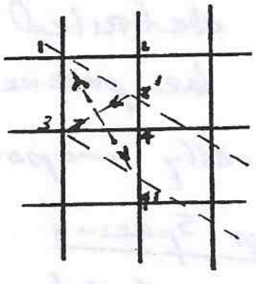


2) Steel – Only transmits uniaxial force
∴ No dowel effect!!! (Will be some in actuality)
from shear resistance in steel

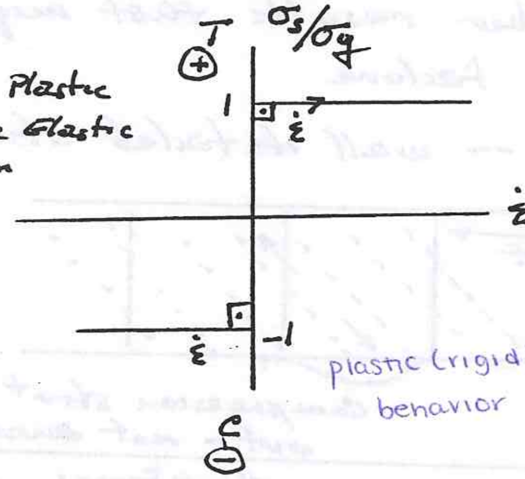
Steel mesh – Ignore welds

No shear resistance of mesh w/o concrete

Diagonals would elongate



Rigid Plastic
Ignore Elastic Portion



plastic (rigid) behavior

$D_{14'}$ elongates
$D_{32'}$ shortens
Means weld not working

3) Failure will be limited by design approach to preclude initial compression failure
∴ Failure will be due to yielding of reinforcement
Under-reinforced section – stirrups should yield before conc crushes
Investigate the limit on f_c so that

$$\sigma_c \leq -f_c \text{ at failure}$$

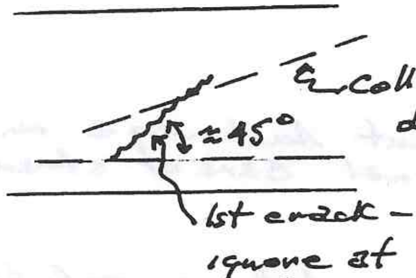
effective concrete stress $= \nu f'_c$

1) Rigid Plastic Theory

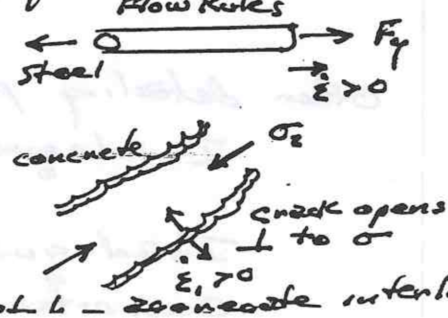
Any elastic or inelastic deformations, or cracks prior to incipient collapse are thus neglected -- redistribution of forces are possible from elastic state – Influence on effective concrete strength must be discussed

you have to shove the compression strut across the cracks – weakens strut

Flow Rules



α < 45° – can form & cross initial crack



..... – aggregate interlock!!

5) Detailing must be well done so that no <u>local</u> failure will occur

↳ you may get failure at lower loads, not a shear failure

when comparing to test results, discard poorly detailed member results that might be premature failure

Especially important -- well detailed stirrups

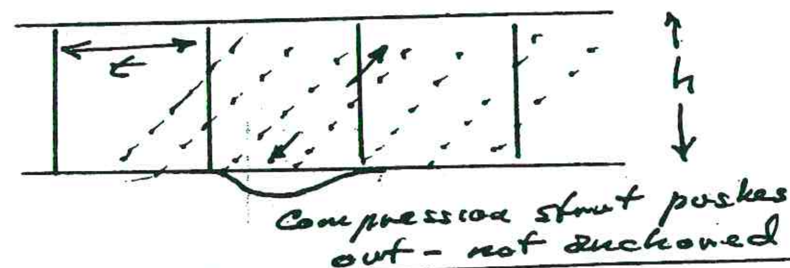
<u>Stirrup Spacing</u>

$t \to h$

$t > h$

As shear increases, amount of stirrups must increase to anchor struts

// ACI ch. 11



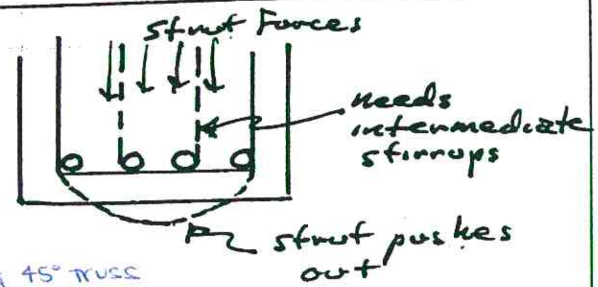
<u>Stirrup Transverse Spacing</u>
- wider beams

$V_n < V_c/2 = \sqrt{f'_c}\,b_w d$ - no stirrups

$\frac{V_c}{2} < V_n < (2\sqrt{f'_c} + 50) b_w d$ - minimum reinf. $s \le d/2$

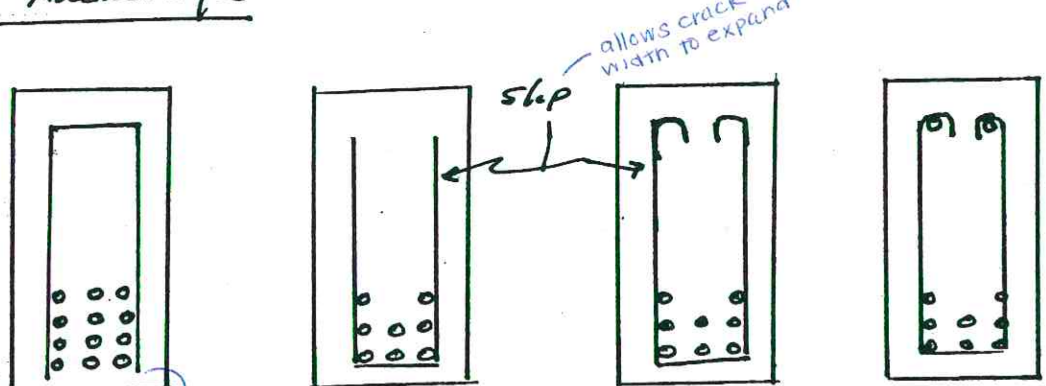
... $< V_n < 6\sqrt{f'_c}\,b_w d$ - size reinforcing using 45° truss $s \le d/2$

$6\sqrt{f'_c}\,b_w d < V_n < 10\sqrt{f'_c}\,b_w d$ - $s \le d/4$



<u>Stirrup Anchorage</u>

Push down hairpin Ugh!



bad poor marginal Best

Other detailing problems:

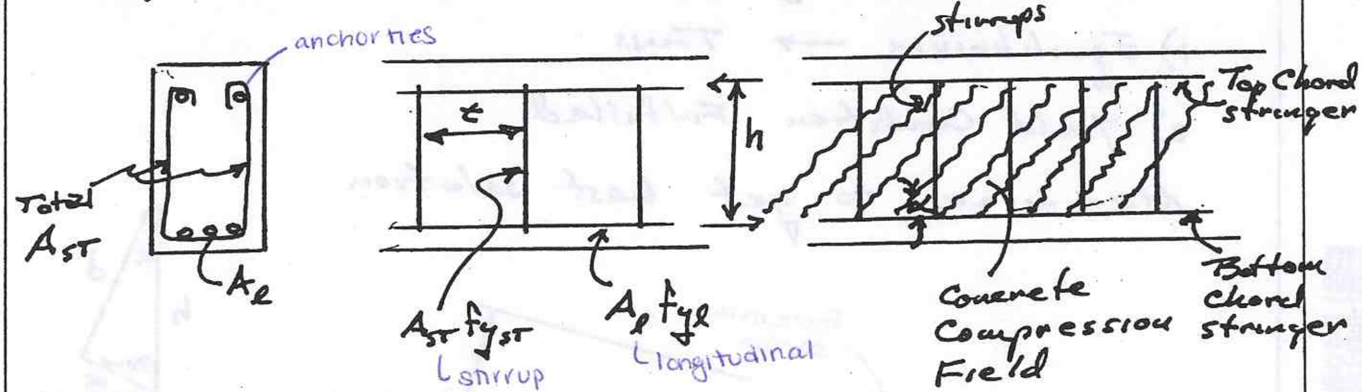
Inadequate Reinforcement Anchorage on Longitudinal Bars or strands

Inadequate Cover

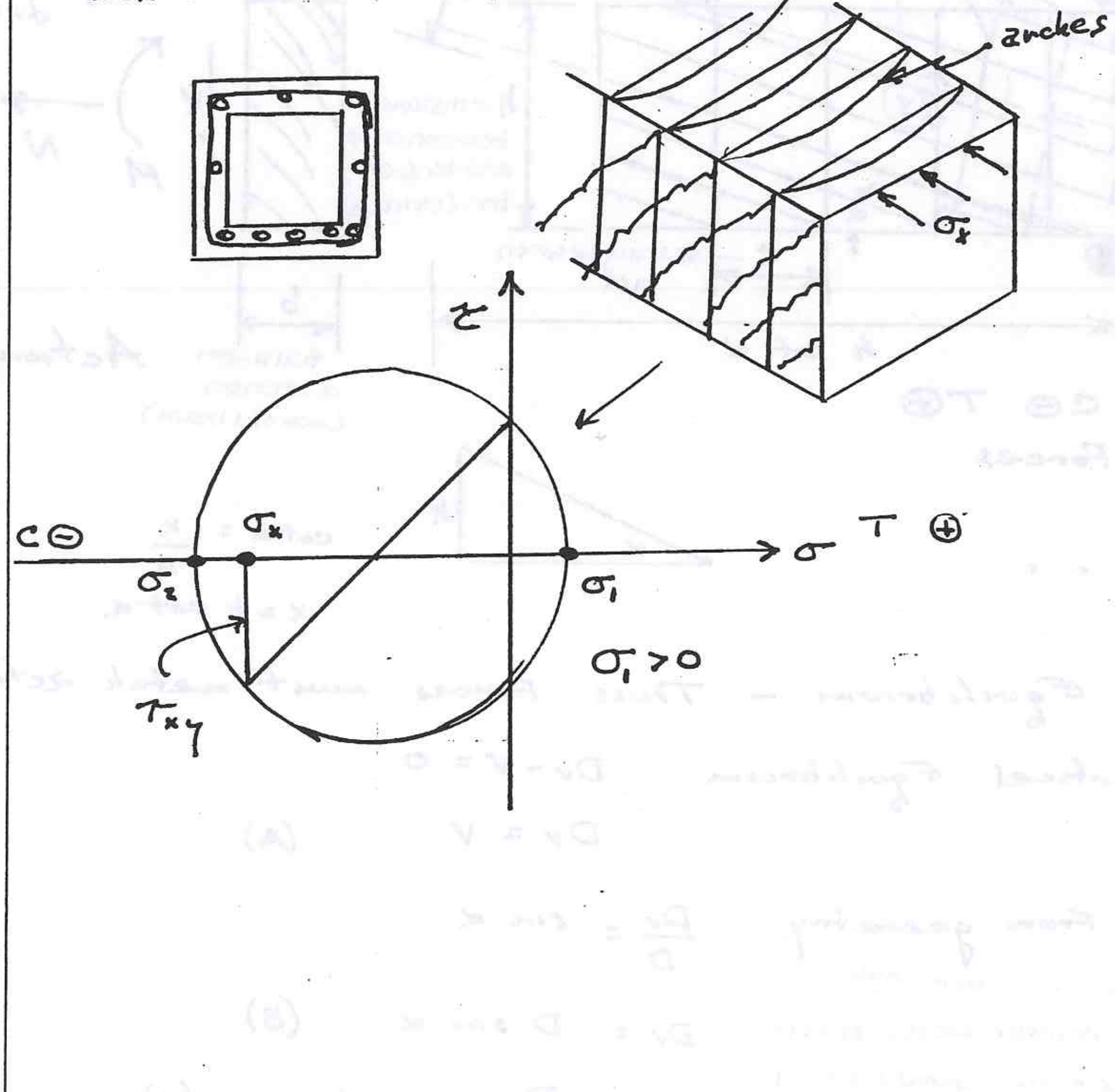
Congestion causing incomplete consolidation

↳ reduced by SCC

PHYSICAL MODEL - TRUSS MODEL



Box Section - Space Truss Model

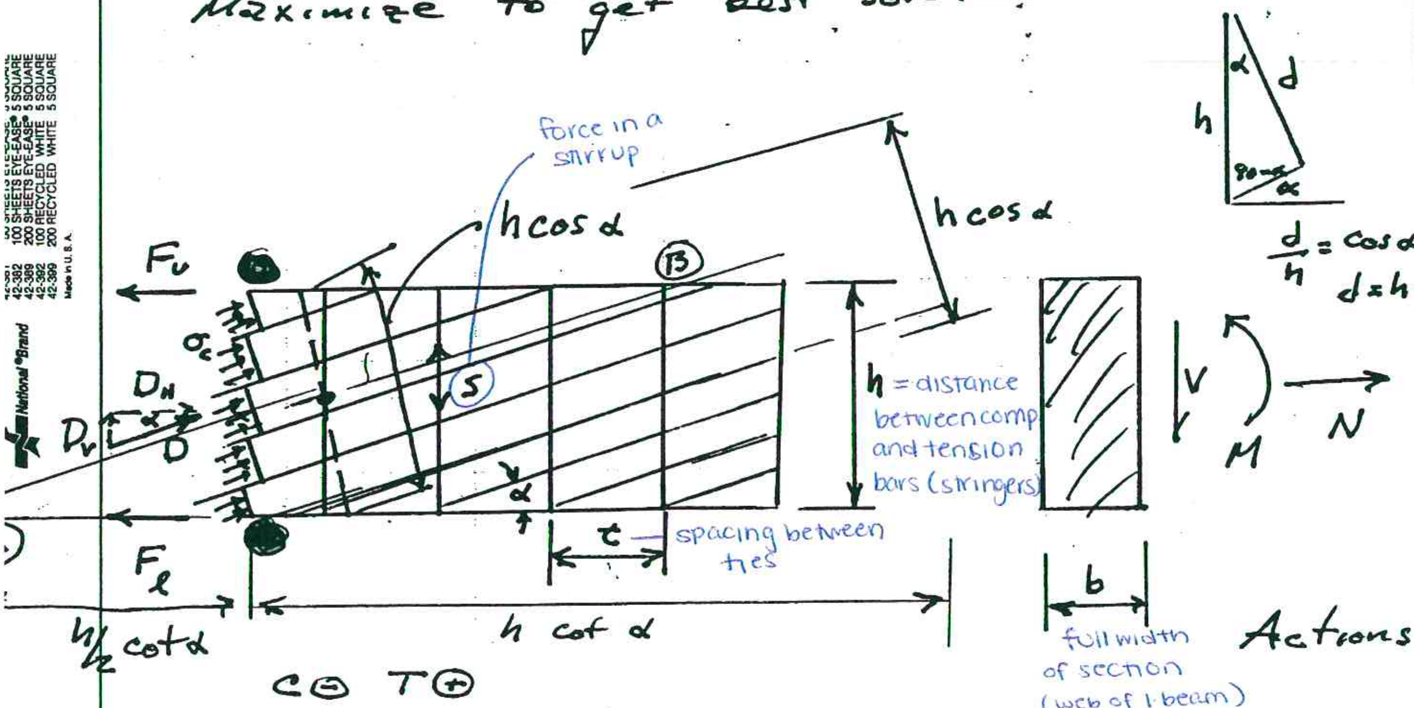


Lower Bound Equilibrium solution

1) Equilibrium → Truss

2) Yield Condition Fulfilled

Maximize to get best solution

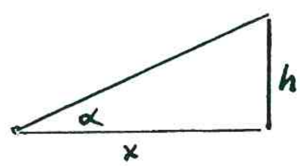


$\frac{d}{h} = \cos\alpha$

$d = h\cos\alpha$

C ⊖ T ⊕

Truss Forces



$\cot\alpha = \frac{x}{h}$

$x = h\cot\alpha$

From Equilibrium – Truss Forces must match actions

Vertical Equilibrium $\quad D_v - V = 0$

$D_v = V \quad$ (A)

From geometry $\quad \frac{D_v}{D} = \sin\alpha$

* α is a variable angle that the designer chooses. Affects the ratio of longitudinal/vertical steel in beam

$D_v = D\sin\alpha \quad$ (B)

$D = \frac{D_v}{\sin\alpha} = \frac{V}{\sin\alpha} \quad$ (1)

Concrete $\quad -D = \sigma_c\, b\, h\cos\alpha \quad$ σ_c is assumed compression

$\sigma_c = \frac{-D}{bh\cos\alpha} = -\frac{V}{bh}\frac{1}{\sin\alpha\cos\alpha} \quad$ (2) $\quad$ Compression is ⊖

Stringers

$$\Sigma M_{(A)} = 0 \;\circlearrowright +$$

shear at section — axial force

$$F_U \cdot h + M - V \frac{h}{2} \cot\alpha - N \frac{h}{2} = 0$$

moment at section

$$F_U = -\frac{M}{h} + \frac{V}{2} \cot\alpha + \frac{N}{2} \qquad (3)$$

$$\Sigma M_{(B)} = 0 \;\circlearrowright +$$

$$-F_L \cdot h + M + V \frac{h}{2} \cot\alpha + N \frac{h}{2} = 0$$

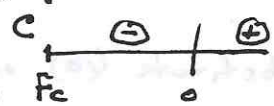
$$F_L = +\frac{M}{h} + \frac{V}{2} \cot\alpha + \frac{N}{2} \qquad (4)$$

Stirrup $\quad V = n\,S \quad$ where n = no. of stirrups $= \dfrac{h \cot\alpha}{t}$

$$V = \frac{h \cot\alpha}{t} S$$

$$S = \frac{V t}{h \cot\alpha} = \frac{V t}{h} \tan\alpha \qquad (5)$$

In order to be statically admissible : to meet yield conditions



$$\left.\begin{array}{ll} \sigma_c \geq -f_c & \text{(must have less compression than } f_c\text{)} \\ \text{upper stringer } U \quad F_U \leq F_{yU} = A_U \, \sigma_{yU} & \\ \text{lower stringer } l \quad F_l \leq F_{yl} = A_l \, \sigma_{yl} & \\ S \leq S_y = A_s \, \sigma_{ys} & \end{array}\right\} \quad (6)$$

Note: $A_{l(x)} \rightarrow F_{yl(x)}$ yield Force can vary along x axis as reinforcement varies

For Combined shear & Bending Case -- no thrust

$N = 0$ -- Only 2 parameters

To see effect of N - Note EQ 3, 4, 5 - Simple Addition

Affects F_l only

Ultimate Load if $F_l = F_{yl}$ (7)

$s = s_y$

$M \to M_p$; $V = V_p$ $N = 0$ (above)

Rewrite (4)

yield of lower stringer

$$F_{yl} = \frac{M_p}{h} + \frac{V_p \cot\alpha}{2} = \frac{M_p}{h} + \frac{V_p}{2\tan\alpha} \quad (8)$$

Rewrite (5)

yield of stirrups

$$S_y = V_p \frac{t}{h} \tan\alpha \quad (9)$$

3 Unknowns M_p, V_p, α -- 2 Equations

Rewrite (9)

$$\tan\alpha = \frac{S_y\ h}{V_p\ t} \quad (10)$$

Substitute (10) in (8)

$$F_{yl} = \frac{M_p}{h} + \frac{V_p}{2}\frac{V_p\ t}{S_y\ h} = \frac{M_p}{h} + \frac{V_p^2\ t}{2 S_y h} \quad (8A)$$

Define "Plastic Moment"

$$V_p = 0 \quad M_{po} = F_{yl}\, h \quad (11) \quad \text{(From 8A)}$$

Define "plastic shear Force"

$$M_p = 0 \quad V_{po} = \sqrt{2 F_{yl} S_y \frac{h}{t}} \quad (12) \quad \text{(From 8A)}$$

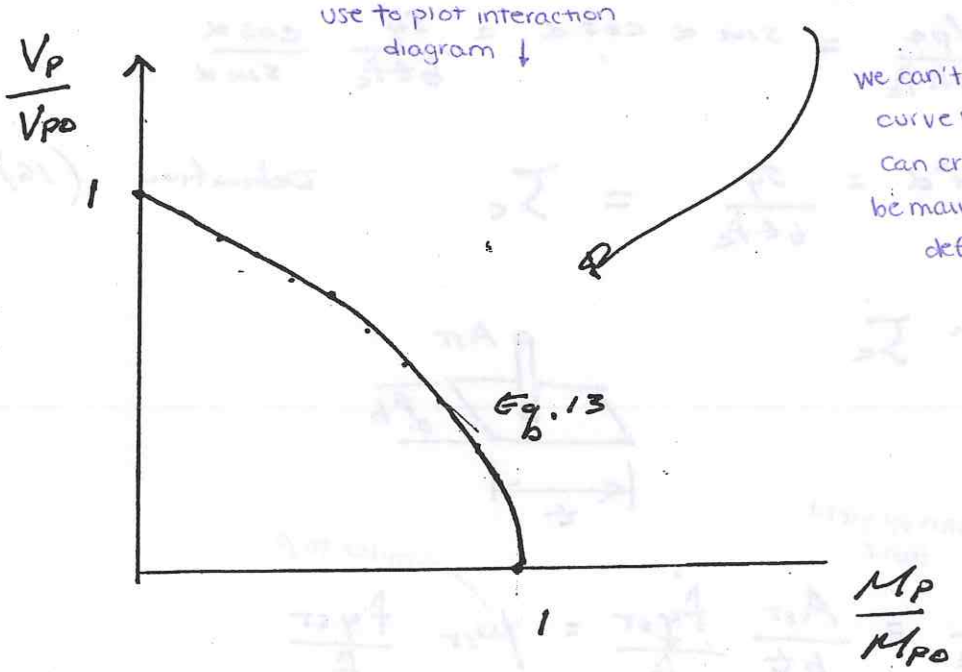
$$V_{po}^2 = 2 F_{yl} S_y \frac{h}{t} \quad (12A)$$

Divide Eq (8A) by F_{yl}

$$\frac{F_{yl}}{F_{yl}} = \underbrace{\frac{M_p}{h F_{yl}}}_{M_{po}} + \underbrace{\frac{V_p^2 t}{2 s_y h F_{yl}}}_{V_{po}^2}$$ (Recalling definitions of M_{po} and V_{po})

$$1 = \frac{M_p}{M_{po}} + \frac{V_p^2}{V_{po}^2} \quad (13)$$ Interaction Equation

use to plot interaction diagram ↓



we can't use the whole curve because the strut can crush (f'_c cannot be maintained through deformation)

Limit imposed by maximum concrete stress

If F_{yl} is increased, M_{po} and V_{po} increase (See Definitions)

A Limit is reached when concrete stress $\sigma_c = -f_c$

This brings on compression failure (in strut)

From Eq (2) with $V = V_p$

$$\sigma_c = -\frac{V_p}{bh}\frac{1}{\sin\alpha\cos\alpha} = -f_c$$ (from Eq. 2)

or letting this failure level be V_{pc}

$$\frac{V_{pc}}{b h f_c} = \sin\alpha\cos\alpha \qquad (14)$$

From (9) $V_{pc} = S_y \frac{h}{t} \frac{1}{\tan\alpha} = S_y \frac{h}{t} \cot\alpha$

Divide by $b h f_c$

$$\frac{V_{pc}}{b h f_c} = \frac{S_y h}{t} \frac{\cot\alpha}{b h f_c} = \frac{S_y}{b t f_c} \cot\alpha \qquad (15)$$

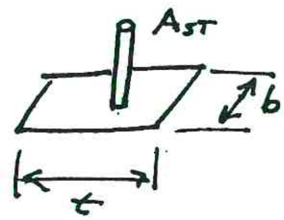
Compare (14) and (15)

$$\frac{V_{pc}}{b h f_c} = \sin\alpha \cos\alpha = \frac{S_y}{b t f_c} \frac{\cos\alpha}{\sin\alpha}$$

So

$$\sin^2\alpha = \frac{S_y}{b t f_c} = \zeta_c \qquad \text{Definition} \qquad (16)$$

Parameter ζ_c



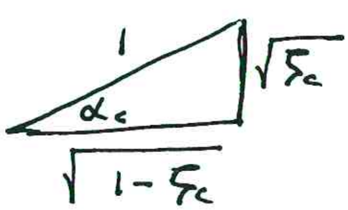
$$\underbrace{\frac{\overset{\text{stirrup yield force}}{S_y}}{b\, t\, f_c}} = \frac{A_{ST}}{b t} \frac{f_{yST}}{f_c} = \mu_{ST} \frac{f_{yST}}{f_c}$$

b: specimen width; t: stirrup spacing; f_c: effective strength

μ_{ST}: similar to ρ

stirrup reinforcement ratio or stirrup index on horiz. plane

From B. Thürlimann, IABSE Colloquium, Copenhagen, May 1979



$$\tan\alpha_c = \frac{\sqrt{\zeta_c}}{\sqrt{1-\zeta_c}} \qquad \cot\alpha_c = \sqrt{\frac{1-\zeta_c}{\zeta_c}}$$

$$\tan^2\alpha_c = \frac{\zeta_c}{1-\zeta_c} = \lambda_c \qquad \text{(Definition)}$$

Recalling (15)

$$\frac{V_{pc}}{bhf_c} = \frac{S_y}{btf_c} \cot\alpha$$

but from (16) $\frac{S_y}{btf_c} = \xi_c$, From Thurlimann $\cot\alpha_c = \sqrt{\frac{1-\xi_c}{\xi_c}}$

so $$\frac{V_{pc}}{bhf_c} = \xi_c\sqrt{\frac{1-\xi_c}{\xi_c}} = \sqrt{\frac{\xi_c^2(1-\xi_c)}{\xi_c}} = \sqrt{\xi_c(1-\xi_c)} \qquad (17)$$

