
Department of Civil, Architectural and Environmental Engineering
University of Texas at Austin
Spring 2006

DYNAMIC RESPONSE OF STRUCTURES
CE 384P – Unique No. 14675

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Class schedule: Tue/Thu 9:30-11am (ECJ 5.410)
Office hours: Mon 2-4pm, Tue 2:30-4pm, Wed 10am-noon
Other times may always be arranged by appointment.

Textbook: A. K. Chopra, "Dynamics of Structures: Theory and Applications to Earthquake Engineering," Prentice Hall, Inc., Second Edition, 2001.
0-13-086973-2

Reference books: R. W. Clough and J. Penzien, "Dynamics of Structures," McGraw-Hill, New York, 1993.
G. C. Hart and K. Wong, "Structural Dynamics for Structural Engineers," John Wiley & Sons, Inc., 2000.
J. W. Tedesco, W. G. McDougal, and C. A. Ross, "Structural Dynamics: Theory and Applications," Addison-Wesley Longman, Inc., 1999.

Course Web site: http://www.ce.utexas.edu/prof/Manuel/Spring2006_CE384P/home.htm

Course objectives

- To learn methods for analyzing structures that are subjected to dynamic excitation (including computation of displacements, forces, stresses, etc.)
- To understand the analytical procedures in a manner that emphasizes physical insights.
- To be able to apply the theory of structural dynamics theory to practical problems, especially in earthquake analysis and design of structures (using appropriate structural idealizations to represent real structures).

Prerequisites

Differential equations; linear algebra; matrix structural analysis; mechanics of materials. Also, some computer programming knowledge (e.g., in C or Fortran) or familiarity with mathematical software packages such as Matlab, Mathcad, Maple, etc. is required.

Course Outline

The following is a tentative list of topics that will be covered in this course:

1. Introduction to the subject of structural dynamics
   Differences between static, quasi-static, and dynamic analyses
   Why study dynamics? When is a dynamic analysis necessary?
2. Single-degree-of-freedom (SDOF) systems
   Formulation of equations