Max Value $$\frac{\partial}{\partial\xi_c}\left(\frac{V_{pc}}{bhf_c}\right) = 0 = \frac{1}{2}\left[\xi_c(1-\xi_c)\right]^{-\frac{1}{2}} \cdot \underbrace{\left[1-2\xi_c\right]}_{\text{Must be } 0}$$

$$1 - 2\xi_c = 0$$

$$\xi_c = 1/2$$

From (16) $\xi_c = \sin^2\alpha_c = 1/2$

$\therefore \alpha_c = \pi/4 = 45°$ This gives max. value

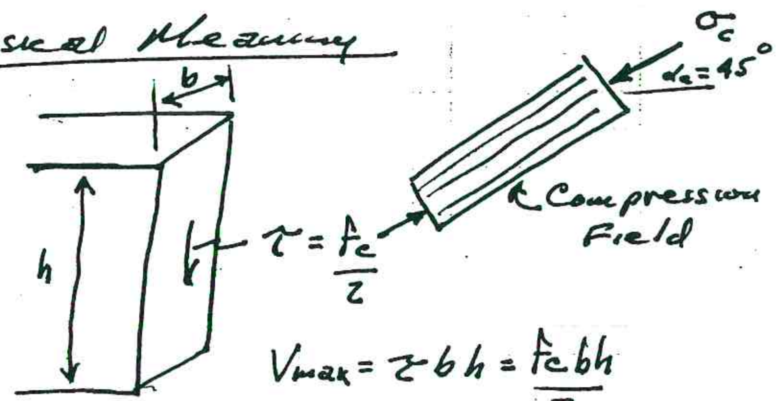
From 17 $$\left(\frac{V_{pc}}{bhf_c}\right)_{max} = \sqrt{\xi_c(1-\xi_c)} = \sqrt{1/2(1-1/2)} = \frac{1}{2}$$

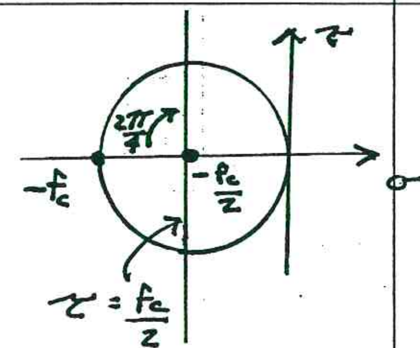
$$(V_{pc})_{max} = 1/2\, bhf_c$$ for $\alpha = 45°$

Abbreviation $(V_{pc})_{max} = V_{max} = 1/2\, bh f_c$ (18)

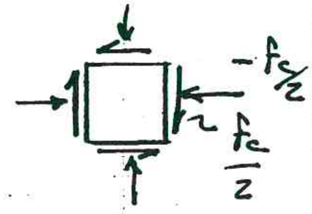
or $2V_{max} = bhf_c$

Physical Meaning





$$V_{max} = \tau b h = \frac{f_c b h}{2}$$



Eq (17) $$\frac{V_{pc}}{b h f_c'} = \sqrt{\xi_c (1 - \xi_c)}$$

Eq (18) $$V_{max} = \tfrac{1}{2}\, b h f_c$$

so $$\frac{V_{max}}{b h f_c} = \tfrac{1}{2} \qquad (18A)$$

Combining (17) & (18A)

$$\therefore \quad \frac{V_{pc}}{V_{max}} = 2\sqrt{\xi_c (1 - \xi_c)} \qquad (19)$$

From (16) $$\xi_c = \frac{s_y}{b t f_c} = \frac{A_{ST}}{b t} \frac{f_{yST}}{f_c} = \mu_{ST} \frac{f_{yST}}{f_c}$$

Also from (16) $$\sin^2 \alpha_c = \xi_c \longrightarrow \alpha_c = \sin^{-1}\sqrt{\xi_c}$$

Limit on Interaction Eq (13)

If $V_p = V_{pc}$, concrete reaches its limiting strength $\sigma_c = -f_c$ --- Hence, interaction ceases -- concrete is at crushing stage and this limits the further yielding of the ductile lower stringer

Eq (13) $$\frac{V_p^2}{V_{po}^2} + \frac{M_p}{M_{po}} = 1$$

$$\frac{V_p}{V_{po}} \longrightarrow \frac{V_{pc}}{V_{po}} = \left(\begin{matrix}\text{from}\\ \text{Eq 19}\end{matrix}\right) = \frac{2V_{max}\sqrt{\xi_c(1-\xi_c)}}{V_{po}} = \sqrt{\frac{4V_{max}^2\,\xi_c(1-\xi_c)}{V_{po}^2}}$$

Using Eq(18), $V_{max} = \frac{1}{2} b h f_c$ Using Eq (12) $V_{po}^2 = 2 F_{yl} S_y \frac{h}{t}$

so $$\frac{V_{pc}}{V_{po}} = \sqrt{\frac{\cancel{4} V_{max} \frac{1}{\cancel{2}} b h f_c}{\cancel{2} F_{yl} S_y \frac{h}{t}}\,\xi_c(1-\xi_c)}$$

$$= \sqrt{\frac{V_{max}\; b\,t\,f_c\,h}{F_{yl}\,S_y\,h}\,\xi_c(1-\xi_c)}$$

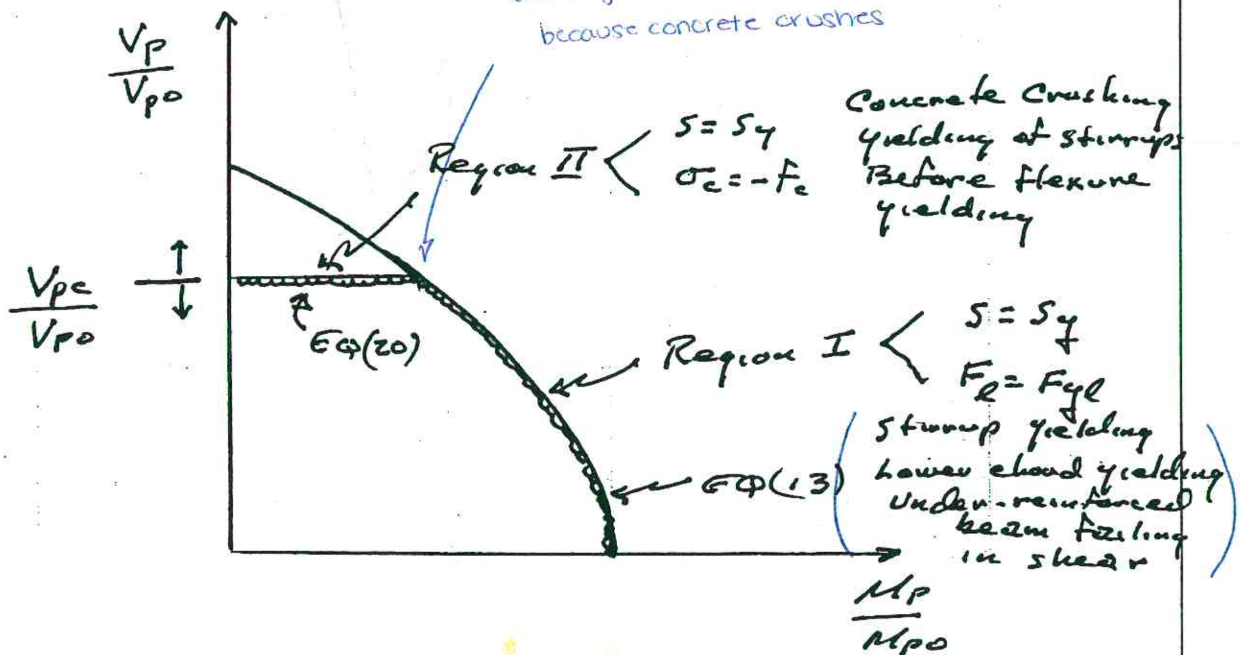
$\frac{1}{2} b h f_c$

$\xi_c = \frac{S_y}{b t f_c}$

but Eq(11) $M_{po} = F_{yl}\, h$, Eq (16) $\frac{S_y}{b t f_c} = \xi_c$

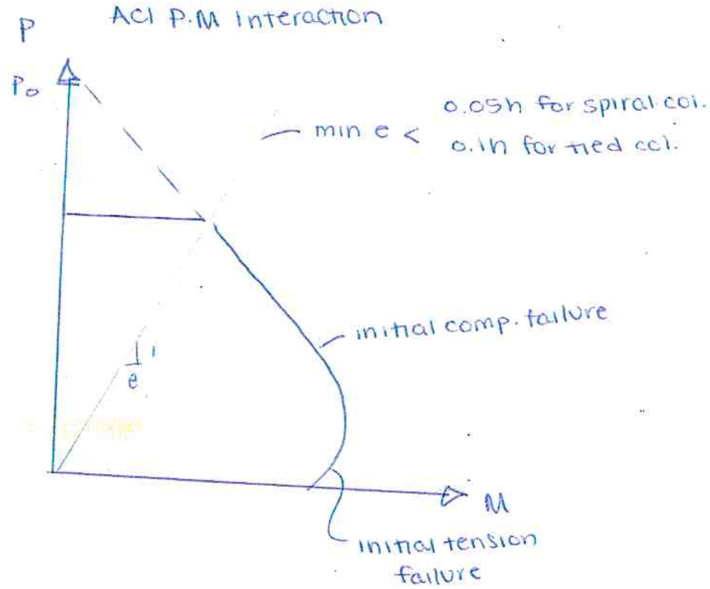
$$\frac{V_{pc}}{V_{po}} = \sqrt{\frac{V_{max}}{M_{po}}\frac{h}{\cancel{\xi_c}}\cancel{\xi_c}(1-\xi_c)} = \sqrt{\frac{V_{max}\,h\,(1-\xi_c)}{M_{po}}} \quad (20)$$

This gives us the limit of $V_p = V_{pc}$ where concrete crushing controls -- no further ductility can't get to lower stringer yield because concrete crushes



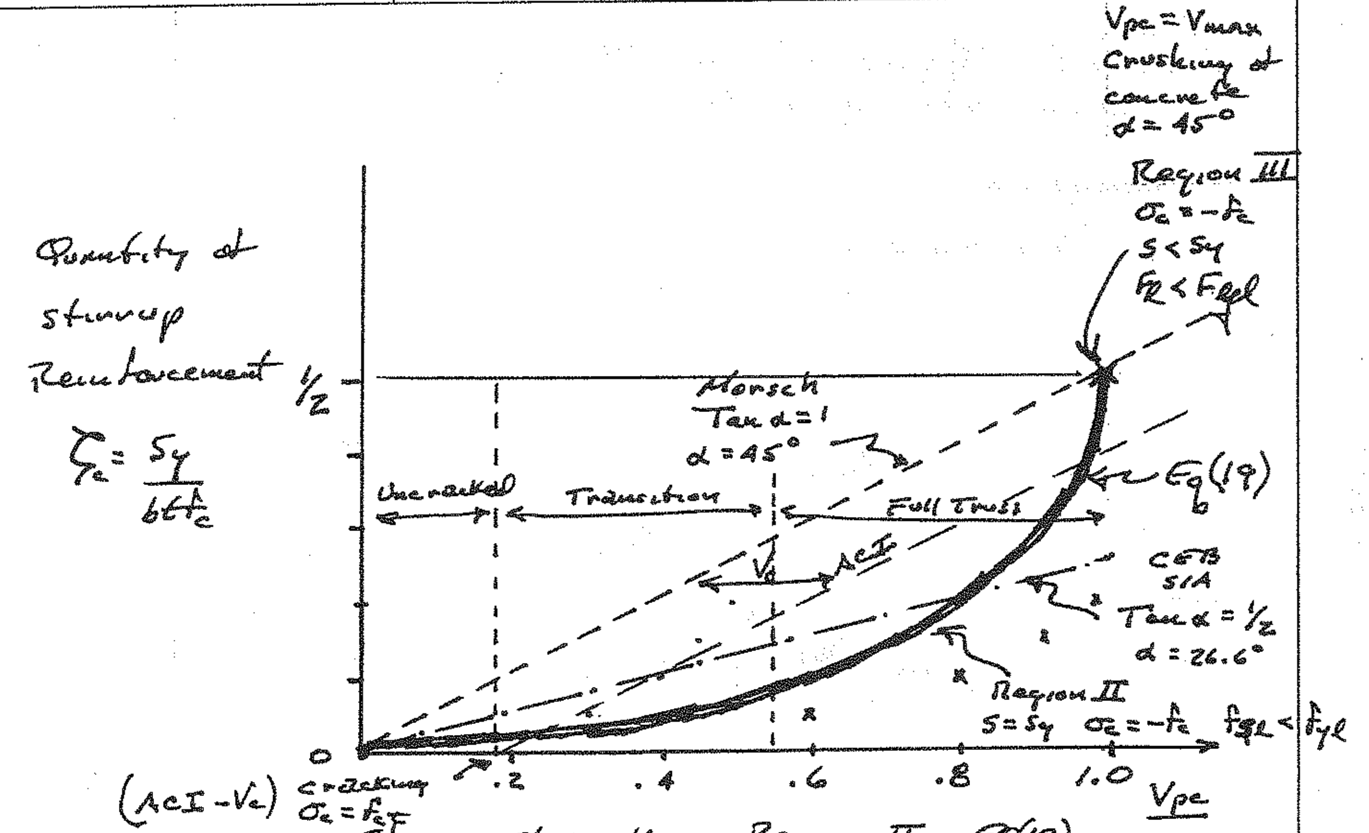
Interaction Diagram - Bending + shear Eq(13)
with compression Limit Eq(20)

ACI P-M Interaction



$P_o = 0.85 f'_c A_c + f_y A_s$

In ACI the maximum cutoff is $10\sqrt{f'_c}\, b_w d$
$2\sqrt{f'_c} - V_c$
$8\sqrt{f'_c} - V_s$
— similar to V_{pc}/V_{pc}



Shear strength in Region II, Eq (19)

Eq (19) No yielding of longitudinal reinf.

$$\frac{V_{pc}}{V_{max}} = 2\sqrt{\psi_c(1-\psi_c)}$$

V_{pc} – shear when concrete crushes

V_{max} – maximum possible value of V_{pc}

$$\psi_c = \frac{S_y}{b t f_c} = \frac{A_{st}}{b t} \cdot \frac{F_{yst}}{f_c}$$

$V_{max} = \frac{1}{2} b h f_c$ (18)

From Eq (15)
$$\frac{V_{pc}}{b h f_c} = \frac{S_y}{b t f_c} \cot\alpha$$

From Eq (18)
$$V_{max} = \tfrac{1}{2} b h f_c$$

$$\therefore \quad \frac{V_{pc}}{V_{max}} = \frac{S_y \cot\alpha}{b t f_c} \frac{b h f_c}{\frac{1}{2} b h f_c} = \frac{S_y}{b t f_c} \frac{2}{\tan\alpha} = \psi_c \frac{2}{\tan\alpha}$$

$$\psi_c = \frac{V_{pc}}{V_{max}} \frac{\tan\alpha}{2}$$ → See dashed lines

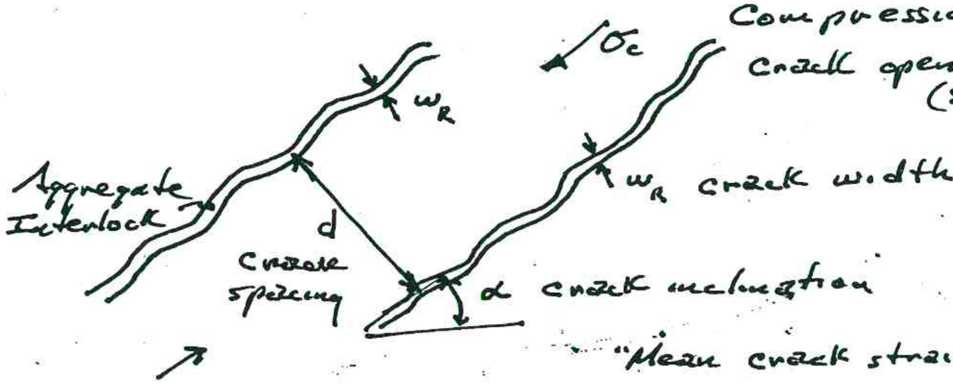
Tan α = 1 - - - - -

Tan α = 1/2 —·—·—·—

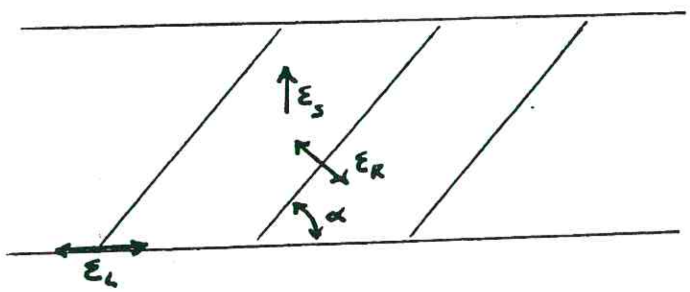
Critical Appraisal of This Approach
using the upper bound approach

Kinematical considerations,

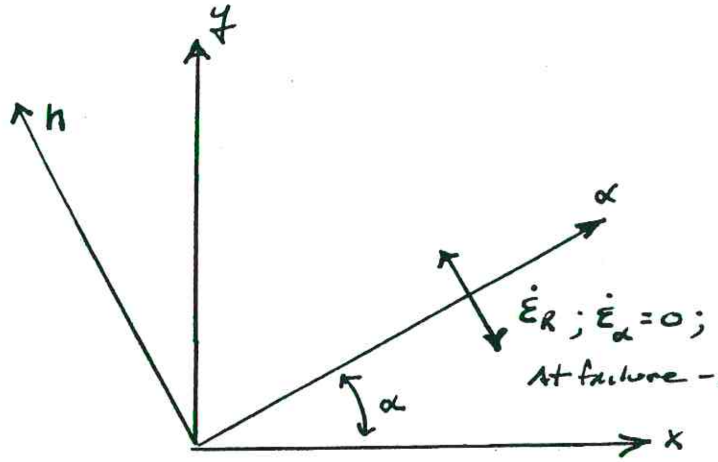
Deformations
Strain Rates
Relative Strain Rates



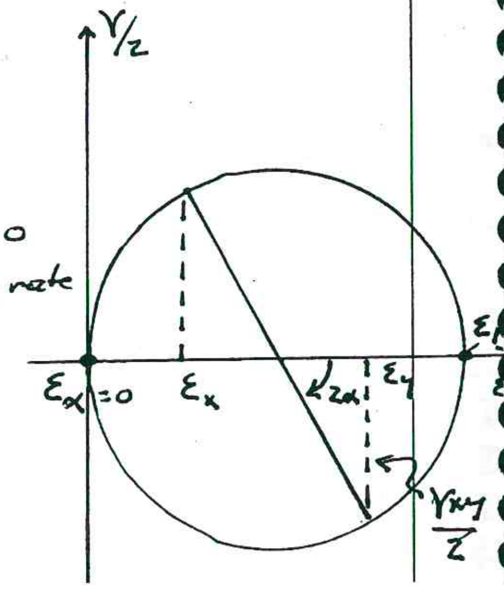
"Mean crack strain" $\varepsilon_R = \frac{w_R}{d}$



* we are constantly looking for a solution to the shear problem



$\dot{\varepsilon}_R$; $\dot{\varepsilon}_\alpha = 0$; $\dot{\gamma}_{R\alpha} = 0$
At failure - strain rate is 0

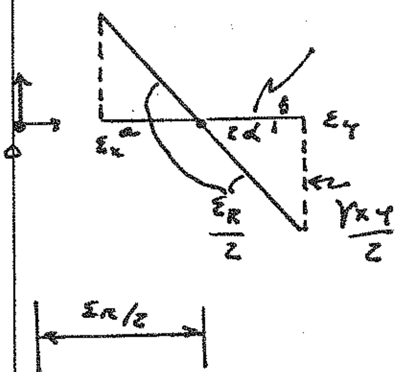


$\therefore$ radius $= \frac{\varepsilon_R}{2}$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

$$\sin^2\alpha = 1 - \cos^2\alpha$$



$$\varepsilon_x = \varepsilon_{R/2} - a \qquad \frac{a}{\varepsilon_R/2} = \cos 2\alpha$$

$$\varepsilon_x = \varepsilon_{R/2} - \frac{\varepsilon_R}{2}\cos 2\alpha = \frac{\varepsilon_R}{2}(1-\cos 2\alpha)$$

$$= \frac{\varepsilon_R}{2}(1-\cos^2\alpha + \sin^2\alpha) = \varepsilon_R \sin^2\alpha = \varepsilon_\ell$$

(longitudinal strain)

$$\varepsilon_y = \frac{\varepsilon_R}{2} + b \qquad \frac{b}{\varepsilon_{R/2}} = \cos 2\alpha$$

$$\varepsilon_y = \frac{\varepsilon_R}{2} + \frac{\varepsilon_R}{2}\cos 2\alpha = \frac{\varepsilon_R}{2}(1+\cos 2\alpha)$$

$$= \frac{\varepsilon_R}{2}(1+\cos^2\alpha - \sin^2\alpha) = \frac{\varepsilon_R}{2}(2\cos^2\alpha) = \varepsilon_R\cos^2\alpha$$

$$= \varepsilon_R\cos^2\alpha = \varepsilon_s \quad \text{(stirrup strain)}$$

$$\frac{\gamma_{xy}}{2} = \frac{\varepsilon_R}{2}\sin 2\alpha = \frac{\varepsilon_R}{2}\,2\sin\alpha\cos\alpha$$

$$= \varepsilon_R \sin\alpha\cos\alpha$$

Thus at collapse conditions:

Stirrup $\varepsilon_s = \varepsilon_R\cos^2\alpha$

Stringer $\varepsilon_\ell = \varepsilon_R\sin^2\alpha$

$$\gamma_{xy} = 2\varepsilon_R\sin\alpha\cos\alpha$$

Theory of Plasticity Conditions

Rigid-Plastic Theory $\longrightarrow$ deals only with strain rates

Total strains (ε_{TOT}) up to point of collapse

$$\varepsilon_{TOT} = \varepsilon_{elastic} + \varepsilon_{inelastic}$$

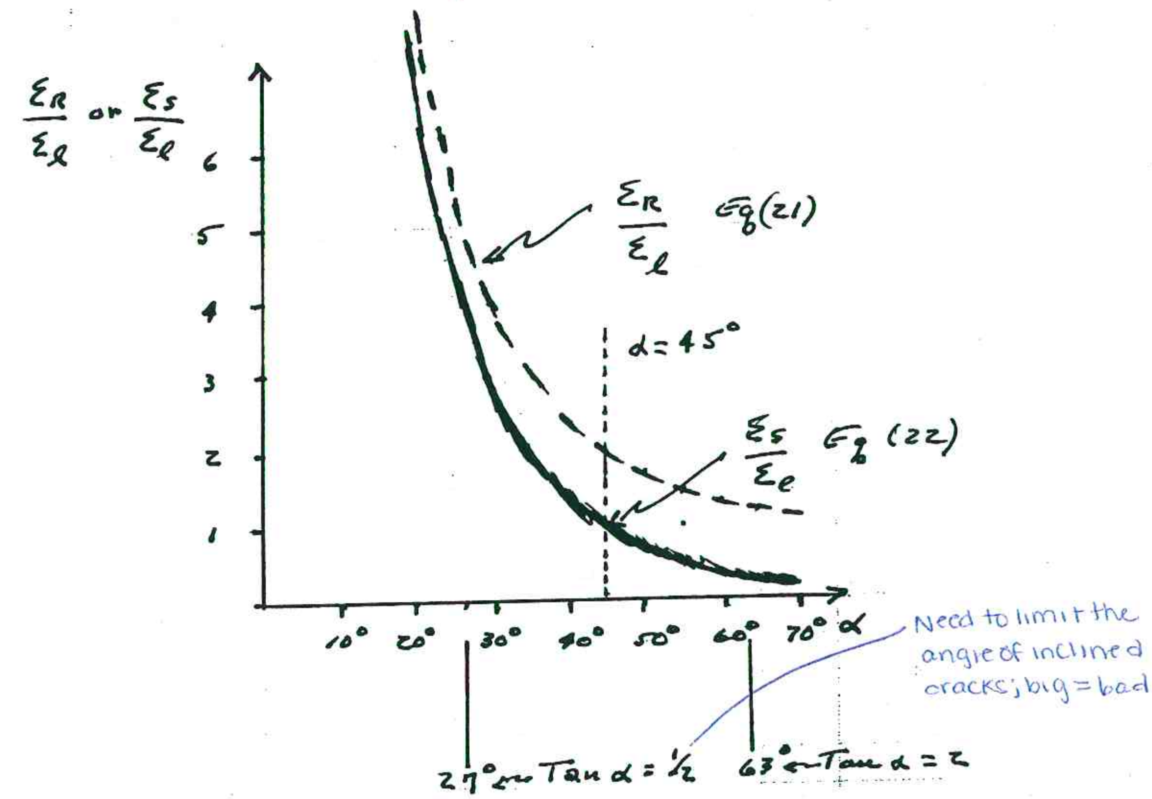
Approximation $\dfrac{\varepsilon_{x\,TOT}}{\varepsilon_{y\,TOT}} \approx \dfrac{\dot\varepsilon_x}{\dot\varepsilon_y}$ Big Assumption

This is a good estimate for previous deformation up to point of collapse

Ratios: $\dfrac{\varepsilon_R}{\varepsilon_\ell} = \dfrac{1}{\sin^2\alpha}$ (21) $\alpha \to 0$ $\dfrac{\varepsilon_R}{\varepsilon_\ell} \to \infty$!

$\dfrac{\varepsilon_s}{\varepsilon_\ell} = \dfrac{\varepsilon_R \cos^2\alpha}{\varepsilon_R \sin^2\alpha} = \cot^2\alpha$ (22) $\alpha \to 0$ $\dfrac{\varepsilon_s}{\varepsilon_\ell} \to \infty$!

Thus there must be some limits on α since strains can't go to infinity



$\varepsilon_\ell \to \varepsilon_y$ as the beam nears flexural failure

as this happens cracks get extremely wide

Probably when $\varepsilon_R \approx 4$ to $5\ \varepsilon_y$ aggregate interlock ceases
Since the new crack and the initial cracks do not always line up, load transfer for σ_c ceases
Failure occurs

Thus there is a practical lower limit on α

Tan α = ½ (or 27°) seems safe from tests

CEB & SIA have used Tan α = 3/5 (or 31°) -- maybe too safe

They often use reciprocal (Tan α = 2 or 63°) for max
but this is unnecessary

From (9) $$V_p = S_y \frac{h}{t} \cot\alpha$$

If $\tan\alpha = 1/2$ at lower α limit

$$V_p = S_y \frac{h}{t} \frac{1}{\tan\alpha} = 2 S_y \frac{h}{t}$$

From (18) when $\sigma_c = -f_c$

$$V_{max} = \tfrac{1}{2} b h f_c$$

$$\therefore \quad \frac{V_p}{V_{max}} = \frac{2 S_y h/t}{\tfrac{1}{2} b h f_c} = \frac{4 S_y}{b t f_c} = 4\xi_c \quad (23)$$

when $\tan\alpha = 1/2$
(see Fig on P 14)

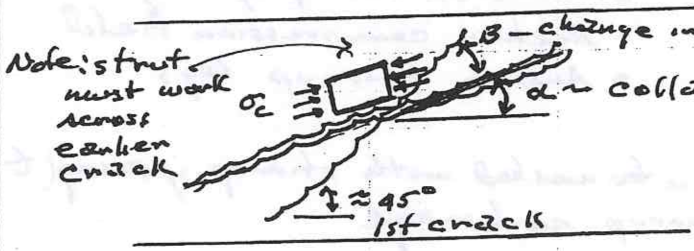
Effective Concrete strength f_c

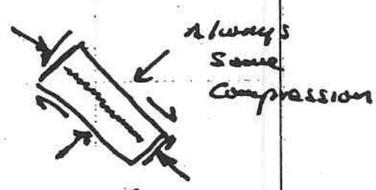
$$\boxed{f_c = \nu_e f_c'}$$

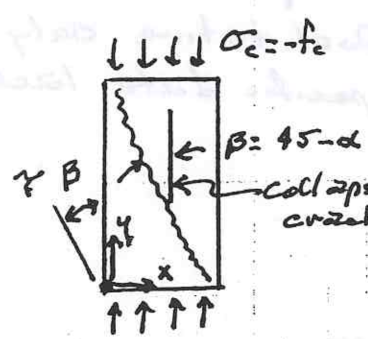
efficiency factor ν_e (nu e)

(Practical Considerations – This is where fudging takes place – this is explanation of rationalizations introduced -- values of ν_e have varied greatly among different authorities)

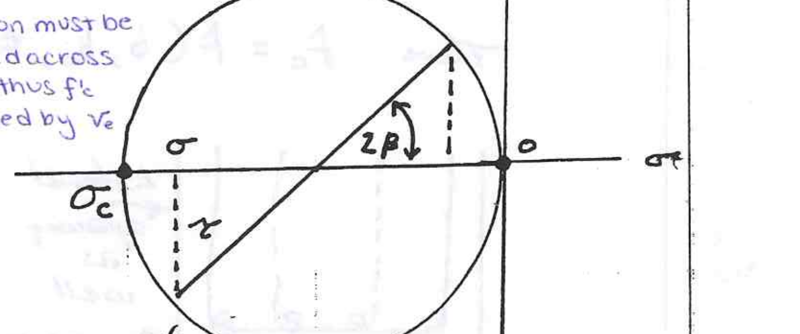
1. Influence of stress-strain history up to failure





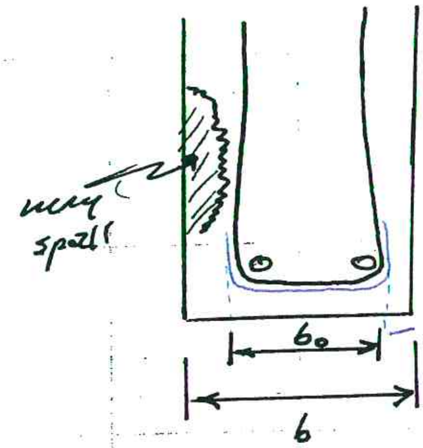


compression must be transferred across a crack; thus f_c' is reduced by ν_e



Thus cracks will diminish the concrete strength contribution

2. Spalling of Concrete Cover



Side face spalling –

should you use b or b_o in strut width

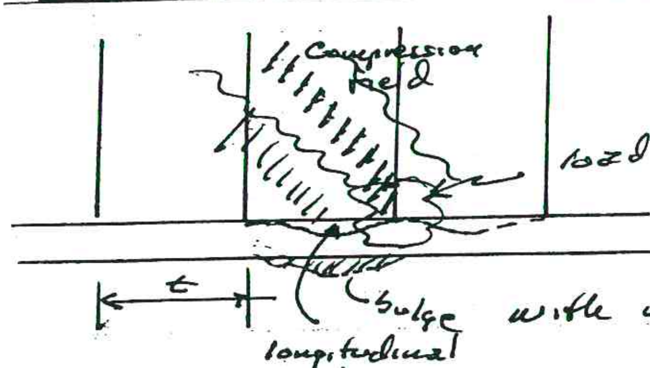
You can do it by either using differing values of b and b_o or by always using b but reducing ν_e when you mean b_o

consider spalling by reducing f_c' further, but use b, not b_o

b is ok for monotonic but b_o is more realistic for cyclic

difference becomes more influential when beam width is smaller; cover stays constant, ratio changes

Bottom face spalling –



load concentration

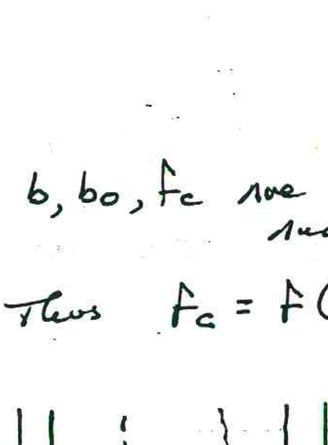
bulge with wide stirrup spacing –

Requires good detailing –

- Minimize stirrup spacing
- Anchor compression field
- Anchor stirrup legs

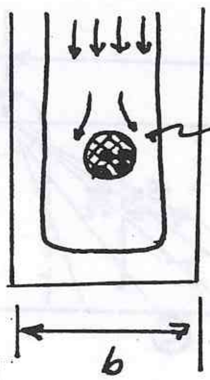
b, b_o, f_c are all intermeshed with stirrup spacing (t) and stirrup anchorage

Thus $f_c = f(b, b_o, t)$ Qualitative only specific data lacking



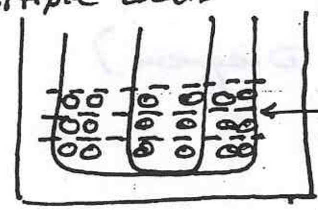
Requires more stirrups

3. Prestressing Ducts in Webs



Poorly grouted – weak f_c'
Well grouted – hard – still 4–5 ksi
web may be 8–10 ksi
grout may be much less, f_c'

multiple ducts



tie together to get compact unit for behavior
ties hold tendons from ripping through sides in tied girders

2 Approaches
1. use $b' < b$
2. use ν_e decreased f_c

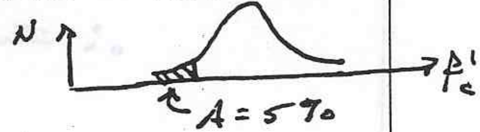
NOT IN CODE; no accounting for weak grout

4. Summary

Basic need for $f_c = \nu_e f_c'$ is a good but conservative value for design

US code handles statistics in a sloppy manner

a) assume f_c' is about 5% fractile -- not bad



Eurocode uses real stats, Gaussian distributions, fractiles...

Thürlimann recommends:

$$f_c = 0.5 \rightarrow 0.6\ f_c'\ \text{cube}$$

$$f_c = 0.6 \rightarrow 0.7\ f_c'\ \text{cylinder}$$

Normal spacing t — close spacing t

Normal spacing – $t_{max} = h/2$ but not more than 30 cm or 12"

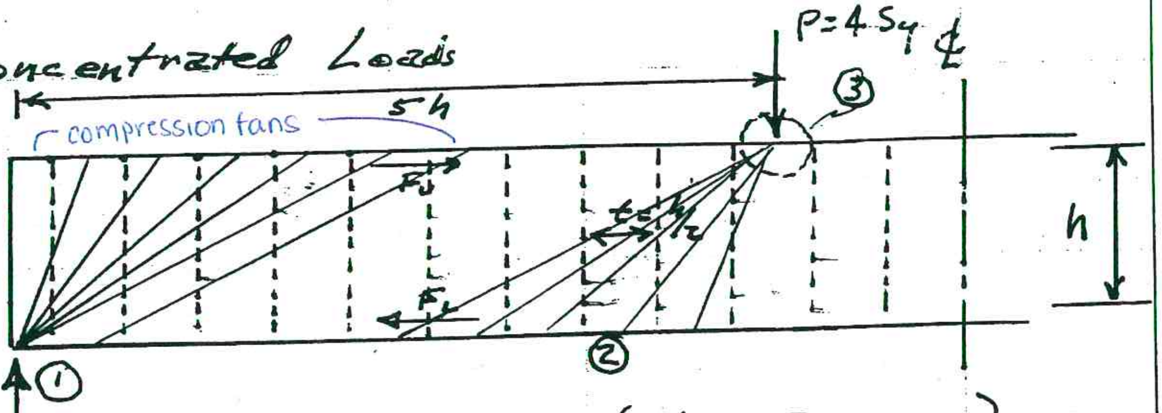
Close spacing – $t_{max} = h/3$ but not more than 20 cm or 8"

(ACI uses 2 values of maximum spacing – $d/2$ and $d/4$ but no absolute dimensions)

for very deep beams, $d/2$, $d/4$ are very large (~5 ft, e.g.)
NOT GOOD; doesn't contain the bottom of the struts

Special Problems :

1) Concentrated Loads



$R = 4 S_y$ $\quad V_p = 4 S_y$ (shear Diagram)

Stirrups : A_{ST} , $S_y = A_{ST} f_{yST}$

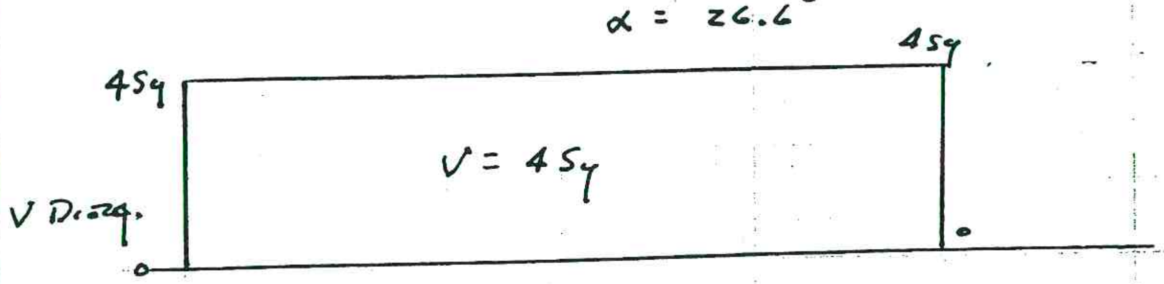
Spacing : $t = h/2$

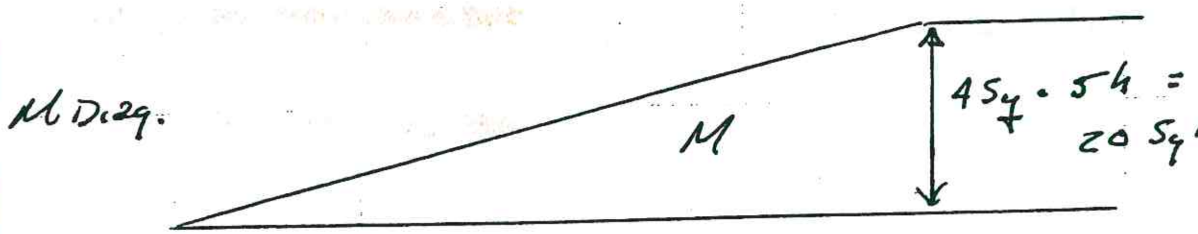
From EQ (9) $\quad S_y = V_p \dfrac{t}{h} \tan\alpha$

$$\tan\alpha = \frac{S_y}{V_p}\cdot\frac{h}{t} = \frac{\cancel{S_y}\ \cancel{h}}{4\cancel{S_y}\ \cancel{h}/2} = \frac{1}{2}$$

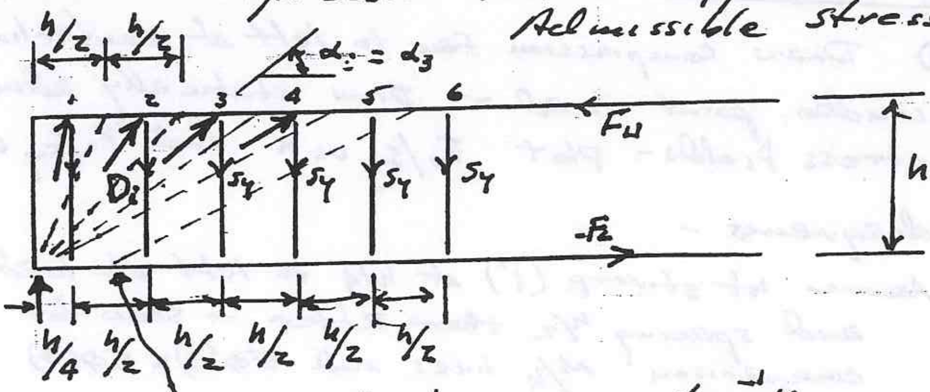
$\therefore$ Inclination $\alpha = \tan^{-1} 1/2$

$\alpha = 26.6°$





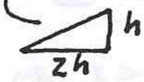
Detail ① Compression Fan at Support – Statically Admissible Stress Field



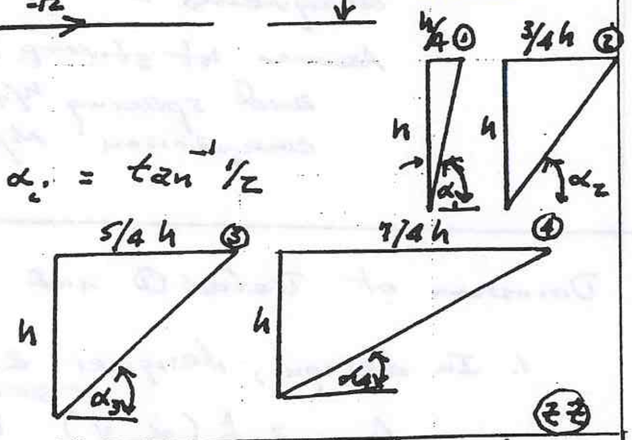
1st point where $\alpha_i = \tan^{-1} 1/2$

Fan Region 2h

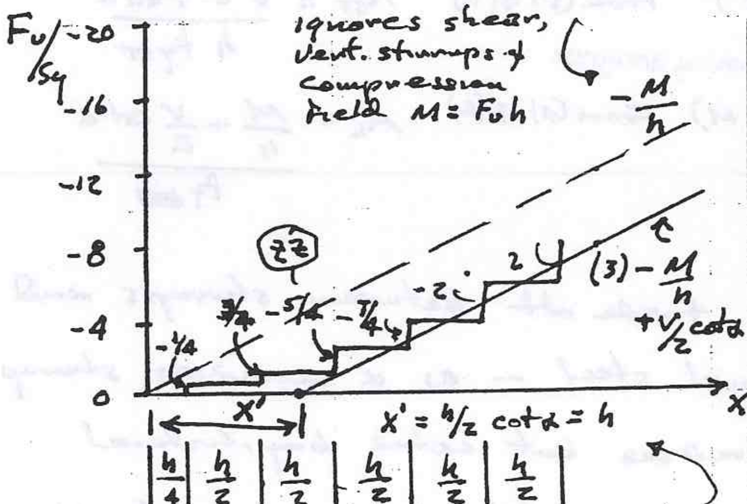
$\tan \alpha = 1/2$



strut shape changes along length of member



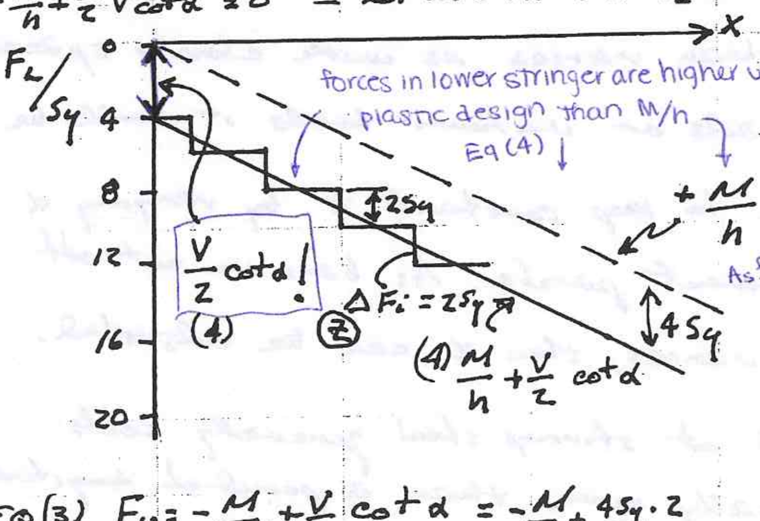
ZZ



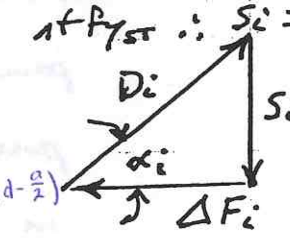
$x' = h/2 \cot\alpha = h$

Node Pt.	$\tan \alpha_i$	Stirrup S_i	D_i	ΔF_i Stringer
1	4	S_y	$1.03 S_y$	$1/4\, S_y$
2	4/3	S_y	$1.25 S_y$	$3/4\, S_y$
3	4/5	S_y	$1.60 S_y$	$5/4\, S_y$
4	4/7	S_y	$2.02 S_y$	$7/4\, S_y$
5, 6, …	1/2	S_y	$2.24 S_y$	$2\, S_y$

$-\frac{M}{h} + \frac{1}{2} V \cot\alpha = 0$ — let $M = Vx' \rightarrow x' = h/2 \cot\alpha$



At failure stirrups are at f_{yst} ∴ $S_i = S_y$

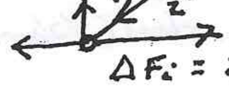


$\sin\alpha_i = \frac{S_i}{D_i}$ ∴ $D_i = \frac{S_i}{\sin\alpha_i}$

$\frac{S_i}{\Delta F_i} = \tan\alpha_i \qquad \Delta F_i = \frac{S_i}{\tan\alpha_i}$ ZZ

EQ (3) $F_U = -\frac{M}{h} + \frac{V}{2}\cot\alpha = -\frac{M}{h} + \frac{4S_y \cdot 2}{2}$

EQ (4) $F_L = +M/h + \frac{V}{2}\cot\alpha = \frac{M}{h} + \frac{4S_y\, 2}{2}$



$\Delta F_i = 2 S_y$ ②

$\frac{M}{h} + \underbrace{\frac{V}{2}\cot\alpha}_{\text{adds to } M/h} = F_L$

Homework Assignment

Detail ② Draw compression fan to left of centerline of load under point load -- Draw statically admissible stress fields - plot F_U/s_y vs x and F_L/s_y vs x diagrams -

Assume 1st stirrup (1') at $h/4$ to left of load point and spacing $h/2$ thereafter -- show for comparison M/h lines and EQ(3) & EQ(4)

Discussion of Detail ① and Detail ②

1. In design, designer assumes an α, can choose t (stirrup spacing)

$$A_{ST} = f(\alpha, V) \quad \text{From (5) \& (6)} \quad A_{ST} = \frac{V\,t\,\tan\alpha}{h\,f_{yST}}$$

shear; chosen by designer

$$A_L = f(\alpha, V, M) \quad \text{From (4) \& (6)} \quad A_L = \frac{\frac{M}{h} + \frac{V}{2}\cot\alpha}{f_{y}}$$

2. Design becomes a trade off between stirrups and longitudinal steel -- as α increases, stirrup steel increases but extra longitudinal steel decreases (if shear is constant). Where shear varies as with closely spaced point loads or uniform loads it would be possible to keep constant t by varying α in different panels. As bars are cut off so A_L varies, then t can be adjusted.

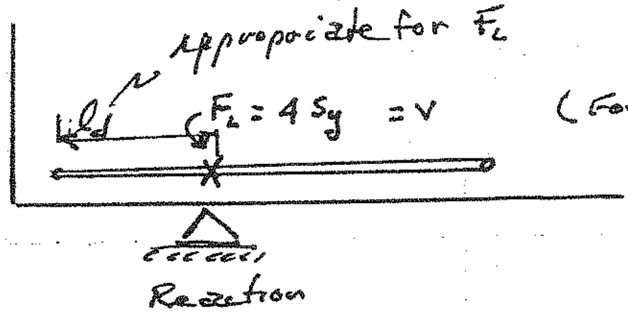
3. Since a pound of stirrup steel generally costs substantially more than a pound of longitudinal steel (labor & fabrication), economy comes with lower α

Discussion of Detail ① and ② cont'd.:

4. ~~For the discrete model shown~~ → ΔF increases in jumps
   However an actual structure is more continuous
   since bond acts between panel points

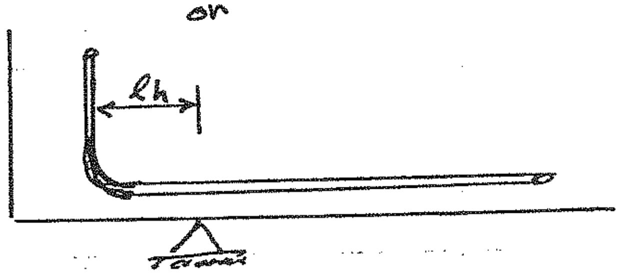
5. Dimensioning the section within $x' = \frac{h}{2}\cot\alpha$ from
   support or concentrated load point --
   Run stirrups up to & past concentrated load or support
   Note: No additional term $\frac{V}{2}\cot\alpha$ at support

   However note: At support the force in the lower or
   tensile stringer is $\frac{V}{2}\cot\alpha$; $F_L = 4 S_y$
   Must have proper end anchorage



(For this example)

or



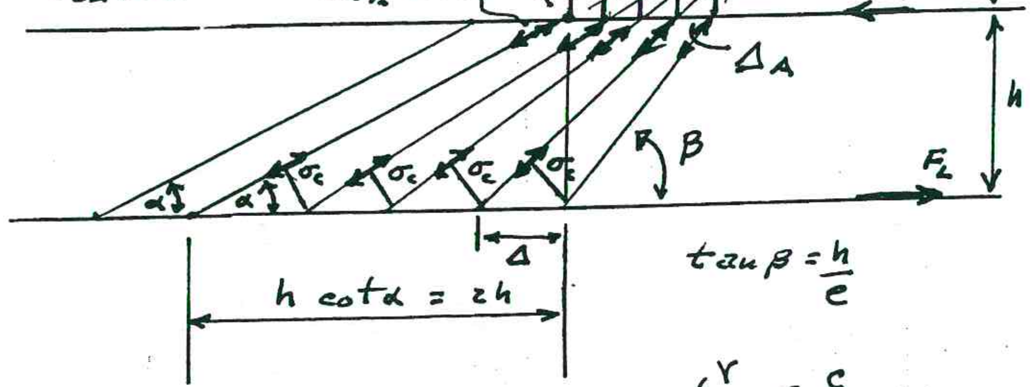
See ACI Code
12.11.1
1/3 As extend into
support 6"
See ACI
Code
12.11.3

$$\boxed{l_d \leq \frac{M_n}{V_u} + l_a}$$

Detail (3): Load P introduced over a finite area, e

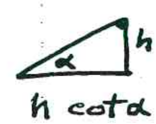
$\Delta_A < \Delta$

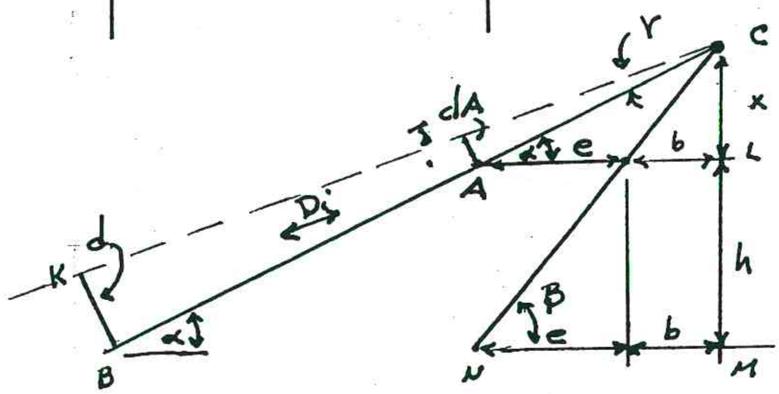
$\sigma_{cA} > \sigma_c$



(Subdivide plate equally to get fan)

$\alpha = \tan^{-1} \frac{1}{2}$





$\tan\alpha = \frac{x}{e+b}$ $\quad x = (e+b)\tan\alpha$

Sim Δ $\quad \Delta_{CAJ} \approx \Delta\, CBK$

$$\frac{dA}{CA} = \frac{d}{CB}$$

$\tan\beta = \frac{x+h}{e+b}$ $\quad x+h = (e+b)\tan\beta$

Sim Δ $\quad \Delta\, CLA \approx \Delta\, CMB$

$$\frac{x}{CA} = \frac{x+h}{CB}$$

$\tan\beta = \frac{h}{e}$ $\quad D_i = \sigma_c\, d = \sigma_{cA}\, d_A$

$$\frac{\sigma_{cA}}{\sigma_c} = \frac{d}{d_A}$$

$$\therefore \frac{dA}{x} = \frac{d}{x+h}$$

$$\frac{\sigma_{cA}}{\sigma_c} = \frac{d}{dA} = \frac{x+h}{x} = \frac{(e+b)\tan\beta}{(e+b)\tan\alpha}$$

$$\frac{d}{dA} = \frac{x+h}{x}$$

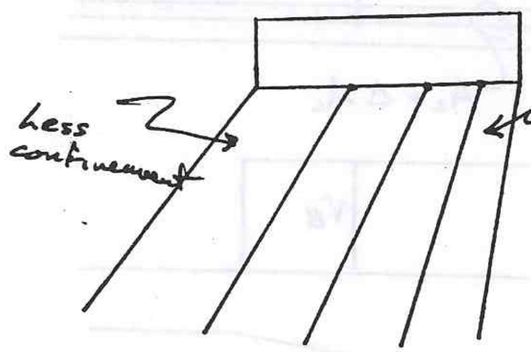
$$\frac{\sigma_{cA}}{\sigma_c} = \frac{\tan\beta}{\tan\alpha} = \frac{h}{e\tan\alpha}$$

$\therefore$ Stress Concentration Factor $\frac{\sigma_{cA}}{\sigma_c} = \frac{d}{dA} = \frac{h}{e}\cot\alpha \quad (24)$

Hence relatively high stress concentrations are possible. This may explain surface failures in tests where small bearings are used

Bearing Plates

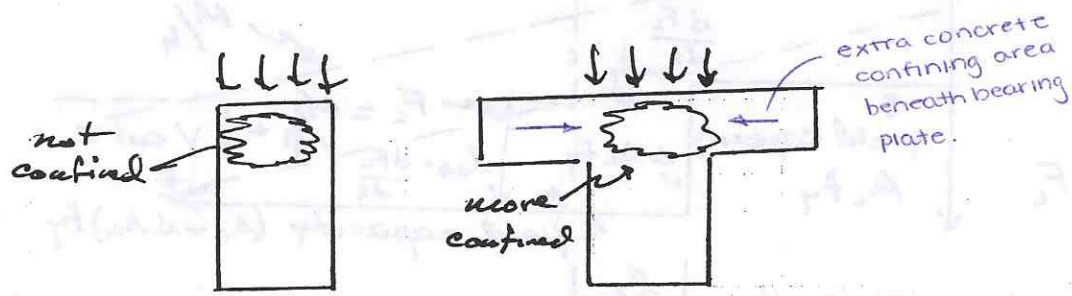
An alternate analysis is possible where one could take some effect of confinement by using unequal areas Δ_A so stress field is non uniform



σ_{BR} increases with confinement

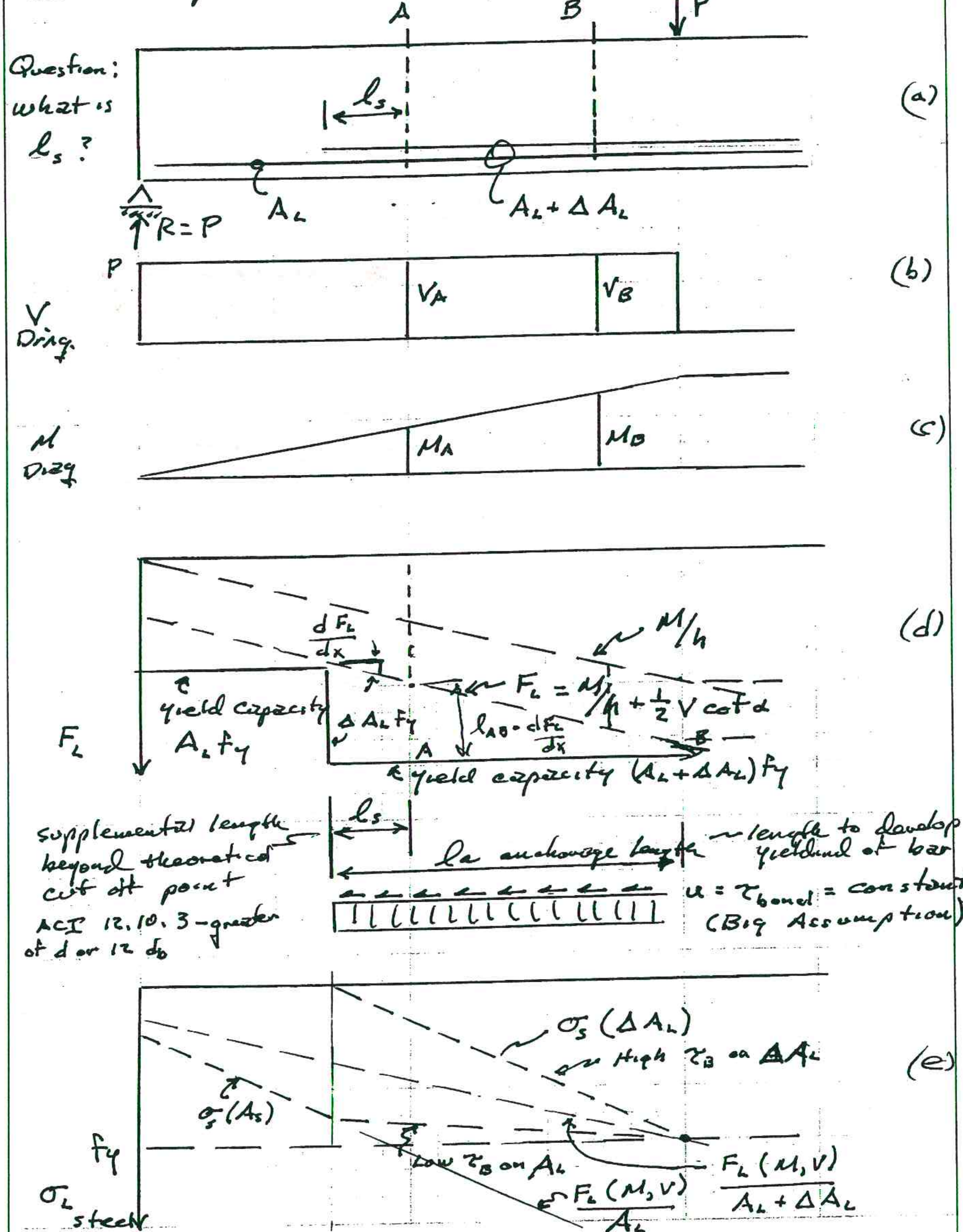
See Ref 9 in Thurlimann paper in IABSE Colloquium Copenhagen May 1979 'Plasticity in Concrete'

Analysis needs to consider 3-D effects



Special Problems

② Change in Area of Longitudinal Reinforcement



$$l_s = l_a - l_{AB}$$

However $l_{AB} \cdot \dfrac{dF_L}{dx} = \Delta A_L f_y$ See Fig (d) P 27

$$l_{AB} = \frac{\Delta A_L f_y}{\frac{dF_L}{dx}}$$

$$\frac{dF_L}{dx} = \frac{d}{dx}\left(\frac{M}{h} + \frac{1}{2} V \cot\alpha\right)$$ (V is constant, α is constant)

$$= \frac{1}{h}\frac{dM}{dx}$$

$$l_{AB} = \frac{\Delta A_L f_y \cdot h}{V}$$

$$= \frac{V}{h}$$

$$\therefore \quad l_s = l_a - \frac{\Delta A_L f_y h}{V} \gg 0$$

See ACI 12.11.3 Eqn 12-2 $\quad l_d \leq \dfrac{M_n}{V_u} + l_a$

Notation change

l_s Thurl = l_a ACI

l_a Thurl = l_d ACI

$$l_a = \frac{M_n}{V_u} + l_s$$

$$l_s = l_a - \frac{M_n}{V_u}$$ $\Delta A_L f_y h$ is M_n for M change

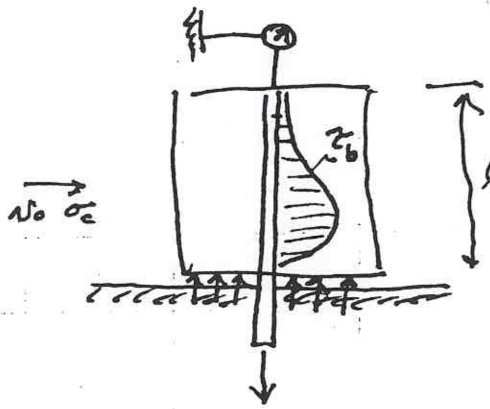
Further Discussion:

τ_b = constant

constant compression field σ_c

Uniform τ_b agrees with $\mu\,\sigma_{cr}$ concept!

Not too bad an assumption



Anchorage length – Pullout test

τ_b not uniform

Compression on face distorts results

Non-uniform τ_b means the pullout test may be too severe -- should be conservative indicator for Compression field

Swiss Specs SIA 162 - Deformed bars w/o hooks

In tensile zone $l_a = 65\phi$

In general use $l_a = 45\phi$

swiss specifications

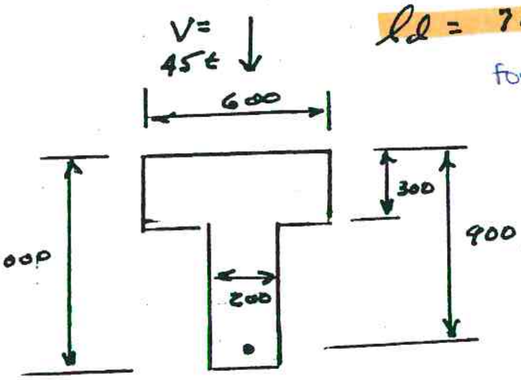
ACI 318-95 - No 7 and larger bars - GR60 $f'_c = 4$

No limit on cover or spacing $\frac{l_d}{d_b} = \frac{3 f_y}{40\sqrt{f'_c}}$ development length

$l_d = 71\phi \approx 65\phi$ (10% diff.)

for $f_y = 60$ KSI, $f'_c = 4$ KSI / high strength concrete shortens development

Example mm.



$A_L = 3\phi\ 28$ mm

$\Delta A_L = 2\phi\ 28$ mm $= 12.3\ cm^2$

$f_y \approx 60$ ksi $\approx 4.6\ \frac{ton}{cm^2}$

$l_a = 65\phi = 182$ cm.

$$l_s = l_a - \frac{\Delta A_L f_y h}{V}$$

$$= 182\,cm - \frac{(12.3)(4.6)(90)}{45} = 69\ cm \quad (27\ in.)$$

ACI - $d = 90$ cm $\quad 12 d_b = 34$ cm $\quad$ 90 cm governs, 30% longer

Generally consistent - little specific test confirmation

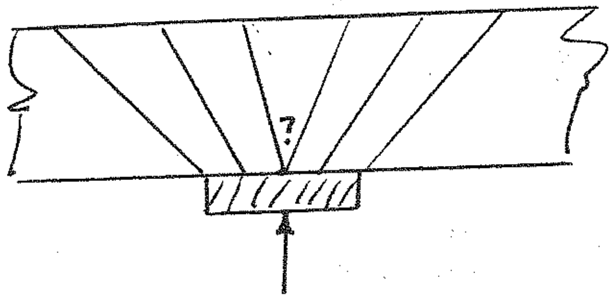
Special Problems:

3. Other Support Details — Bonus Homework Problems

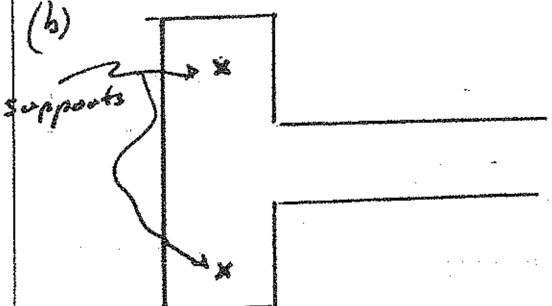
Assume any α
Plot FL, FU

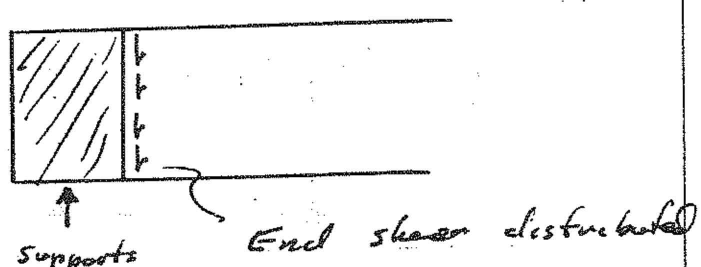
Intermediate Support

(a)



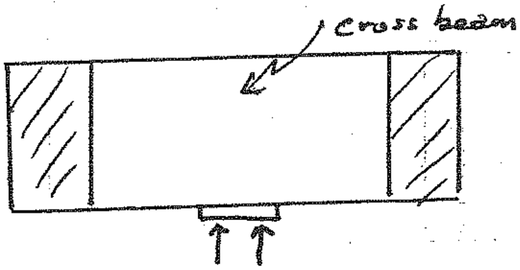
(b)



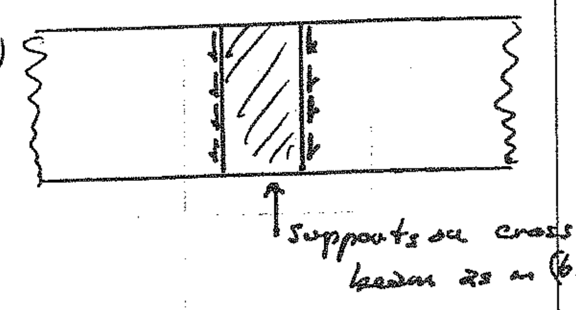


Plan   Elev.

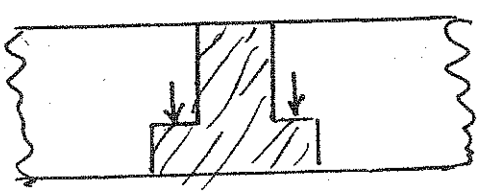
(c)



(d)



(e)



Loads on cross inverted T bars

Analysis vs. Design

Analysis: Given – Cross section, Span, Supports, Loading, Longitudinal Reinforcing, Transverse Reinforcing including spacing

Find – M_{po} (11), V_{po} (12), V_{max} (18), V_{pe} (14), Interaction (13) & (2?) ↓ set V_p, M_p
Check f_c – fans, webs –

EQ 2 – σ_c – check f_c
EQ 3 – F_U – check capacity
EQ 4 – F_L – check capacity
| EQ (6) statically admissable
EQ 5 – Unique α when all reinforcement set

Design or Dimensioning:

Given – M, V, f'_c, f_y, Span, Supports, Loading

Select – b, h (b may have to iterate – σ_c)

Find – Longitudinal Reinforcement
Transverse Reinforcement A_{ST}, t

Procedure: select a tan α based on economy, constructability, sentimentality,

Check σ_c either directly or by V_{max}
change b as needed

Select A_{sl} to meet EQ (3)-(4)
Select A_{ST}, t to meet EQ 5 for α desired

Check Interaction EQ 13 & revise as may be necessary

Check fans, etc to correctly detail discontinuity areas

Additional Complications requiring modifications

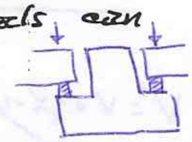
(a) Hanging loads (Tension chord loads)
(b) Distributed loads
(c) Concentrated loads
(d) change in stirrup Reinforcement
(e) Variable Depth Members
(f) Brackets & Corbels
(g) Walls – Shear walls
(h) Deep Beams

(a) Hanging loads

loads applied to the bottom chord of a truss system (or shear reinf.)

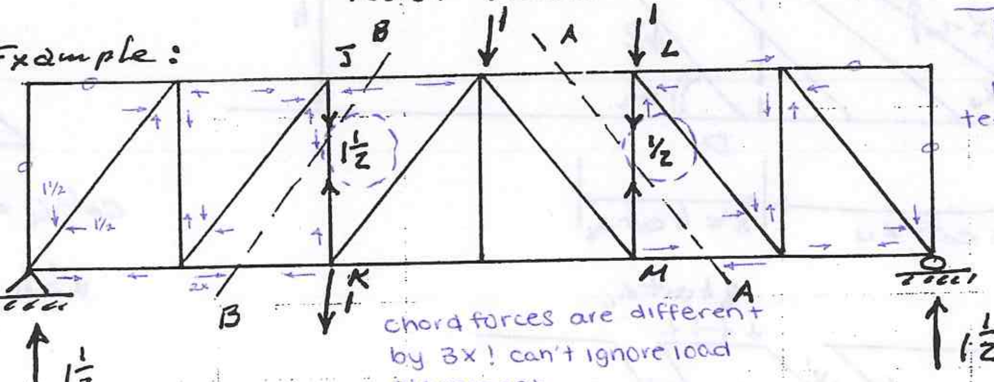
Basic Truss System Assumes Loads At upper Panel Points

Must add additional hangers to "pick-up" bottom chord loads to hang from upper chord point at load application where compression field diagonals can resist them



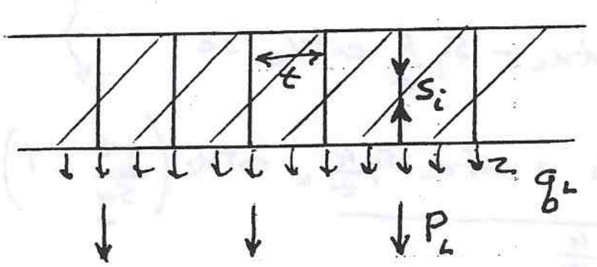
inverted tee (bent)

Truss Example:



chord forces are different by 3x! can't ignore load placement

Note: Section AA and BB are basically the same except that lower panel load at K causes member JK to have to "pick up" load from K to J
Contrast with LM where load is already at top



Stirrup Force S:

normal stirrup force for V with load on top

$$S_i = S_i(V) + \Delta S_i = S_i(V) + \underbrace{q_L t + P_L}_{\text{Hanging Force}}$$

Hence – Additional "pick up" stirrup steel

$$\Delta A_{ST_i} = \frac{(q_L t + P_L)}{f_{y_{ST}}}$$

All loads have to be introduced into top chord!!

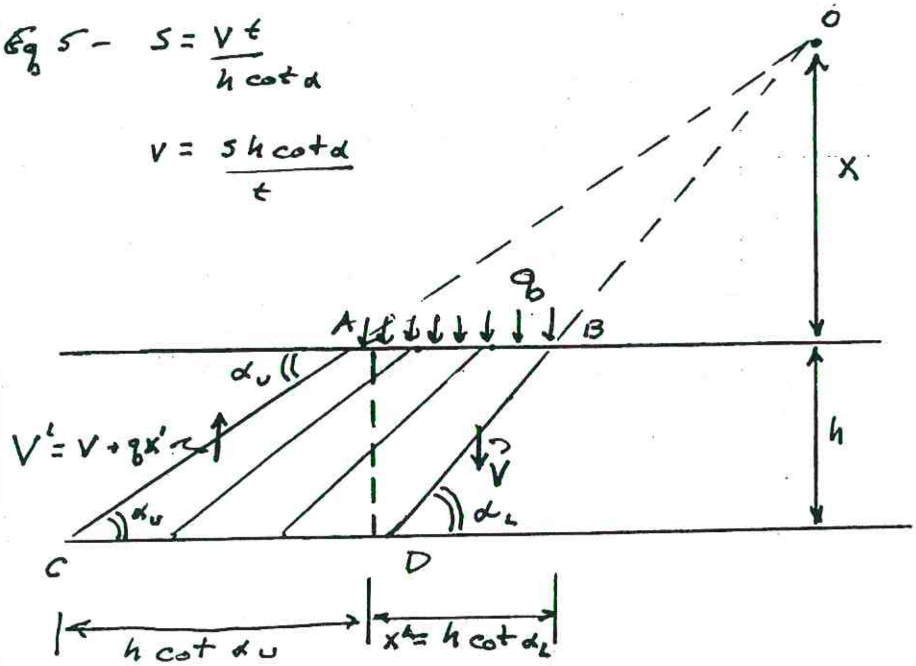
(b) Distributed Loads:

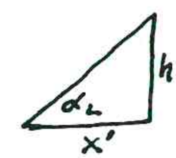
Assume constant stirrup size and spacing
S_y = constant t = constant   (This may be changed between points)

Lost of admissible stress field in Region I $\sigma_c \geq f_c$
i.e. stirrups yield, long. reinf yields
no limit on α

Eq 5 – $s = \dfrac{Vt}{h\cot\alpha}$

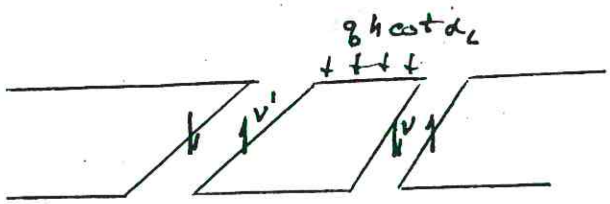
$V = \dfrac{s h \cot\alpha}{t}$





$\cot\alpha_L = \dfrac{x'}{h}$

$x' = h\cot\alpha_L$



Σ Vert $F = 0 \uparrow$

Note – slow change in compression field angle

$$V' - q_b h\cot\alpha_L - V = 0$$

$$S_y \frac{h}{t}\cot\alpha_U - q_b h\cot\alpha_L - S_y\frac{h}{t}\cot\alpha_L = 0$$

$$\cot\alpha_U = \frac{\cot\alpha_L\, q_b h + \cot\alpha_L\, S_y \frac{h}{t}}{S_y \frac{h}{t}} = \cot\alpha_L\left(\frac{q_b t}{S_y} + 1\right) \quad (26)$$

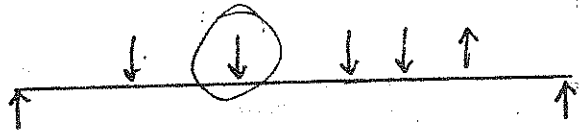
Δ's $OAB \approx OCD$

$$\frac{x}{x+h} = \frac{h\cot\alpha_L}{h\cot\alpha_U} = \frac{\cot\alpha_L}{\cot\alpha_U} = \frac{1}{\left(\frac{q_b t}{S_y}+1\right)} = \frac{1}{\left(\frac{q_b t + S_y}{S_y}\right)} = \frac{S_y}{S_y + q_b t}$$

$\therefore x S_y + x q_b t = x S_y + h S_y$

$x q_b t = h S_y$ so $\dfrac{x}{h} = \dfrac{S_y}{q_b t}$

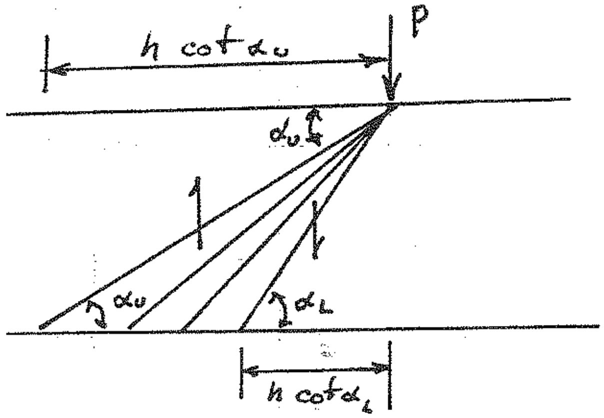
(c) Series of Concentrated Loads



Assume: S_y = constant

t = constant

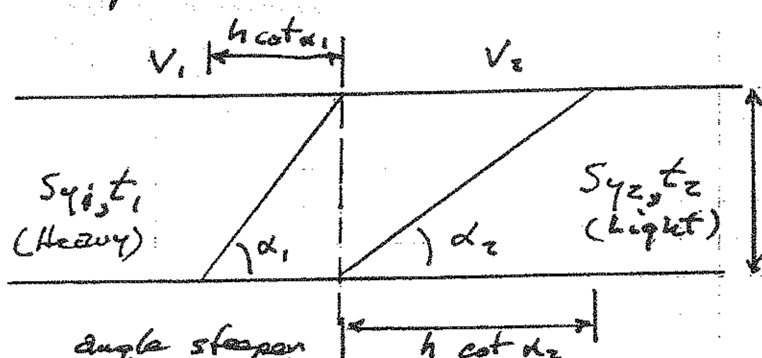
$\therefore S_y/t$ = constant



$$\Sigma V = 0 \quad S_y \frac{h}{t} \cot\alpha_U - P - S_y \frac{h}{t} \cot\alpha_L = 0$$

$$\cot\alpha_U = \cot\alpha_L + \frac{Pt}{S_y h} \qquad (27)$$

(d) Change in Stirrup Reinforcement



Assume $\frac{S_{y1}}{t_1} > \frac{S_{y2}}{t_2}$

(Heavy) (Light)

Also Assume $V_1 = V_2$

angle steeper with heavier stirrups

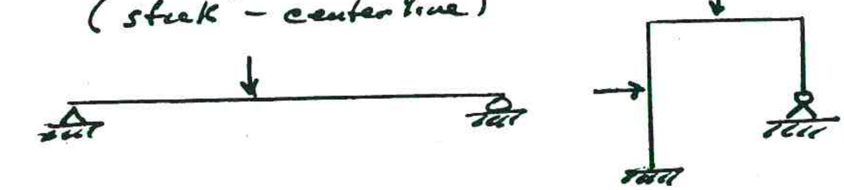
$$\Sigma V = 0 \quad \frac{S_{y1}}{t_1} h \cot\alpha_1 = \frac{S_{y2}}{t_2} h \cot\alpha_2$$

$$\cot\alpha_1 = \frac{S_{y2}\, t_1}{S_{y1}\, t_2} \cot\alpha_2 \qquad (28)$$

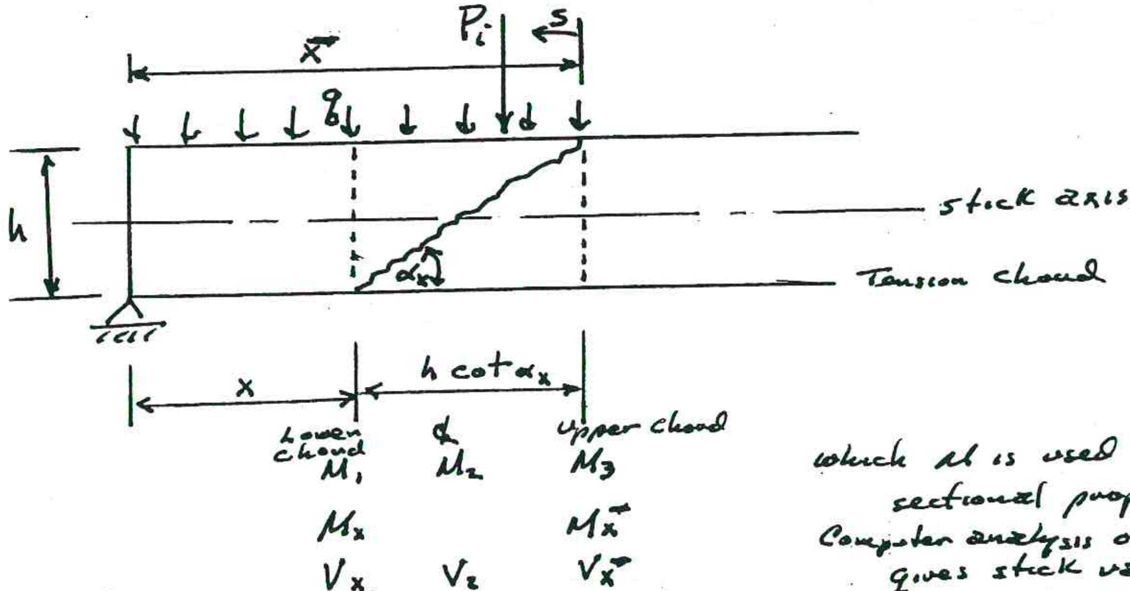
→ Side Issue

Appropriate Sectional Forces for Proportioning

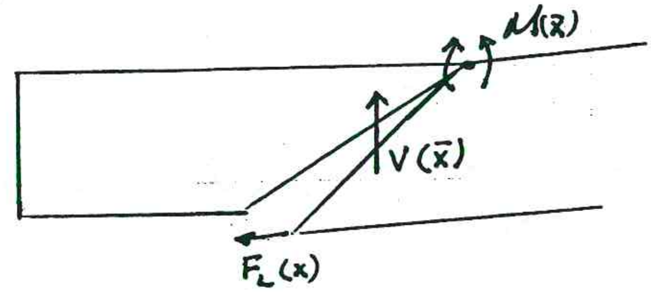
Basic Problem – Beam Theory is only 1 dimensional
(stick – centerline)



Actual beam is two-dimensional structure
and cracks are often inclined



which M is used in sectional proportioning?
Computer analysis output gives stick values M_2, V_2



Basic Rule – Calculate along loaded (top) chord
$V_{\bar{x}}$ $M_{\bar{x}}$

$$V_{\bar{x}} = V_x - \sum^{s} P_i - \int^{s} q \, ds$$

$$M_{\bar{x}} = M_x + V_x \cdot h \cot \alpha_x - \sum^{s} P_i s_i - \int^{s} q \, s \, ds$$

$$F_{L(x)} \, h + \tfrac{1}{2} V(\bar{x}) \, h \cot \alpha_x = M_{\bar{x}}$$

$$F_{L(x)} = \frac{M_{\bar{x}}}{h} - \frac{1}{2} V_{\bar{x}} \cot \alpha_x \qquad (29)$$

Special Cases

① $V = \text{Constant} \quad \therefore \; V_x = V_{\bar{x}} = V \quad q_b = 0 \quad P_i = 0$

$$M_{\bar{x}} = M_x + V\,h\cot\alpha_x$$

$$F_{L(x)} = \frac{M_x}{h} + \frac{1}{2} V \cot\alpha_x \qquad \text{(See EQ 4)}$$

For many practical cases the distributed dead load can be ignored so $V \approx$ constant and thus is sufficiently accurate

② $q_b = \text{constant} \quad P_i = 0$

$$V_{\bar{x}} = V_x - \int^s q_b\,ds = V_x - q_b\,h\cot\alpha_x$$

$$M_{\bar{x}} = M_x + V_x\,h\cot\alpha_x - \int^s q_b\,s\,ds$$

$$\int^s q_b\,s\,ds = q_b\frac{s^2}{2}$$

$$F_{L(x)} = \frac{M_{\bar{x}}}{h} - \frac{1}{2}V_{\bar{x}}\cot\alpha_x$$

$$= \frac{M_x}{h} + \frac{V_x \cancel{h}\cot\alpha_x}{\cancel{h}} - \frac{q_b s^2}{2h} - \frac{1}{2}V_x\cot\alpha_x + \frac{1}{2}\cot\alpha_x\,q_b\,h\cot\alpha_x$$

$$= \frac{M_x}{h} + \frac{V_x\cot\alpha_x}{2} - \frac{q_b s^2}{2h} + \frac{q_b\,h^2\cot^2\alpha_x}{2h}$$

but $s = h\cot\alpha_x$

$$\therefore F_{L(x)} = \frac{M_x}{h} + V_x\frac{\cot\alpha_x}{2}$$

General case :

$$F_L(x) = \frac{M_{\bar{x}}}{h} - \frac{1}{2} V_{\bar{x}} \cot \alpha_x$$

$$= \frac{M_x}{h} + V_x \cot\alpha_x - \frac{\sum^s P_i s_i}{h} - \frac{\int^s q\, s\, ds}{h} - \frac{1}{2} V_x \cot\alpha_x$$

$$+ \tfrac{1}{2} \cot\alpha_x \left[+ \sum^s P_i + \int^s q\, ds \right]$$

$$= \frac{M_x}{h} + \frac{V_x \cot\alpha_x}{2} - \frac{1}{h}\left[\sum^s P_i s_i + \int^s q\, s\, ds \right]$$

$$+ \tfrac{1}{2} \cot\alpha_x \left[\sum^s P_i + \int^s q\, ds \right]$$

$$= \frac{M_x}{h} + \frac{V_x \cot\alpha_x}{2} - R$$

where R is the Remainder term

$$R = + \tfrac{1}{h} \left[\sum^s P_i s_i + \int^s q\, s\, ds \right] \cancel{+} \mp \tfrac{1}{2} \cot\alpha_x \left[\sum P_i + \int^s q\, ds \right]$$

Let $z = \dfrac{s}{h \cot\alpha_x}$ $\quad \dfrac{dz}{ds} = \dfrac{1}{h \cot\alpha_x}$ $\quad$ so $ds = dz\, h \cot\alpha_x$

for $0 < z < 1$

$$R = \frac{\sum^s P_i s_i \cot\alpha_x}{h \cot\alpha_x} + \frac{\int^s q\, s\, ds \cot\alpha_x}{h \cot\alpha_x} - \frac{\cot\alpha_x}{2} \sum P_i - \frac{\cot\alpha_x}{2} \int q\, ds$$

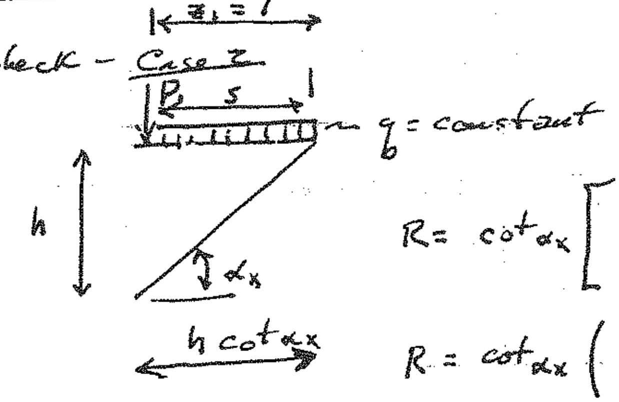
$$R = \sum P_i z_i \cot\alpha_x + \int q\, z\, dz\, h \cot^2\alpha_x - \cot\alpha_x \frac{\sum P_i}{2} - \frac{\cot\alpha_x}{2} \int q\, dz\, h \cot\alpha_x$$

$$R = \cot\alpha_x \left[\left(\sum P_i z_i + h \cot\alpha_x \int q\, z\, dz \right) - \frac{1}{2} \left(\sum P_i + h \cot\alpha_x \int q\, dz \right) \right] \quad (30)$$

$$F_L(x) = \frac{M_x}{h} + \frac{V_x \cot\alpha_x}{2} - R \quad (31)$$

Check - Case 1 - $P_i = 0 \quad q = 0 \qquad R = 0 \qquad$ O.K.

check - Case 2



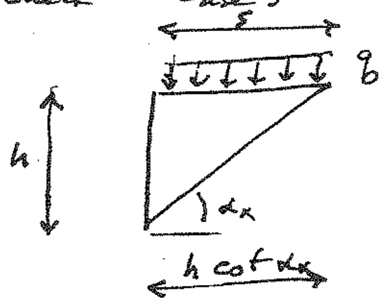
$z = \frac{s}{h \cot\alpha_x} \quad \therefore z_1 = \frac{h\cot\alpha_x}{h\cot\alpha_x} = 1$

$\int q z\, dz = \frac{q}{b}\frac{z^2}{2} = \frac{1}{2}\frac{q}{b} \quad \int q\, dz = \frac{q}{b}\cdot 1$

$R = \cot\alpha_x \left[\left(P_1 \cdot 1 + h\cot\alpha_x \frac{q/b}{2} \right) - \frac{1}{2}\left(P_1 + h\cot\alpha_x \frac{q}{b} \right) \right]$

$R = \cot\alpha_x \left(\frac{P_1}{2} \right)$! No $\frac{q}{b}$ term !

check - Case 3



$$M_{\bar{x}} = M_x + V_x h\cot\alpha_x - \int^{s} q s\, ds$$
$$= M_x + V_x h\cot\alpha_x - \frac{q}{b}\frac{s^2}{2}$$
$$= M_x + V_x h\cot\alpha_x - \frac{q/b}{2}(h\cot\alpha_x)^2$$

$$V_{\bar{x}} = V_x - \int \frac{q}{b}\, ds = V_x - \frac{q}{b}s = V_x - \frac{q}{b} h\cot\alpha_x$$

From (29)

$$F_{Lx} = \frac{M_{\bar{x}}}{h} - \frac{1}{2}V_{\bar{x}}\cot\alpha_x$$
$$= \frac{M_x}{h} + \frac{V_x h\cot\alpha_x}{h} - \frac{q/b}{2h}(h\cot\alpha_x)^2 - \frac{1}{2}\cot\alpha_x\left(V_x - \frac{q}{b}h\cot\alpha_x\right)$$
$$= \frac{M_x}{h} + V_x \frac{\cancel{h}}{\cancel{h}}\cot\alpha_x - \frac{q/b}{2h}(h\cot\alpha_x)^2 - \tfrac{1}{2}\cot\alpha_x V_x + \frac{q}{2}\frac{h^2}{h}\cot^2\alpha_x$$
$$= \frac{M_x}{h} + \frac{V_x\cot\alpha_x}{2}$$

$\therefore$ You can use the vertical section at the tension steel as long as you include the V_x component for the lower stringer force as long as there is no concentrated load -- distributed load does not effect same $R = 0$ for $\frac{q}{b}$ term

Maximum error for concentrated loads - P_i

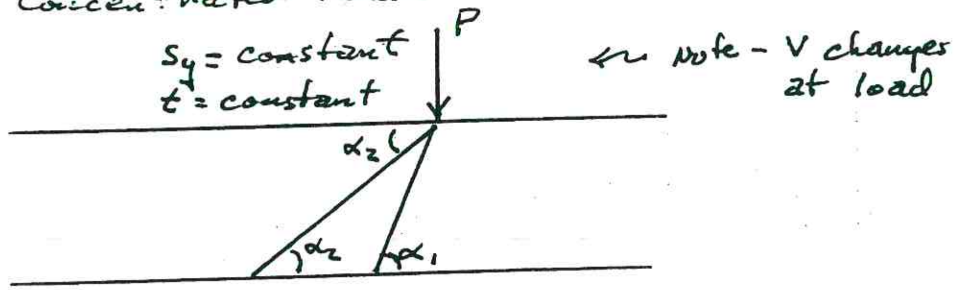
$z_i = 1 \longrightarrow R = \cot\alpha_x\left(P - \frac{1}{2}P\right) = \frac{P}{2}\cot\alpha_x$ — Neglecting R safe

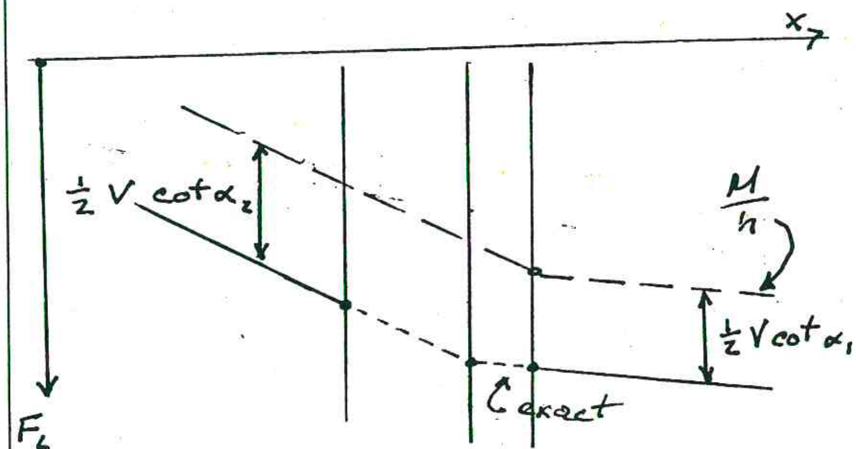
$z_i = 0 \longrightarrow R = \cot\alpha_x\left(0 - \frac{1}{2}P\right) = -\frac{P}{2}\cot\alpha_x$ — Unsafe

Use $M(x)$, $V(x)$, α_x for dimensioning except when conc load.

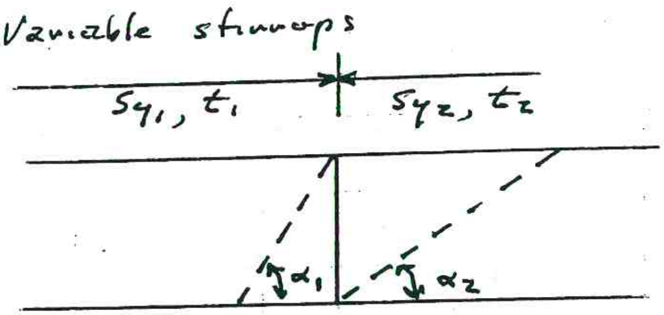
F_L Diagrams for cases (c) and (d)

(c) Concentrated loads



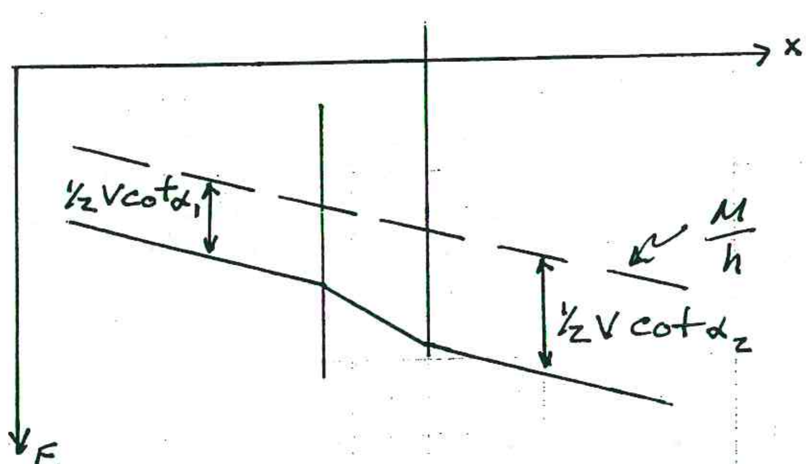


(d) Variable stirrups



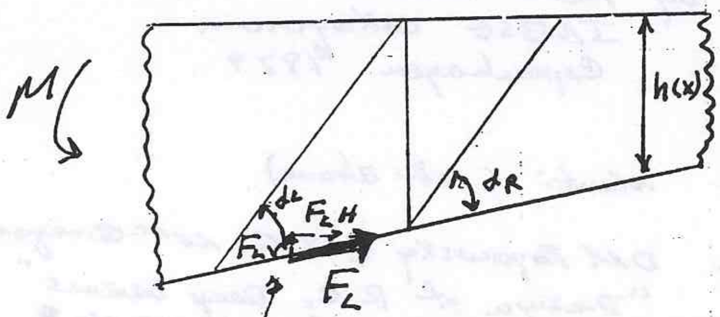
$\alpha_2 < \alpha_1$

$\cot \alpha_2 > \cot \alpha_1$



e) Variable Depth Members $h(x)$

See Dissertation ETHZ P. Müller

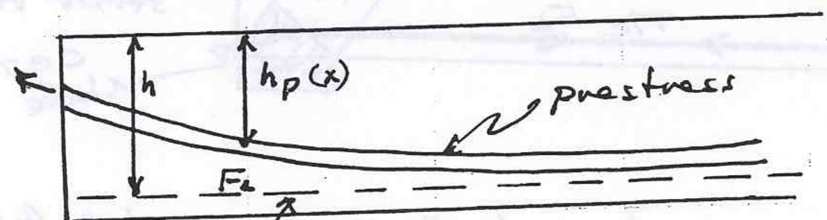


The vertical component of the inclined stringer force resists part of shear

this results in a variable α_x since stirrups are not only factor in resisting shear

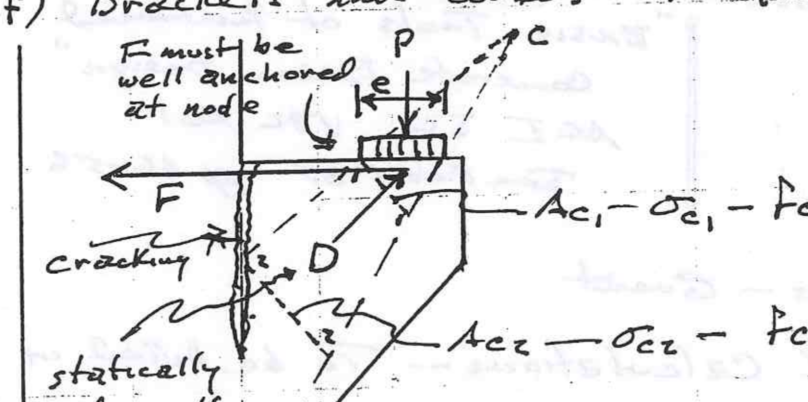
The same general technique can be extended to draped post-tensioning cables

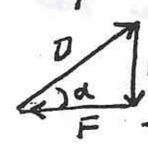
This is $\underline{V_p}$ of ACI Code



example of longitudinal reinforcement that is at an incline.

f) Brackets and Corbels – Theory of Plasticity





$\tan\alpha = \dfrac{P}{F}$

$F = P\cot\alpha$

If local confinement under load $f_{c1} > f_{c2}$

If no local confinement under load $f_{c1} = f_{c2}$

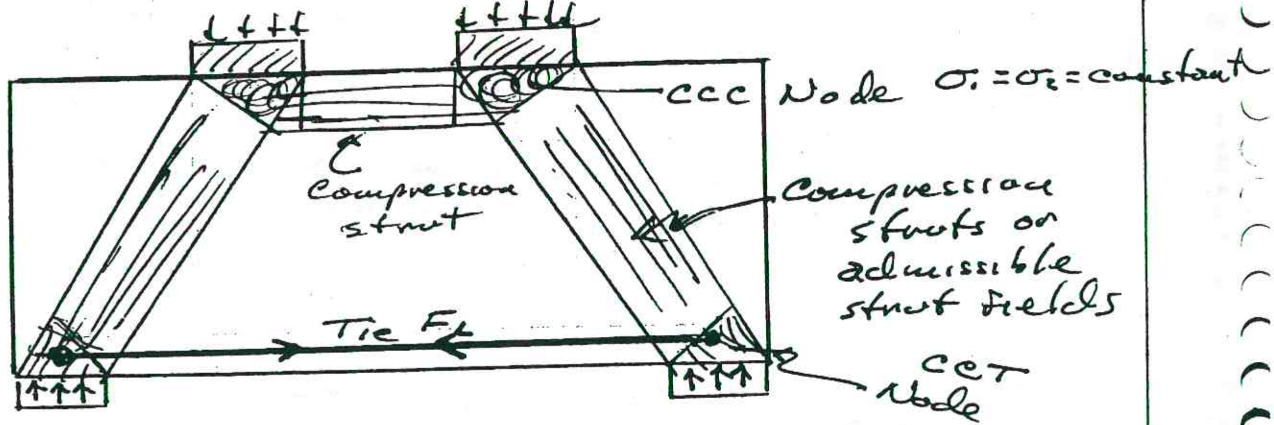
σ_{c1} critical

Note – at service loads needs additional "secondary steel" ... anchored :

g) Shear walls – See "Plastic Analysis of RC Shear Walls" by Peter Marti IABSE Colloquium Copenhagen 1979

h) Deep Beams see Marti (ref above)

see DM Rogowsky & J.G. MacGregor "Design of R.C. Deep Beams" Concrete International V8 No 8 Aug 1986 pp 46-58



Thurlimann approach is lacking in careful definition of nodes – anchorage – confinement – complicated

see for example – Peter Marti "BASIC Tools of Reinforced Concrete Beam Design" ACI JL V82 No1 Jan Feb 1985 pp 46-56

General Ideas – Great

Details & Calculations -- To be filled in

4) Torsion (Circulatory Torsion)

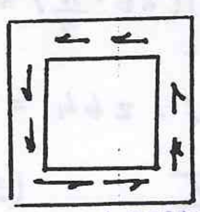
See: (1) B. Thurlimann – "Torsional Strength of R.C. and P.C. Beams – CEB Approach" – ACI Symposium on Shear & Torsion, 1976, Philadelphia

(2) B. Thurlimann "Plastic Analysis of R.C. Beams" IABSE Colloquium "Plasticity in Concrete" Copenhagen May 1979

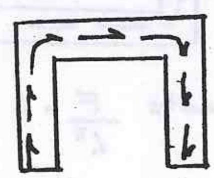
General: Will end up with same type diagrams except with T/T_{max} in place of V/V_{max}

2 Basic Types of Torsion

Circulatory Torsion (St Venant Torsion) | Warping Torsion (Vlassov Torsion)



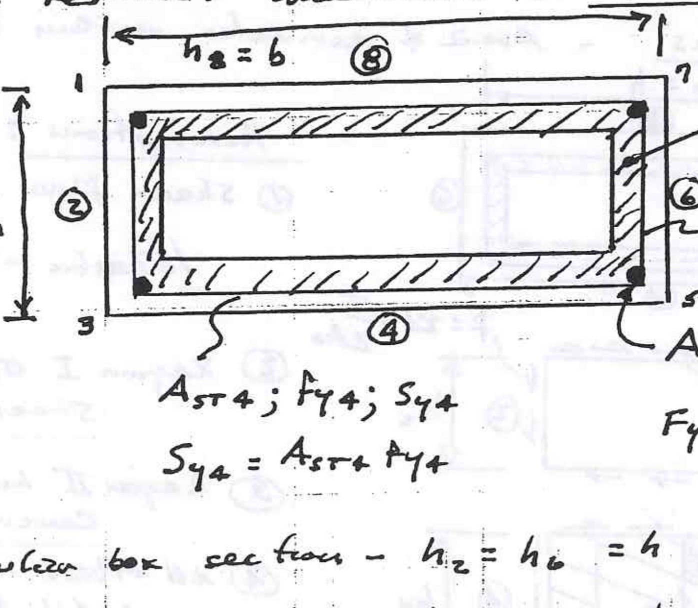
closed section



open section

Initially restrict discussion to circulatory torsion

Number: corners – odd, sides – even



A_o = enclosed area (From theory it is area inside bars)

u = circumference or perimeter of enclosed area

A_5; f_{y5}; F_{y5}

$F_{y5} = A_5 f_{y5}$

A_{ST4}; f_{y4}; S_{y4}

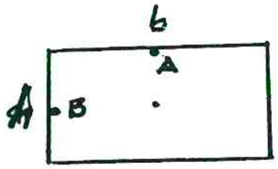
$S_{y4} = A_{ST4} f_{y4}$

In rectangular box section – $h_2 = h_6 = h$

$h_4 = h_8 = b$

Conventional Torsion Analysis

$$\tau = \frac{Tc}{J}$$



$$J = I_p = \frac{1}{12} bh^3 + \frac{1}{12} hb^3$$

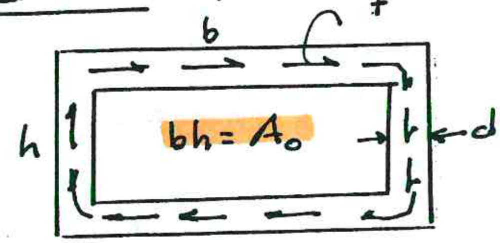
$$= \frac{bh}{12}(h^2 + b^2)$$

$I_p = 1/12\,bh^3$ in each direction

$$\tau_A = \frac{Tc}{J} = \frac{T \cdot h/2}{\frac{bh}{12}(h^2+b^2)} = \frac{T}{\frac{b}{6}(h^2+b^2)} \qquad c = \frac{h}{2}$$

$$\tau_B = \frac{T \cdot \frac{b}{2}}{\frac{bh}{12}(h^2+b^2)} = \frac{T}{\frac{h}{6}(h^2+b^2)} \qquad c = \frac{b}{2}$$

Shear Flow $\bar{f}$



(Think of thin ring)

Think of thin ring

$$\bar{f} = \tau d$$

$$T = \Sigma (\tau d)(\text{length})(\text{arm})$$

$$= \tau d \left[\left(2b \cdot \frac{h}{2}\right) + \left(2h \cdot \frac{b}{2}\right) \right]$$

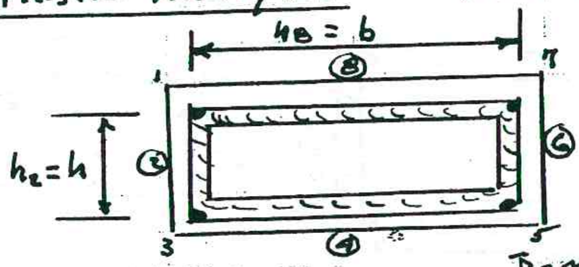
Note $\bar{f}$ units $\frac{F}{L^2} \cdot L = \frac{F}{L}$

$$= \tau d \cdot 2bh = \tau d \cdot 2A_o$$

$$\bar{f} = \tau d = \frac{T}{2A_o} \qquad (32)$$

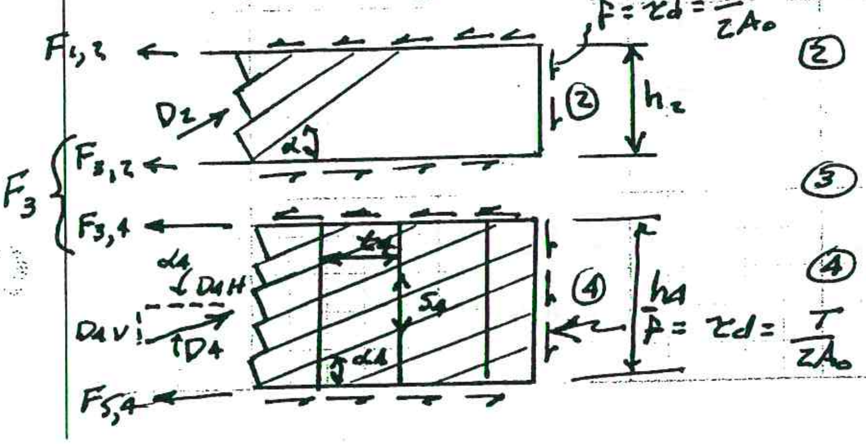
Plastic Analysis

– Area of perimeter within reinforcement



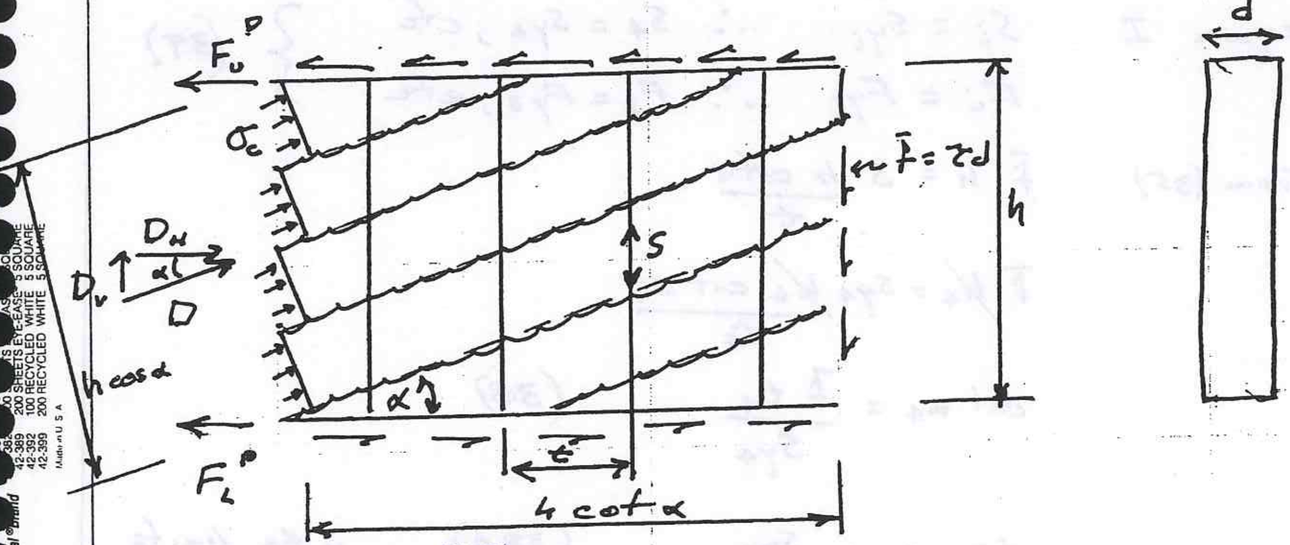
Assumptions:

1. Shear flow $f = \tau d = \frac{T}{2A_o}$ (Elastic – Thin wall section)



2. Region I $\sigma_c > -f_c$ Steel yields
3. Region II limit $\sigma_c = -f_c$ Concrete crushes
4. All others as before in M & V by plasticity

Admissible Stress Field - Static Solution - Truss Model



shear flow

$$\bar{f} = \tau d = \frac{T}{2A_0} \qquad (32)$$

Concrete:
As in (1) Vertical Equilibrium

$$D_V = D\sin\alpha \qquad \therefore \quad D = \frac{D_V}{\sin\alpha}$$

$$D_V = \bar{f} h$$

$$D = \frac{D_V}{\sin\alpha} = \frac{\bar{f}h}{\sin\alpha} \qquad (33)$$

$$-D = \sigma_c \, d \cdot h\cos\alpha$$ σ_c is assumed a compression; compression is $\ominus$

As in (2) $$\sigma_c = \frac{-D}{d h\cos\alpha} = -\frac{\bar{f}h}{d h \cos\alpha \sin\alpha}$$ But $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= \frac{-\bar{f}h \cdot 2}{d h \sin\alpha\cos\alpha \cdot 2} = -\frac{2\bar{f}}{d \sin 2\alpha} \qquad (34)$$

Stirrup:
As in (5)

number of stirrups cut

$$\bar{f} h = S\left[\frac{h\cot\alpha}{t}\right] \quad (35)$$ (Cutting inclined free body $\perp$ to α see p6 - $\frac{h\cot\alpha}{t}$ is no. of stirrups cut)

longitudinal steel $$F_U^P + F_L^P = D_H = D\cos\alpha \overset{(33)}{=} \bar{f}h\frac{\cos\alpha}{\sin\alpha} = \bar{f}h\cot\alpha$$

Assuming equal split top & bottom

$$F_U^P = F_L^P = \frac{\bar{f}h}{2}\cot\alpha \qquad (36)$$

shear flow adds force in longitudinal bars!

Collapse Load -- steel yields $s = s_y \quad F = F_y$

Region I $\left.\begin{array}{l} S_i = S_{y_i} \quad \therefore \ S_4 = S_{y4}, \text{ etc} \\ F_i = F_{y_i} \quad \therefore \ F_3 = F_{y3}, \text{ etc} \end{array}\right\} \quad (37)$

From (35) $\quad \bar{f}\, h = S \dfrac{h \cot\alpha}{t}$

$$\bar{f}\, \cancel{h_4} = S_{y4} \frac{\cancel{h_4} \cot\alpha_4}{t_4}$$

$$\cot\alpha_4 = \frac{\bar{f}\, t_4}{S_{y4}} \qquad (38)$$

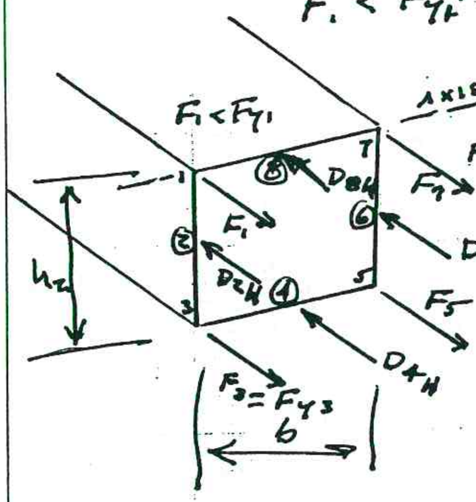
$$\tan\alpha_4 = \frac{S_{y4}}{\bar{f}\, t_4} \qquad (38A)$$

$\therefore \alpha_4$ limits $0 < \alpha_4 < \frac{\pi}{2}$!!

This is similar to Eq(10) -- α is established by shear reinforcement

Maximum possible $\bar{f}_p \rightarrow \bar{f}_{max}$ will occur when all longitudinal steel yields except for compression chord steel -

$F_1 < F_{y1}$, $F_7 < F_{y7}$ (Bending compression keeps comp chord steel at lower tension)



Lower bound - Max $T \rightarrow$ Max $\bar{f}$ ($\bar{f}_{max}$)

If you have a very mixed loading and section you must try varied axes of rotation as this assumes beam will rotate about axis 1-7

Moment equilibrium about axis 1-7

$$M_{17} = 0 = h_z(F_{y3} + F_{y5}) - D_{2H}\frac{h_z}{2} - D_{6H}\frac{h_z}{2} - D_{4H}\, h_z$$

where $D_{2H} = D_2 \cos\alpha_2 = \dfrac{\bar{f}\, h_z}{\sin\alpha_2} \cos\alpha_2 = \bar{f}\, h_z \cot\alpha_2 = \bar{f}\, h_z \dfrac{\bar{f}\, t_z}{S_{y2}}$

(33) (38)

$$M_{17} = 0 = \cancel{h_2}(F_{y3}+F_{y5}) - \bar{F} h_2 \frac{\bar{F} t_2}{s_{y2}} \cancel{\frac{h_2}{2}} - \frac{\bar{F} h_6 \bar{F} t_6}{s_{y6}} \frac{h_2}{2}$$

$$- \frac{\bar{F} h_4 \bar{F} t_4}{s_{y4}} \cancel{h_2}$$

But
$h_2 = h = h_6 \quad h_4 = b$

$$F_{y3} + F_{y5} = \bar{F}^2 \frac{h}{2} \frac{t_2}{s_{y2}} + \bar{F}^2 b \frac{t_4}{s_{y4}} + \bar{F}^2 \frac{h}{2} \frac{t_6}{s_{y6}}$$

$$F_{y3} + F_{y5} = \bar{F}^2 \left(\frac{h t_2}{2 s_{y2}} + \frac{b t_4}{s_{y4}} + \frac{h t_6}{2 s_{y6}} \right)$$

but from (32) $\bar{F} = \dfrac{T}{2A_o}$ so $T = \bar{F}\, 2A_o$

Plastic Torsional Moment – T_{po} :

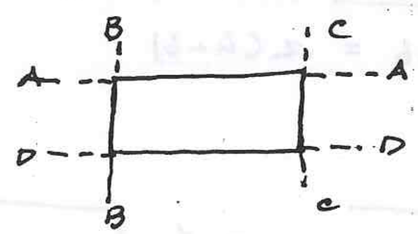
$$\boxed{\begin{aligned} T_{po} &= \bar{F}\, 2A_o \\ T_{po} &= 2A_o \sqrt{\frac{F_{y3}+F_{y5}}{\frac{h}{2}\left(\frac{t_2}{s_{y2}}+\frac{t_6}{s_{y6}}\right) + b\frac{t_4}{s_{y4}}}} \end{aligned}} \quad (39)$$

stringer forces

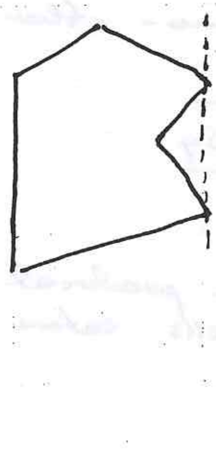
geometry, stirrup forces

plastic torsional capacity

This applies generally to box section but one may need to check varied axes – AA, BB, CC, DD



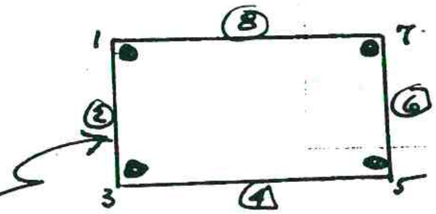
On a more general cross section there may not always be a compression stringer



Special Case - box section where S_y/t is constant around cross section - No Bending

① $F_{yL} < F_{yu}$

$$T_{po} = 2 A_o \sqrt{\frac{F_{y3} + F_{y5}}{\frac{h}{2}\left(\frac{t_2}{s_{y2}} + \frac{t_6}{s_{y6}}\right) + b\frac{t_4}{s_{y4}}}}$$



$F_{y1} \doteq F_{y7} = F_y$ upper $= F_{yu}$

$F_{y3} \doteq F_{y5} = F_y$ lower $= F_{yL}$

$\frac{S_{yi}}{t_i}$ is constant around cross section

$$T_{po} = 2 A_o \sqrt{\frac{2 F_{yL}}{\frac{h}{2}\left(\frac{2t}{s_y}\right) + b\frac{t}{s_y}}}$$

$$T_{po} = 2 A_o \sqrt{\frac{2 F_{yL}}{\frac{t}{s_y}(h+b)}}$$

much simpler

but perimeter $= u = 2h + 2b = 2(h+b)$

$$\therefore \; h + b = \frac{u}{2}$$

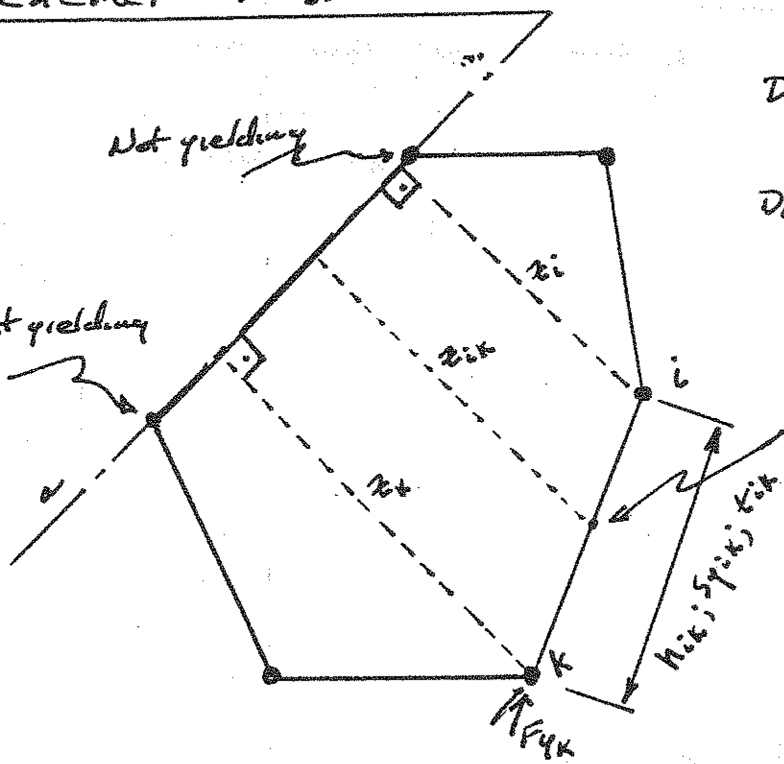
So $$T_{po} = 2 A_o \sqrt{\frac{2 F_{yL}}{\frac{t}{s_y}\frac{u}{2}}} = 2 A_o \sqrt{\frac{4 F_{yL} S_y}{u t}} \quad (40A)$$

② $F_{yu} < F_{yL}$ -- Same procedure - other axis - No Bending

$$T_{po} = 2 A_o \sqrt{\frac{4 F_{yu} S_y}{u t}} \quad (40B)$$

Torsional failure will generally be produced by the compression stress limit in the side walls before the longitudinal stringers can yield

General Torsion Case



$$D_{ik} = \frac{1}{\sin\alpha} \bar{f} h \quad (33)$$

$$D_{ik,h} = D_{ik}\left[h_{ik} \parallel \text{to axis}\right] = D_{ik}\left[h_{ik} \cdot \cos\alpha_{ik}\right]$$

$$D_{ik,h} = \bar{f} \cdot h_{ik} \cot\alpha_{ik}$$

eq(38) — $\cot\alpha_i = \dfrac{\bar{f}\, t_i}{s_{yi}}$

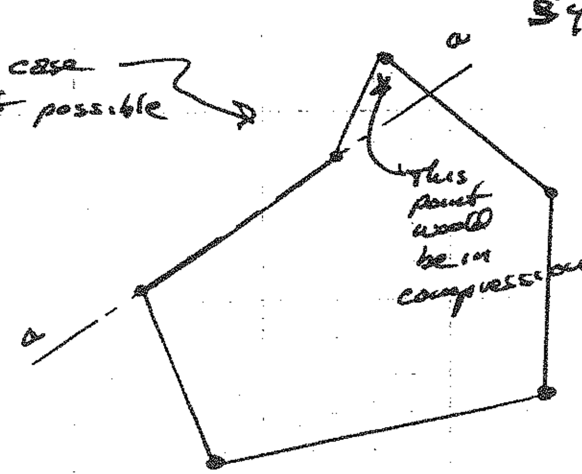
$$D_{ik,h} = \bar{f}\, h_{ik} \cot\alpha_{ik} = \bar{f}\, h_{ik} \frac{\bar{f}\, t_{ik}}{s_{yik}} = (\bar{f})^2 \frac{h_{ik}\, t_{ik}}{s_{yik}}$$

$$M_{a-a} = \Sigma F_{yi} z_i - (\bar{f})^2 \Sigma \frac{h_{ik}\, z_{ik}\, t_{ik}}{s_{yik}} \qquad \text{where } (\bar{f})^2 = \left(\frac{T}{2A_o}\right)^2 \text{ from } (32)$$

Plastic Torsion Strength — No Bending Applied ∴ $M_{a-a} = 0$

$$\left|T_{po}\right|_{min} = 2A_o \sqrt{\frac{\Sigma F_{yi} z_i}{\Sigma \dfrac{h_{ik} z_{ik} t_{ik}}{s_{yik}}}} \quad (41)$$

This case is not possible



— would result in longitudinal stringer in compression which is not minimum torque with no bending applied

Limitation – Concrete Capacity (See EQ (14))

Region I – Stirrups and longitudinal reinf. yield but $|\sigma_c| < |f_c|$ or $\sigma_c > -f_c$

Region II – Stirrups yield and concrete crushes

What if concrete crushes? So $\sigma_c = -f_c$ in $\underline{\underline{\text{one}}}$ side wall – concrete crushes in $\underline{\text{one}}$ side wall -- (no index i_K used for simplicity)

From EQ (34) $\sigma_c = \dfrac{-\bar{f}}{d \sin\alpha \cos\alpha} = -f_c$

Let $\bar{f} = \bar{f}_{pc}$ for plastic crushing of concrete

$$\bar{f}_{pc} = f_c\, d \sin\alpha \cos\alpha = \tfrac{1}{2} f_c d \sin 2\alpha \qquad (42)$$

But from Eq (38) $\bar{f} = \dfrac{S_{y_i} \cot\alpha_i}{t_i} = \dfrac{S_y}{t} \cot\alpha \qquad (43)$

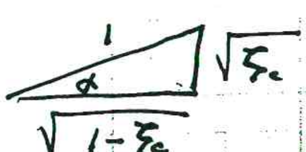
So $\bar{f}_{pc} = \dfrac{S_y}{t} \cot\alpha$

∴ Rewriting EQ (42) and combining with EQ (43)

$$\bar{f}_{pc} = f_c\, d \sin\alpha \cancel{\cos\alpha} = \frac{S_y}{t} \frac{\cancel{\cos\alpha}}{\sin\alpha}$$

So $\sin^2\alpha = \dfrac{S_y}{t d f_c} = \zeta_c \quad (44)$ (Note – since web width d is same as beam width b, this is basically the same Eq as Eq (16) on p 8)

The same physical explanation and relationship exists as on bottom of p 9 for (16) ∴ $0 \leq \zeta_c \leq 1$



$\tan^2\alpha_c = \dfrac{\zeta_c}{1-\zeta_c} = \lambda_c$ (same Definition as on p 9)

$$\cot\alpha_c = \sqrt{\frac{1-\zeta_c}{\zeta_c}}$$

Now rewriting EQ 43 and substituting for $\cot\alpha_c$

$$\bar{f}_{pc} = \frac{S_y}{t}\cot\alpha = \frac{S_y}{t}\sqrt{\frac{1-\xi_c}{\xi_c}} \qquad (43A)$$

If EQ(43A) is divided by $d f_c$ and remembering from EQ(44)

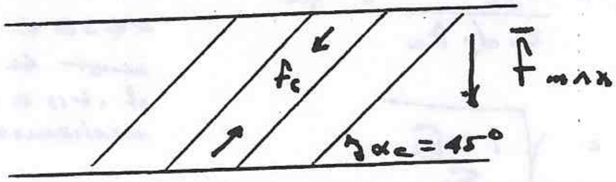
that $\frac{S_y}{t\,d\,f_c} = \xi_c$, then

$$\frac{\bar{f}_{pc}}{d f_c} = \frac{S_y}{t\,d\,f_c}\sqrt{\frac{1-\xi_c}{\xi_c}} = \xi_c\sqrt{\frac{1-\xi_c}{\xi_c}} = \sqrt{\xi_c(1-\xi_c)} \qquad (45)$$

(Compare with Eq 17 on p.10)

As shown on P10, the maximum value of $\xi_c = \frac{1}{2}$ for which $\alpha_c = \frac{\pi}{4} = 45^\circ$

$$\therefore \bar{f}_{max} = d f_c \sqrt{(\tfrac{1}{2})(1-\tfrac{1}{2})} = \underline{\tfrac{1}{2} d f_c} = \bar{f}_{max} \qquad (46)$$



$\therefore$ Reference Values

From Eq 32 $\quad \bar{f} = \frac{T}{2A_o}$ so $\quad T_{max} = 2A_o \bar{f}_{max} = 2A_o \frac{1}{2} d f_c$

$$T_{max} = A_o\, d\, f_c \qquad (47)$$

$$\frac{T_{pc}}{T_{max}} = \frac{2A_o\bar{f}_{pc}}{2A_o\bar{f}_{max}} = \frac{2A_o\bar{f}_{pc}}{A_o\,d\,f_c} = \frac{2\bar{f}_{pc}}{d\,f_c} = \frac{2S_y\cot\alpha}{t\,d\,f_c}$$

$$\therefore \frac{2\bar{f}_{pc}}{d f_c} = 2\frac{S_y}{t\,d\,f_c}\cot\alpha = 2\xi_c\sqrt{\frac{1-\xi_c}{\xi_c}} = 2\sqrt{\xi_c(1-\xi_c)}$$

$\xi_c \cdot \sqrt{\frac{1-\xi_c}{\xi_c}}$

$$\frac{T_{pc}}{T_{max}} = 2\sqrt{\xi_c(1-\xi_c)} \qquad (48)$$

See EQ 19 on P 11

$\xi = \frac{S_y}{t \cdot d \cdot f_c}$

Collapse Modes – Pure Torsion

1. Region I : a) All stirrups yielding $S_i = S_{yi}$

b) All longitudinal steel yielding except for the two adjacent stringers through which the moment axis passes (check around periphery to find critical case)

c) $\sigma_c > -f_c$ (no concrete crushing)

d) mechanism forms – hence collapse load

From Eq(39)

$$T_{po} = 2\bar{f}_p A_o$$

(T_{po} = Plastic Torsional Moment)
($\bar{f}_p$ = plastic shear flow)

Check for each wall in turn:

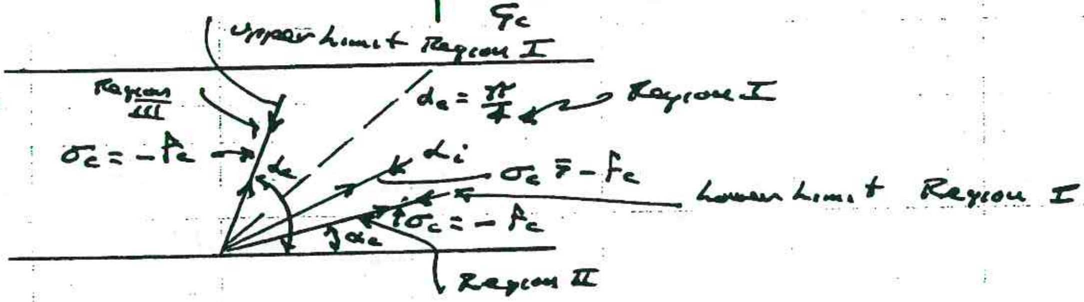
Eq(38) $$\cot\alpha_i = \frac{\bar{f}\,t_i}{S_{yi}}$$ (Inclination of the compression field in each wall – set by stirrup amount)

Eq(44) $$\sin^2\alpha_c = \frac{S_{yi}}{t_i d_i f_c} = \xi_c$$

Note the α in Eq 38 & Eq 44 must be compatible if it is a valid mechanism

$$\cot\alpha_c = \sqrt{\frac{1-\xi_c}{\xi_c}}$$

α_c Bounds



Therefore

$$\alpha_c \lesssim \alpha \lesssim \frac{\pi}{4}$$ Increase in longitudinal steel is no help -- Increase stirrups

$$\pi/4 \lesssim \alpha \lesssim \alpha_c$$ Increase in stirrup steel is no help -- Increase longitudinal steel

(49)

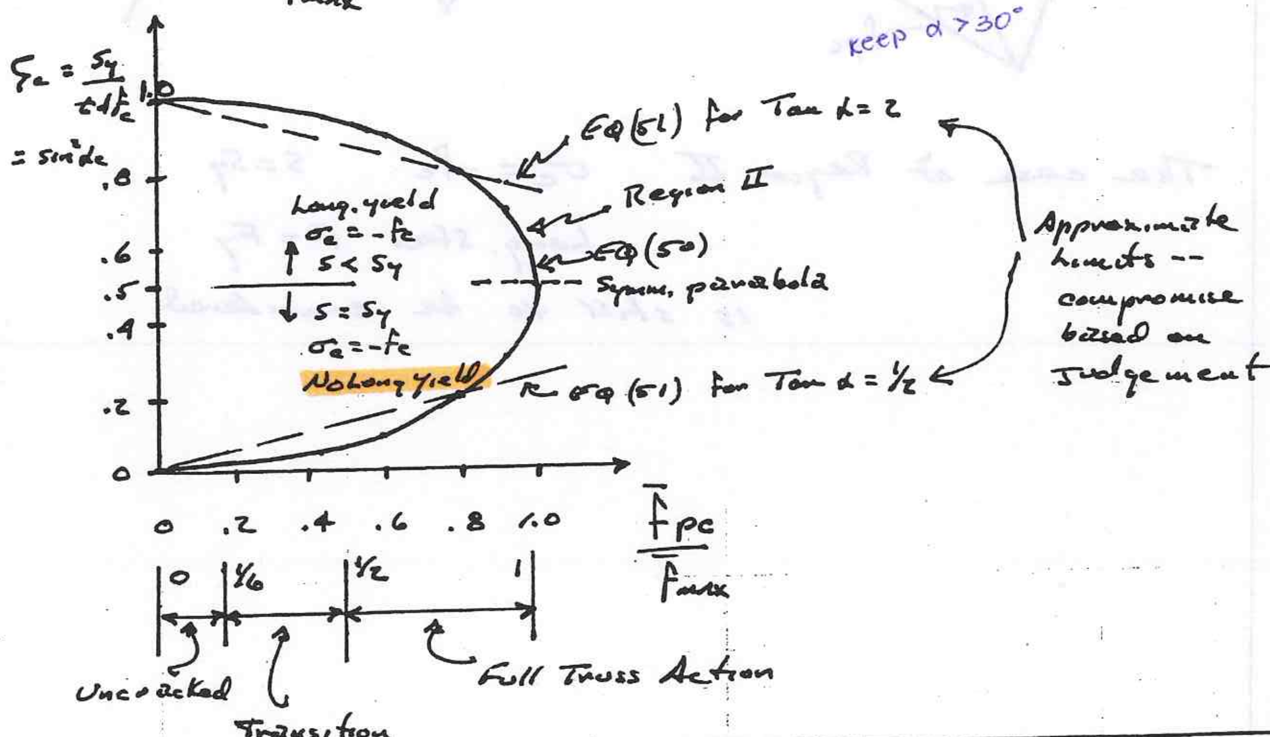
Diagram:

$$\frac{Eq(45)}{Eq(46)} \qquad \frac{\bar f_{pc}}{\bar f_{max}} = \frac{\bar f_{pc}}{\frac{1}{2} d f_c} = \frac{\sqrt{\xi_c(1-\xi_c)}\; d f_c}{\frac{1}{2}\, d f_c} = 2\sqrt{\xi_c(1-\xi_c)} \quad (50)$$

$$\frac{Eq(43)}{Eq(46)} \qquad \frac{\bar f_p}{\bar f_{max}} = \frac{S_y \cot\alpha}{t \frac{1}{2} d f_c}$$

but $\xi_c = \dfrac{S_y}{t d f_c}$ (44)

$$\frac{\bar f_p}{\bar f_{max}} = 2\, \xi_c \cot\alpha \quad (51)$$

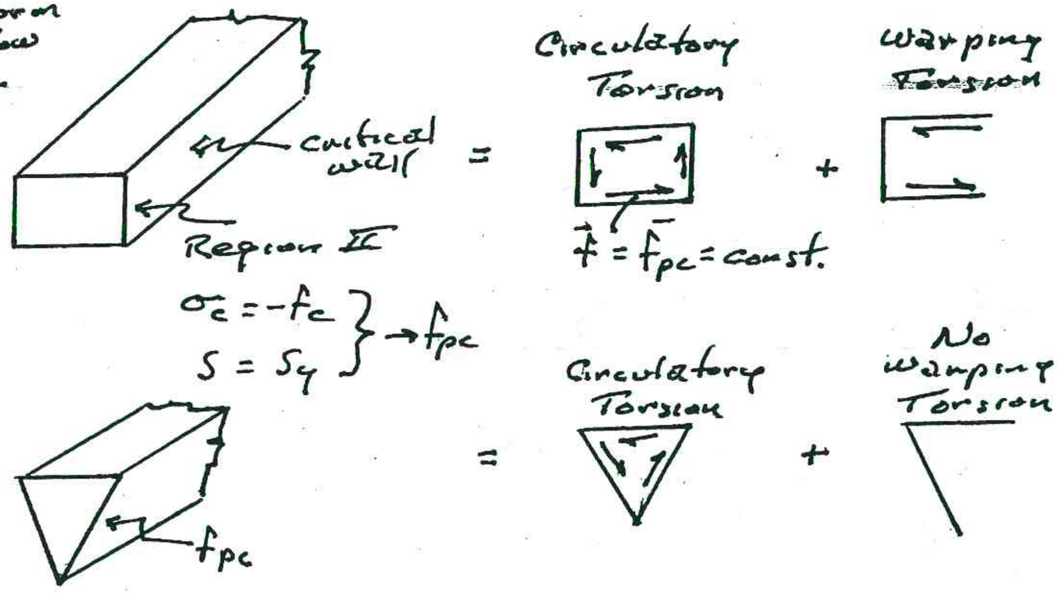


	Region I		Region II	
	$\alpha_c < \alpha \le \frac{\pi}{4}$	$\frac{\pi}{4} \le \alpha \le \alpha_c$	$0 < \alpha_c \le \pi/4$	$\pi/4 \le \alpha_c \le \frac{\pi}{2}$
Concrete	$\sigma_c > -f_c$	$\sigma_c > -f_c$	$\sigma_c = -f_c$	$\sigma_c = -f_c$
Stirrups	$S = S_y$	$S = S_y$	$S = S_y$ in one wall	No yielding in one or more
Long. Reinf	Yielding	Yielding	Not yielding	Yielding

Region I - Mechanism hence collapse load

Region II - Lower bound - only plastic flow in some parts
Incomplete Mechanism

Non Uniform shear Flow



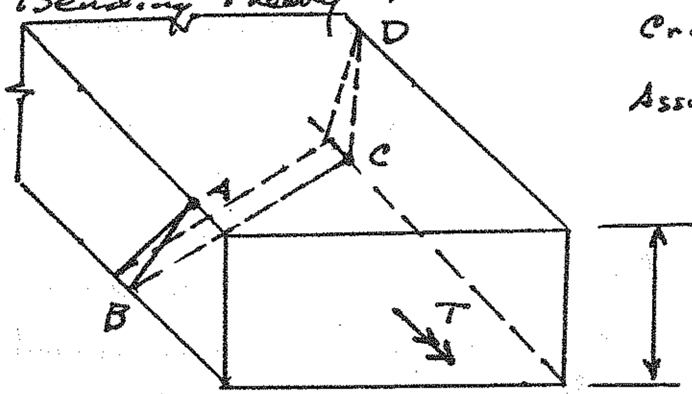
The case of Region II $\sigma_c = -f_c$ $S = S_y$

Long. steel $F = F_y$

is still to be considered

Kinematic Solution – Upper Bound Approach

Skew Bending Theory ?

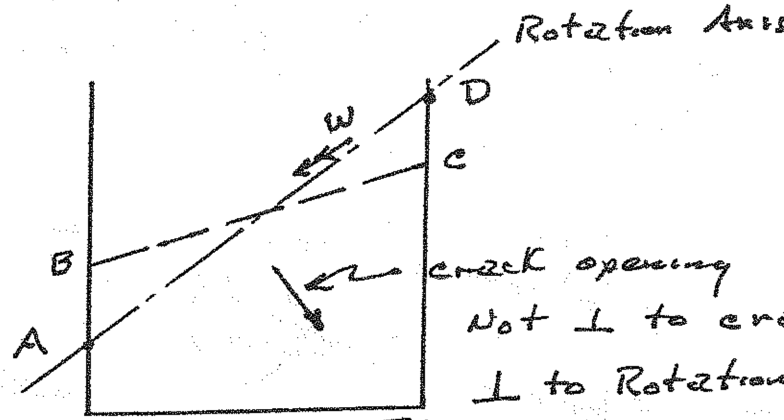


Crack along A-B-C-D

Assume: Steel Yielding

No concrete failure

Rotation about axis A-D

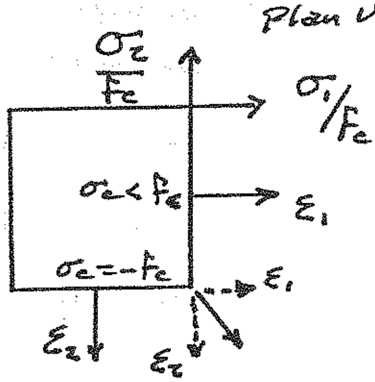


crack opening w_h

Not $\perp$ to crack

$\perp$ to Rotation Axis

Plan View – Top



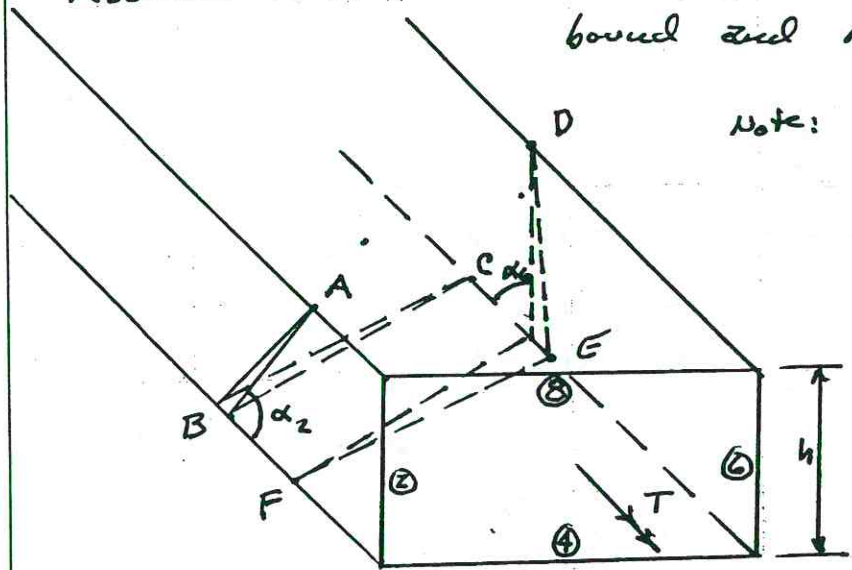
Velocity field is not admissible if it is assumed $\sigma_c \gg -f_c$. If $\sigma_c = -f_c$ then crushing of concrete occurs and there is a dissipation term $W_{i\,con} = \int_B^C f_c \cdot \varepsilon_2 \cdot dV$

However such a dissipation term is neglected in skew bending

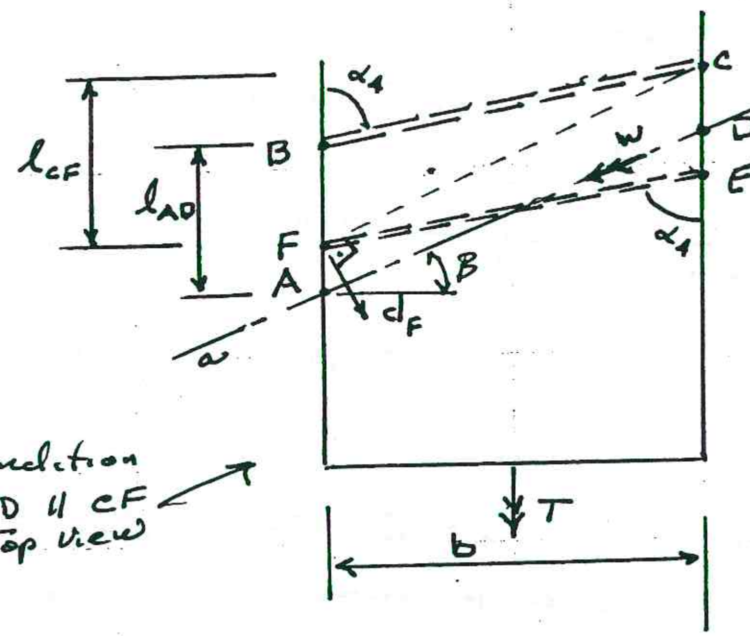
RESULT – NOT A THEORY AT ALL – Just a tool in the right direction. By "fudging" or "juggling" with test results some agreement can be found. However mismatch in crack opening and rotation Axis above shows a basic error! Abandon

Kinematically Admissible Velocity Field

Assume 2 cracks A-B-C and D-E-F - Make upper bound and lower bound converge



Note: For a mechanism, 2 cracks are sufficient - discontinuous field; in an actual test there would be a series of parallel cracks in a continuous field



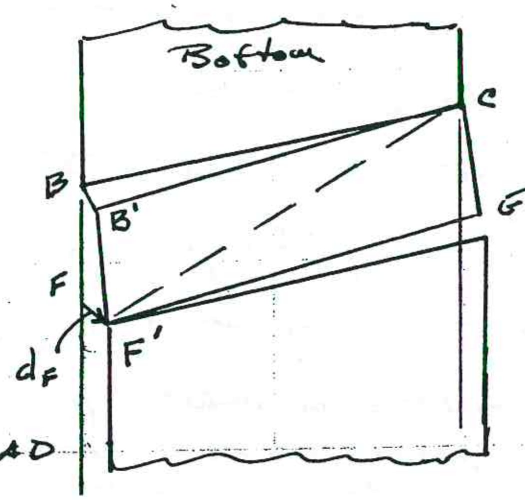
Axis of rotation - aa

Cracks must be parallel for the Kinematic condition

Condition AD ∥ CF Top View

Kinematic condition - Rotation axis AD is parallel to line CF

∴ Displacement of F ⊥ AD !
Displacement of F around C !



Line CF ∥ to AD

$$l_{CF} = \underbrace{l_{AD}}_{\text{From C to A}}$$

$$l_{CF} = b\cot\alpha_4 + h\cot\alpha_2 - l_{AD} \underbrace{+\, h\cot\alpha_6 + b\cot\alpha_4}_{\text{From F to D}}$$

$$\cot B = \frac{l_{AD}}{b}$$

$d_F \perp AD$
Hence $CF \parallel AD$

$$l_{AD} + l_{AD} = 2b\cot\alpha_4 + h\cot\alpha_2 + h\cot\alpha_6$$

$$l_{AD} = b\cot\alpha_4 + \frac{h}{2}(\cot\alpha_2 + \cot\alpha_6)$$

$$\cot\beta = \frac{l_{AD}}{b} = \cot\alpha_4 + \frac{1}{2}\frac{h}{b}(\cot\alpha_2 + \cot\alpha_6) \quad (52)$$

Upper Bound occurs when work is a minimum

Minimum condition gives the requirement that shear flow $\bar{F} = \frac{T}{2A_0} =$ constant !!

Hence this is proof of previous assumption

For further development see B. Thürlimann, IABSE Colloquium, "Plasticity in Concrete", Copenhagen May 1979, p11.

Length of Mechanism

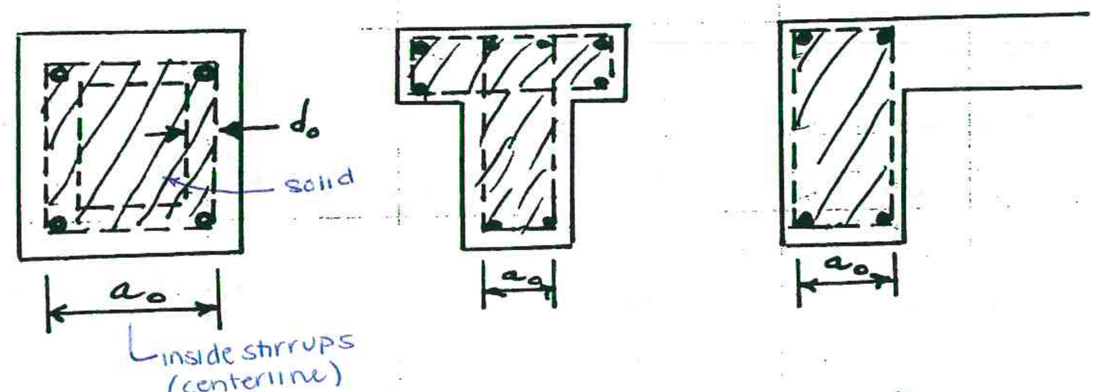
$$\left.\begin{array}{lll} & l_{AC} = h\cot\alpha_2 + b\cot\alpha_4 & \text{If } \cot\alpha_2 > \cot\alpha_6 \\ \text{or} & l_{DF} = b\cot\alpha_4 + h\cot\alpha_6 & \text{If } \cot\alpha_6 > \cot\alpha_2 \end{array}\right\} (53)$$

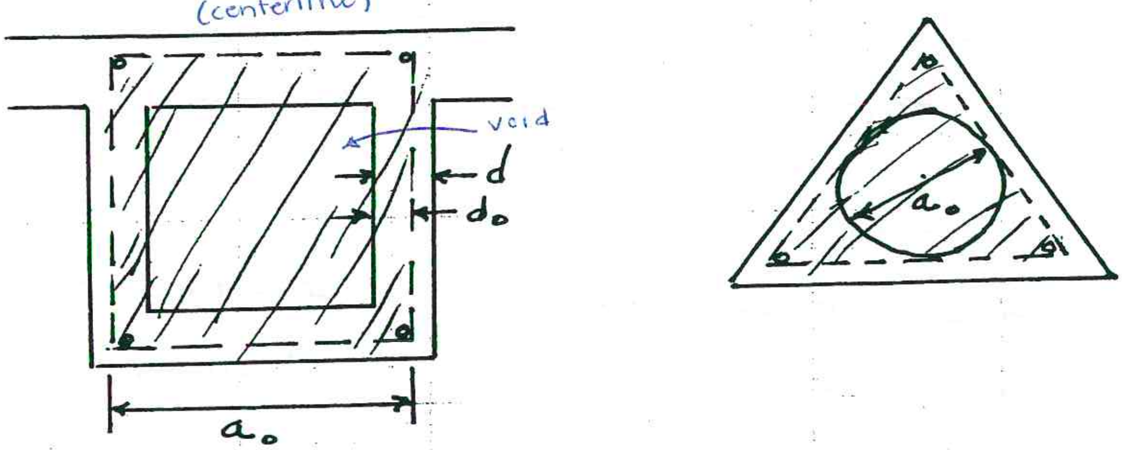
If the test length of a beam is smaller than l_{AC} (or l_{DF} when second case governs), restraints are imposed such that a higher resistance will result (due to higher warping torsion!!) Many reported tests do not meet this condition -- You must be very careful in the interpretation of test results when flat angles occur since l_{AC} may be long!

Solid Cross Sections

From experimental evidence you may assume an equivalent effective box section with wall thickness d_o

steel cage and confined concrete shell make up compression field!





Enclosed Area = A_o

Smallest Dimension = a_o

Effective wall thickness

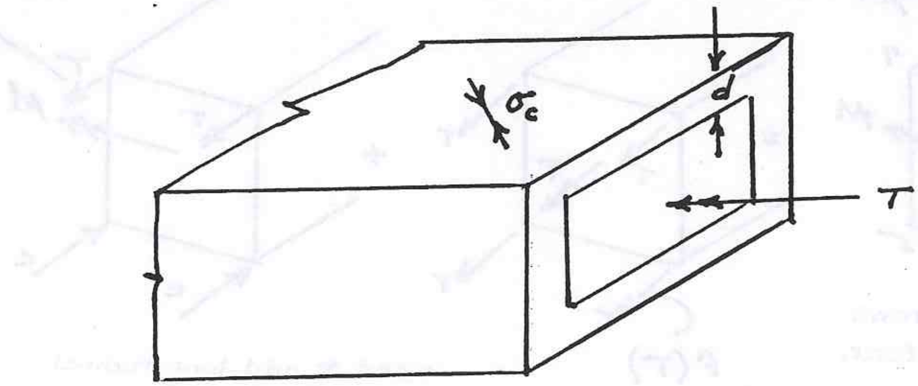
$$d_o = a_o/8$$

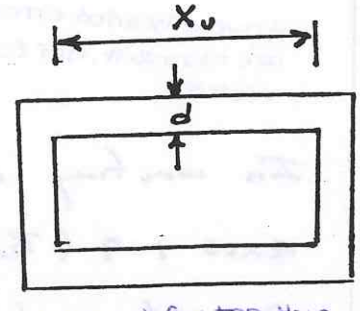
For box sections – smaller of

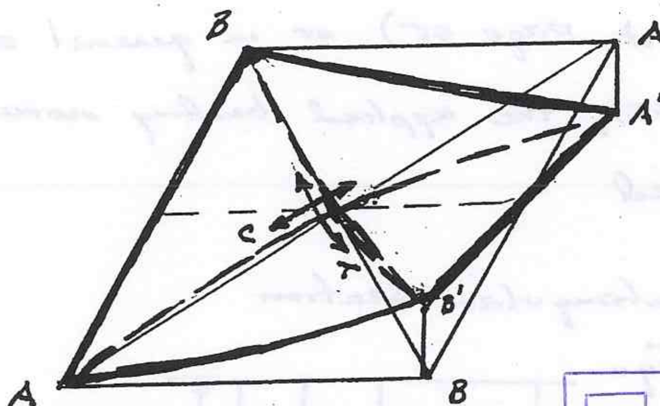
$$d_o = a_o/8 \text{ or } d$$

(54)

Distortion of Slender Walls





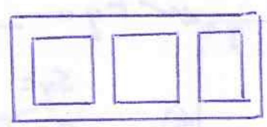


watch out for too thin walls - local buckling can occur!

$\frac{X_u}{d} \geq 12$ to 15 = bad!

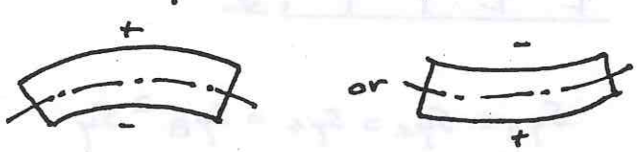
AASHTO l_0/t ← thickness; unsupported length

≥ 18, BAD.



add internal stiffeners

In very slender plates, secondary bending

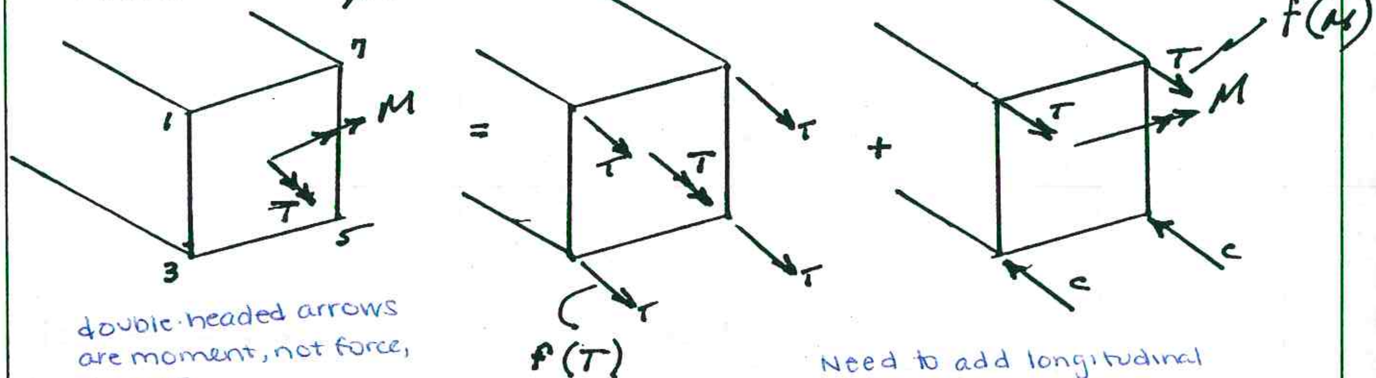


can exist and be superimposed on membrane stresses (in-plane stress) due to compression field. This can also lead to misinterpretation of test results.

σ_c total = σ_c membrane + σ_c secondary $\approx$ 2-3 σ_c has been observed. Andy Taylor (UT-PhD) has shown that above $X_u/d \approx 12$-15 significant local buckling decreases capacity of thin wall hollow piers.

(5) Combined Torsion and Bending

Basic Concept:

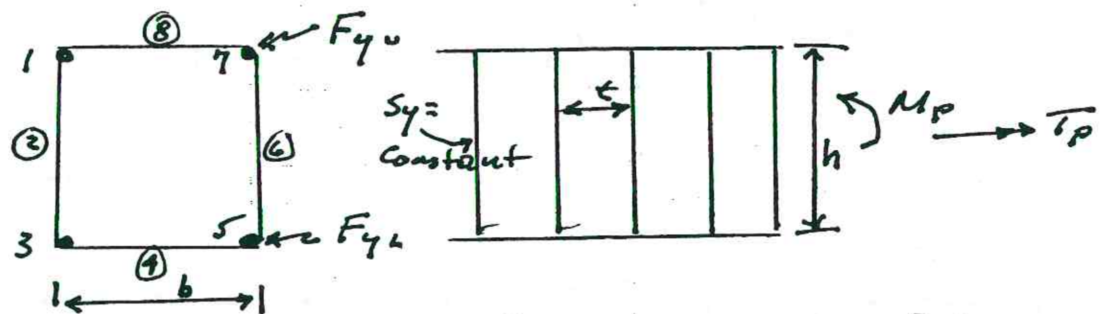


double headed arrows are moment, not force, vectors

Need to add longitudinal steel for torsion

In writing equation for Moment equilibrium about axis 1-7 (Bottom of Page 45) or in general case for $M_{a\text{-}a}$ (Page 48), the applied bending moment M has to be included

Special case: Rectangular section



$F_{y1} = F_{y7} = F_{yu}$

$F_{y3} = F_{y5} = F_{yL}$

$F_{yL} > F_{yu}$ (General with Bending Reint. For ⊕ M)

$S_{y2} = S_{y4} = S_{y6} = S_{y8} = S_y$ (General for closed stirrups)

$A_o = bh$

$U = 2(h+b)$

Collapse load occurs at two adjacent stringers yield, for example 3 & 5 -- hence $F_3 = F_5 = F_{yL}$

Part of yield force ηF_{yL} is used to carry M_p: $0 \le \eta \le 1$

Thus $M_p = 2h\eta F_{yL}$

Remainder of $F_{yL} = (1-\eta)F_{yL}$ is available for T_p

From Eq (40) on page 47 for Torsion alone

$$T_{po} = 2A_o\sqrt{\frac{4F_{yL}S_y}{ut}} \qquad (40A)$$

so we can rewrite as

$$T_{po} = 2A_o\sqrt{\frac{4(1-\eta)F_{yL}S_y}{ut}}$$

$A_o = bh$
$u = 2(h+b)$

Introducing the reference values

Page 7 "Plastic Moment" $M_{po} = 2F_{yL}h \qquad (11)$

Page 46-47 "Plastic Torsional Moment" $T_{po} = 2A_o\sqrt{\frac{4F_{yu}S_y}{ut}}$ (40B)

(Note - Because $F_{yu} < F_{yL}$ in this case, failure in pure torsion would occur because of yielding of upper stringers)

From line (1) above $\eta = \dfrac{M_p}{2hF_{yL}}$

∴ From Eq(11) above $\eta = \dfrac{M_p}{M_{po}}$

Thus
$$\frac{T_p}{T_{po}} = \frac{\cancel{2A_o}\sqrt{\frac{4(1-\eta)F_{yL}S_y}{ut}}}{\cancel{2A_o}\sqrt{\frac{4F_{yu}S_y}{ut}}}$$

so
$$\left(\frac{T_p}{T_{po}}\right)^2 = \frac{\frac{4(1-\eta)F_{yL}S_y}{ut}}{\frac{4F_{yu}S_y}{ut}} = (1-\eta)\frac{F_{yL}}{F_{yu}}$$

since $\eta = \dfrac{M_p}{M_{po}}$, $\left(\dfrac{T_p}{T_{po}}\right)^2 = \left(1-\dfrac{M_p}{M_{po}}\right)\dfrac{F_{yL}}{F_{yu}}$ or $\left(\dfrac{T_p}{T_{po}}\right)^2\dfrac{F_{yu}}{F_{yL}} + \dfrac{M_p}{M_{po}} = 1$

Yielding of lower stringers:

$F_3 = F_5 = F_{yL}$

$$\left(\frac{T_p}{T_{po}}\right)^2 \frac{F_{yU}}{F_{yL}} + \frac{M_p}{M_{po}} = 1 \qquad (55)$$

interaction of moment, torsion

Note that if applied bending moment M_p is negative, the upper stringers could become critical since the tension due to a negative bending moment on the top stringers would add to the tension on those stringers due to torsion

Given again $F_{yL} > F_{yU}$ but F_{yU} may govern due to $\ominus M$

$M_p = -2h\gamma F_{yU}$ where $(1-\gamma) F_{yU}$ is available for T_p

$M_{po} = 2 F_{yL} h$ (Definition)

$$T_p = 2A_o \sqrt{\frac{4(1-\gamma) F_{yU} s_y}{ut}}$$

$$T_{po} = 2A_o \sqrt{\frac{4 F_{yU} s_y}{ut}} \qquad M_p = 0$$

Tpo occurs when there is no moment

From equation for M_p above, $\gamma = -\dfrac{M_p}{2h F_{yU}}$

$$\left(\frac{T_p}{T_{po}}\right)^2 = \frac{(2A_o)^2\left(\sqrt{\frac{4(1-\gamma) F_{yU} s_y}{ut}}\right)^2}{(2A_o)^2\left(\sqrt{\frac{4 F_{yU} s_y}{ut}}\right)^2} = 1-\gamma$$

$$\left(\frac{T_p}{T_{po}}\right)^2 = 1-\gamma = 1 - \left(-\frac{M_p}{2h F_{yU}}\right) = 1 + \frac{M_p}{\cancel{2h} F_{yU}} \frac{\cancel{2} F_{yL} \cancel{h}}{M_{po}}$$

$$\left(\frac{T_p}{T_{po}}\right)^2 = 1 + \frac{F_{yL}}{F_{yU}} \frac{M_p}{M_{po}} \qquad (56)$$

Yielding of Upper Stringers

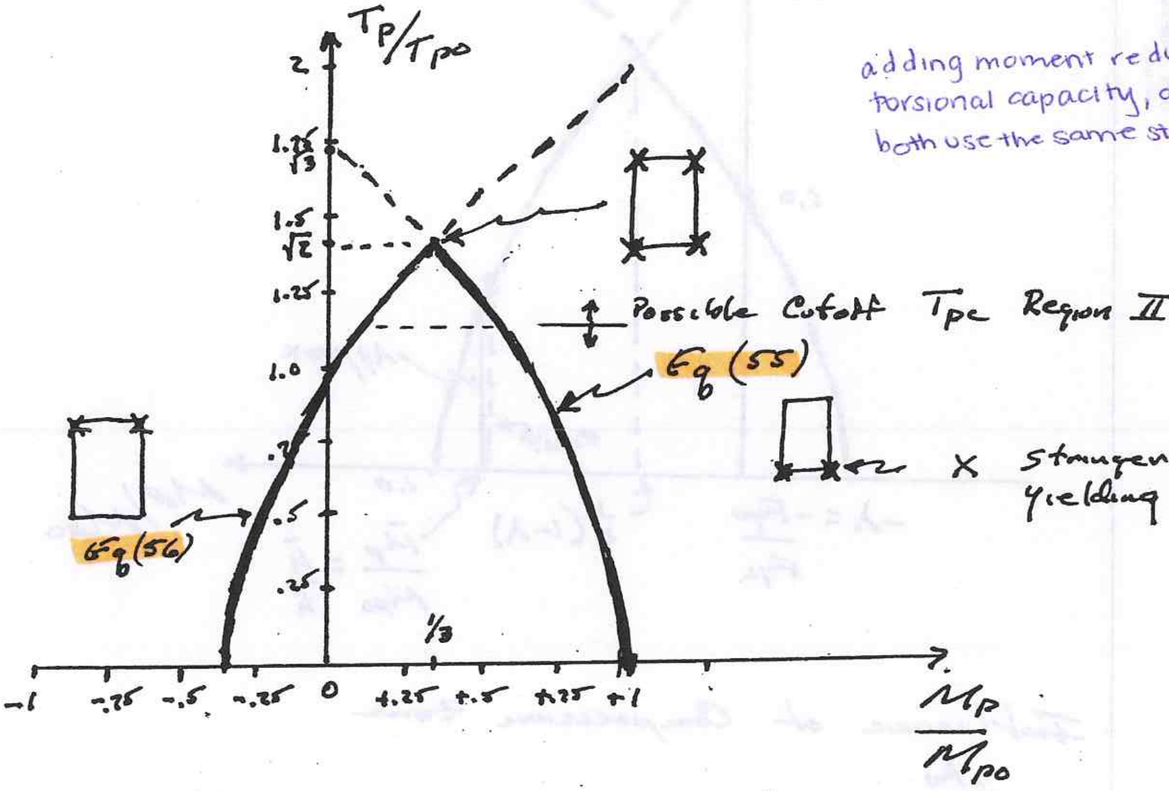
$F_1 = F_7 = F_{yU}$

Example – $F_{yL}/F_{yv} = 3$ depends on capacity!

Eq (55) $\left(\frac{T_p}{T_{po}}\right)^2 \frac{1}{3} + \frac{M_p}{M_{po}} = 1 \quad \therefore \quad \frac{T_p}{T_{po}} = \sqrt{3\left(1 - \frac{M_p}{M_{po}}\right)}$

Eq (56) $\left(\frac{T_p}{T_{po}}\right)^2 = 1 + 3\frac{M_p}{M_{po}} \quad \therefore \quad \frac{T_p}{T_{po}} = \sqrt{1 + \frac{3M_p}{M_{po}}}$

adding moment reduces torsional capacity, as both use the same steel



This same solution is given Kinematically

Every ratio F_{yL}/F_{yv} results in a new interaction curve

Cutoff T_{pc} depends on ratio $\xi_c = S_y/t d f_c$. For $\alpha_c < \pi/4$ stirrup reinforcement is governing

Other cutoff for $\alpha_c > \pi/4$ if longitudinal reinforcement is governing -- this occurs with very strong stirrups

$$\frac{T_{pc}}{T_{po}} = \frac{2 A_o F_{pc}}{T_{po}} = \frac{2 A_o f_c d \sin\alpha \cos\alpha}{2 A_o \sqrt{\frac{4 F_{yv} S_y}{v t}}} = \frac{f_c d \sin\alpha \cos\alpha}{2\sqrt{\frac{F_{yv} S_y}{v t}}}$$

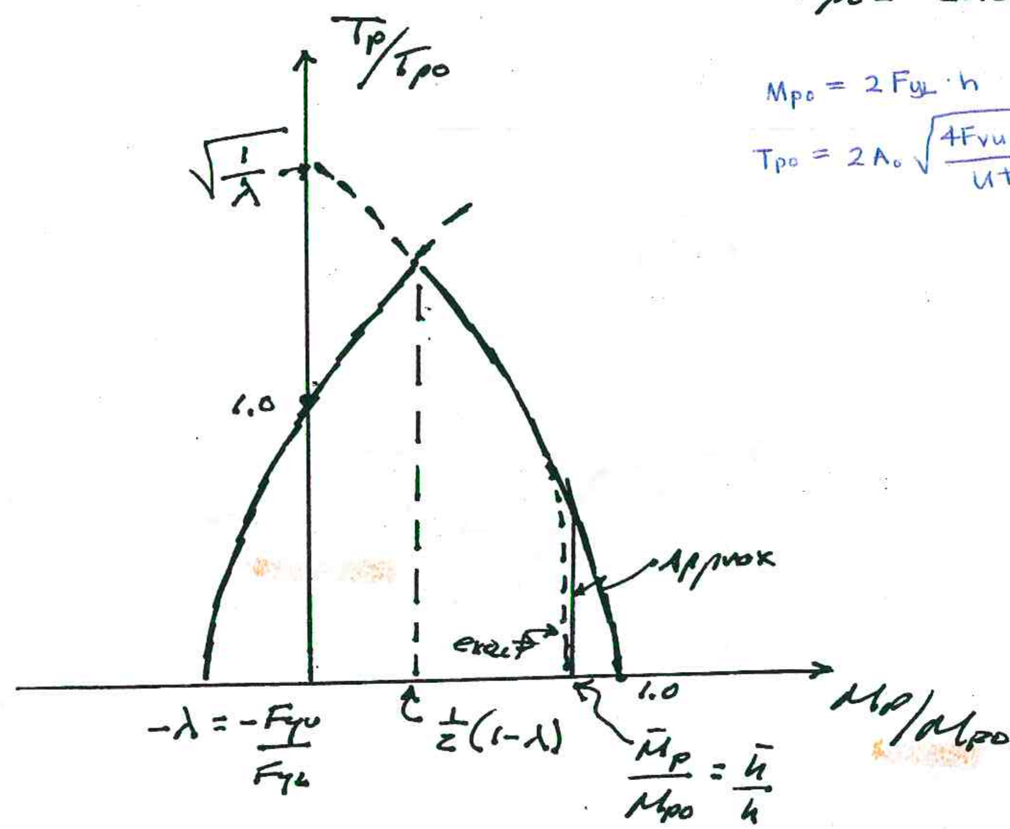
General Case: $F_{yu}/F_{yL} = \lambda \leq 1$ (otherwise denominator F_{yL}/F_{yu})

$$M_{po} = 2 F_{yL}\, h \quad (7)$$

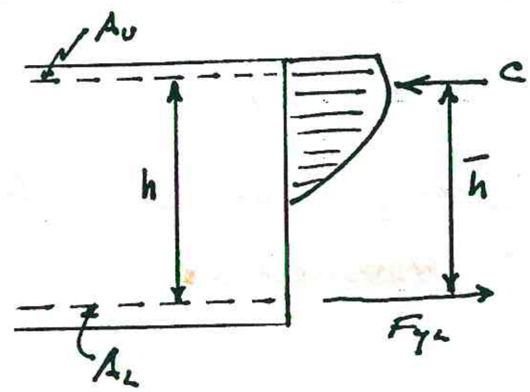
$$T_{po} = 2A_o\sqrt{\frac{4F_{yu}S_y}{ut}} \quad ($$

$$M_{po} = 2F_{yL}\cdot h$$

$$T_{po} = 2A_o\sqrt{\frac{4F_{vu}S_v}{ut}}$$



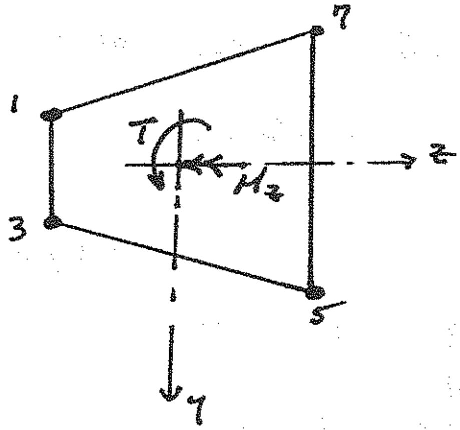
Influence of Compression Zone

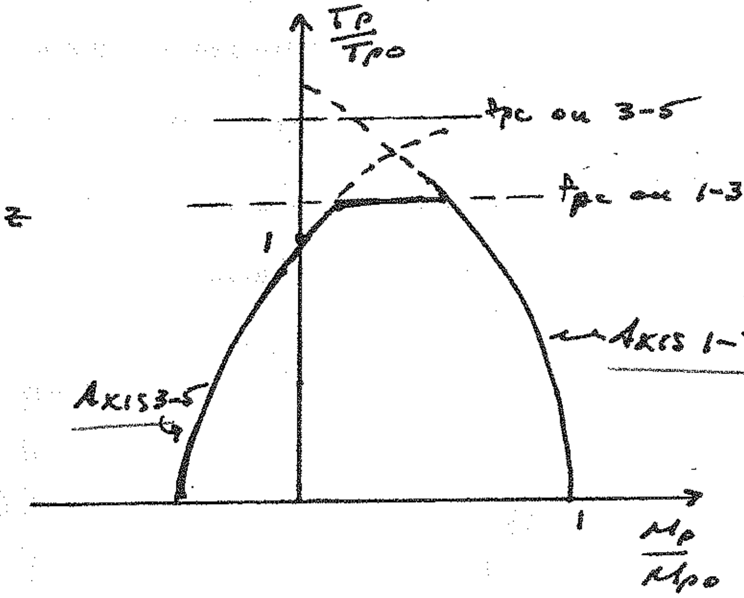


$\bar{h} < h \qquad \bar{M}_{po} < M_{po}$

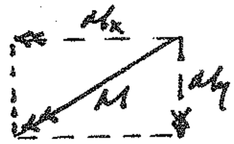
$$\frac{\bar{M}_{po}}{M_{po}} < 1$$

Irregular Cross Section -- General Case





New Interaction Curve for each combination

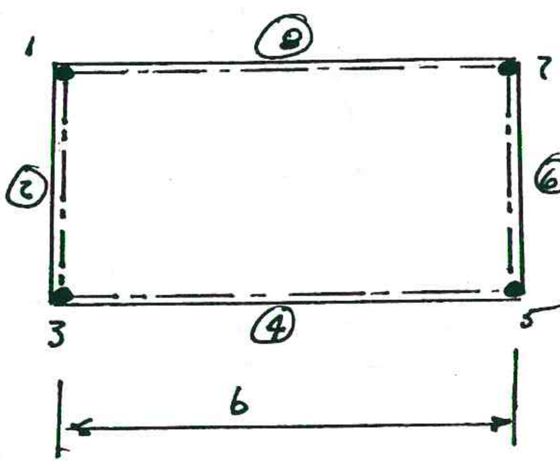


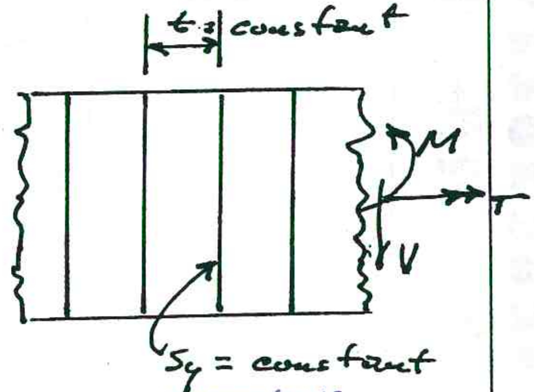
M_x / M_y = constant

ACI Chapter 11

Note: For most general case axial compression or axial tension can be included -- stringer forces will be affected -- See EQ (3) and EQ (4)

For simplicity reasons this presentation will be limited to a rectangular box section:

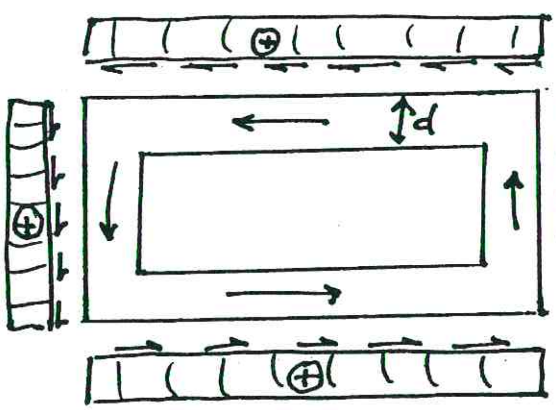


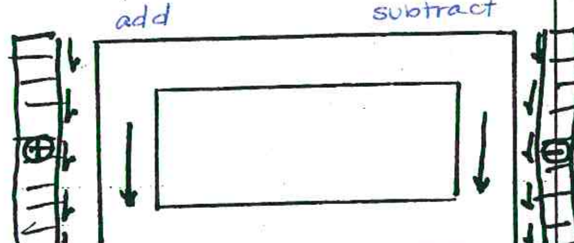


$A_o = bh$ (area) $\quad v = 2(h+b)$ (perimeter, P_h in ACI)

$F_{y1} = F_{y7} = F_{yu}$

$F_{y3} = F_{y5} = F_{yL} > F_{yu}$ (Positive Moment Bending Reinf.)





one web is more critical than other

Eq (32) $\quad \bar{f}(T) = (\tau d) = \dfrac{T}{2A_o}$ $\qquad \bar{f}(V) = \pm \dfrac{V}{2h}$ in vertical walls only

shear flow in walls

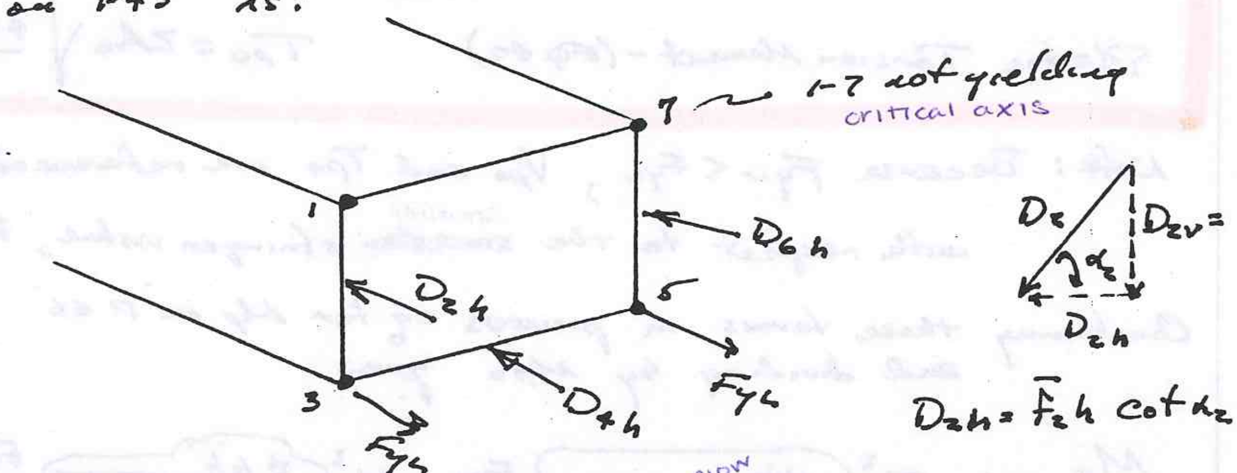
critical!! $\bar{f}_{②} = \dfrac{T}{2A_o} + \dfrac{V}{2h}$ $\qquad \bar{f}_{⑥} = \dfrac{T}{2A_o} - \dfrac{V}{2h}$ $\Big\}$ (5)

$\bar{F}_{④} = \bar{F}_{⑥} = \dfrac{T}{2A_o}$

Shear Flow due to Torsion $\qquad$ Shear Flow due to Shear

At collapse there will be yielding at all stringers except for two stringers on ~~centroid~~ critical axis. For most cases all stirrups will also yield

The torsion moment about axis 1-7 is given beginning on P45 as:



$D_{zh} = \bar{F}_z h \cot\alpha_z$

Inclination α_z from EQ (35) with $s = s_y$ (or EQ (58))

shear flow

$$\cot\alpha_z = \frac{\bar{F}_z\, t}{s_y} \qquad (58)$$

spacing

f_y of stirrups

α can now be determined from actual stirrup geometry.

Therefore

$$M_{1-7} = M_p = 2F_{yL}h - \bar{F}_2 h\cot\alpha_2\cdot\frac{h}{2} - \bar{F}_4 b\cot\alpha_4\cdot h - \bar{F}_6 h\cot\alpha_6\frac{h}{2}$$

Replacing $\cot\alpha$ by EQ (58) and $\bar{F}$ by EQ (57) gives

$$M_p = 2hF_{yL} - \left(\frac{T_p}{2A_0}+\frac{V_p}{2h}\right)^2\cdot\frac{h^2}{2}\cdot\frac{t}{s_y} - \left(\frac{T_p}{2A_0}\right)^2 b\frac{h\,t}{s_y} - \left(\frac{T_p}{2A_0}-\frac{V_p}{2h}\right)^2\cdot\frac{h^2}{2}\cdot\frac{t}{s_y}$$

$$M_p = 2hF_{yL} - \frac{th}{s_y}\left[\left(\frac{T_p}{2A_0}\right)^2\underbrace{\left(\frac{h}{2}+b+\frac{h}{2}\right)}_{h+b=u/2} + \left(\frac{V_p}{2h}\right)^2\left(\frac{h}{2}+\frac{h}{2}\right)\right]$$

Introducing Previously Defined Terms

Perimeter $\quad u = 2(h+b)$

Plastic Moment – (EQ 11 × 2) ← 2 stringers $\quad M_{po} = 2h F_{yL}$

Plastic shear Force – (EQ 12 × 2) ← 2 webs $\quad V_{po} = 2\sqrt{2 F_{yu} S_y \frac{h}{t}}$

Plastic Torsion Moment – (EQ 40) $\quad T_{po} = 2A_o \sqrt{\frac{4 F_{yu} S_y}{ut}}$

Note: Because $F_{yu} < F_{yL}$, V_{po} and T_{po} are referenced with respect to the ~~smaller~~ smaller stringer value, F_{yu}

Combining these terms in previous eq for M_p on P 66 and dividing by M_{po} gives

$$\frac{M_p}{M_{po}} = 1 - T_p^2 \underbrace{\frac{u+h}{(2A_o)^2 \cdot 2S_y \cdot 2h F_{yL}}}_{1/T_{po}^2} \frac{F_{yu}}{F_{yL}} - V_p^2 \underbrace{\frac{t\,h}{(2h)^2 \cdot S_y \cdot 2h F_{yL}}}_{1/V_{po}^2} \frac{F_{yu}}{F_{yL}}$$

$F_3 = F_5 = F_{yL}$

This gives

positive bending/moment

$$\boxed{\left(\frac{T_p}{T_{po}}\right)^2 + \left(\frac{V_p}{V_{po}}\right)^2 = \frac{F_{yL}}{F_{yu}}\left(1 - \frac{M_p}{M_{po}}\right)} \qquad (60)$$

If the moment M is relatively small or negative, the upper stringers will yield $F_1 = F_7 = F_{yu}$. For this case the moment must be taken with respect to axis 3-5

The Result is

$$\left(\frac{T_p}{T_{po}}\right)^2 + \left(\frac{V_p}{V_{po}}\right)^2 = 1 + \frac{F_{yL}}{F_{yu}} \frac{M_p}{M_{po}} \qquad (61)$$

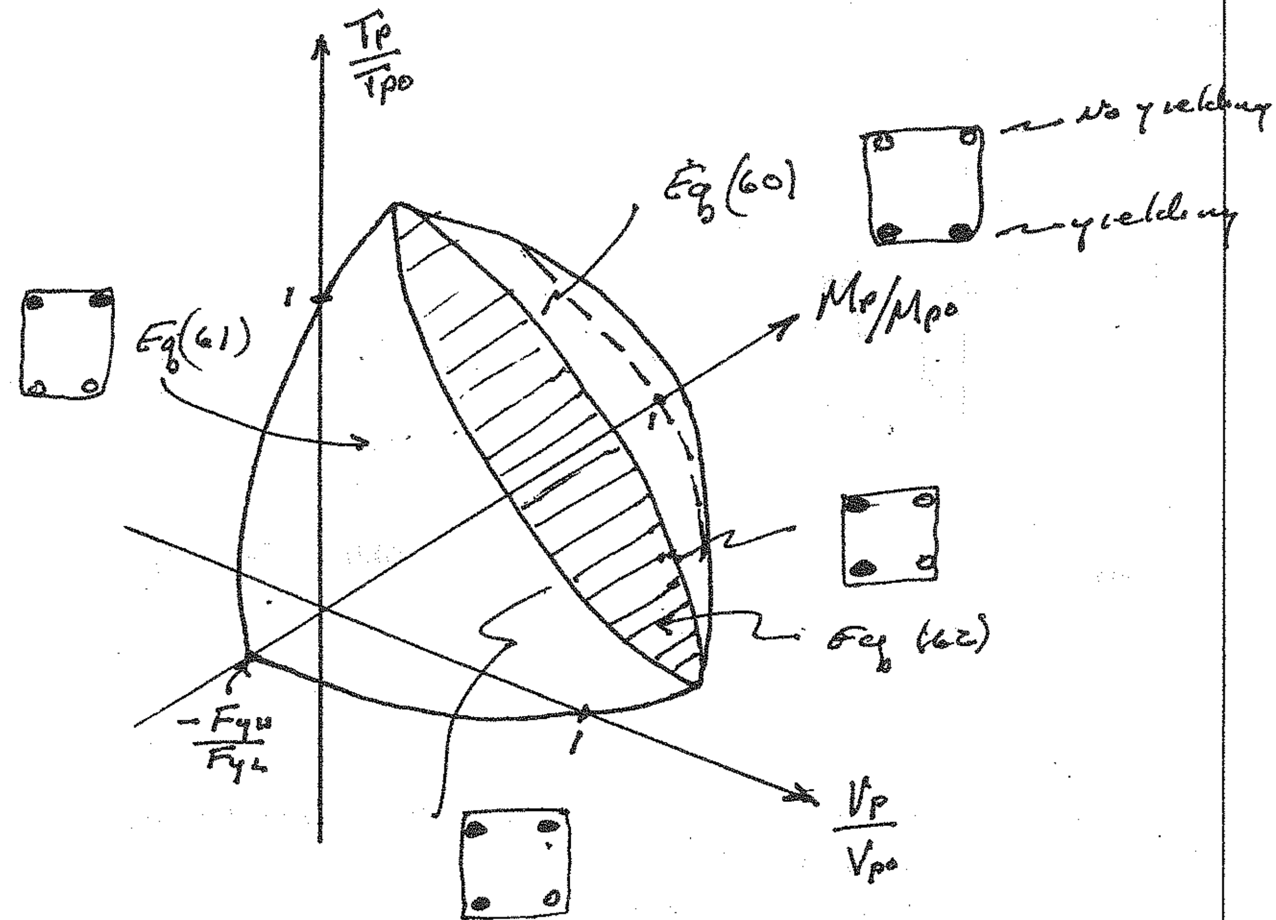
small / negative moment

No similar equation in the ACI code (moment, shear, AND torsion).

Note: if $\frac{M_p}{M_{po}} = \frac{-F_{yu}}{F_{yL}}$, right side $=0$

Additionally it is possible that stringers 1 and 3 could yield since $\bar{F}(T)$ and $\bar{F}V$ are additive on side (2). The corresponding moment equation would be written with respect to axis 5-7. This yields:

$$\left(\frac{T_p}{T_{po}}\right)^2 + \frac{2\,T_p V_p}{T_{po} V_{po}} \cdot \sqrt{\frac{2h}{w}} + \left(\frac{V_p}{V_{po}}\right)^2 = \frac{1}{2}\left(\frac{F_{yL}}{F_{yu}} + 1\right) \quad (62)$$



Note: Assumes an underreinforced section such that $\sigma_c > -f_c$. As before limits for the case $\sigma_c = -f_c$ are needed. In design this is not a problem since it can be checked that $\sigma_c > -f_c$ -- If not section is changed to increase wall size or f_c' is increased

Concrete Stress σ_c

Box Section – Use Eq 2, 34

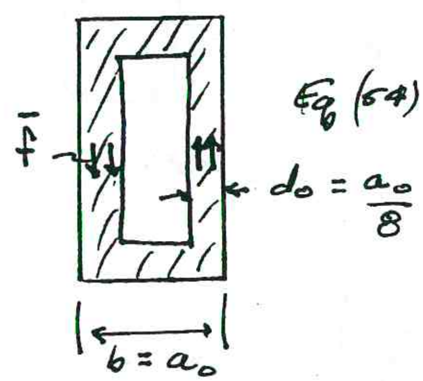
$$\sigma_c = -\frac{\bar{f}}{d}\frac{1}{\sin\alpha\cos\alpha} = -\frac{2\bar{f}}{d}\frac{1}{\sin 2\alpha} \geqq -f_c \qquad (63)$$

d = wall thickness – must be $\geqq d_o$ – See Eq (54)

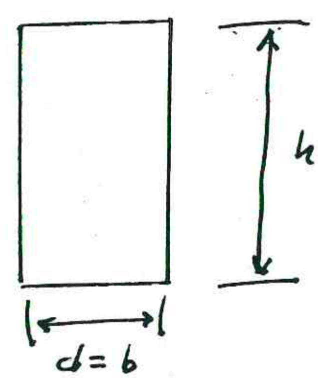
Eq (57) $\quad \bar{f} = \bar{f}(T) + \bar{f}(V) = \frac{T}{2A_o} \pm \frac{V}{2h}$

Solid Sections

Torsion



Shear



Eq (34) $\quad \sigma_c = -\frac{\bar{f}}{d_o}\frac{1}{\sin\alpha\cos\alpha}$

Eq (2) $\quad \sigma_c \doteq \frac{-V}{bh}$

Conditions $\qquad \sigma_c(T) + \sigma_c(V) \geqq -f_c \qquad (64)$

Note: Torsion and Shear are both statically admissible stress fields. If superposition does not violate yield conditions $\sigma_c \geqq -f_c$ gives a lower bound. No upper bound solutions for M, T, V combination have yet been developed

Bending & Torsional Stiffness have been studied by Thurlimann & by Collins and Mitchell. However they are not directly related to rigid-plastic theory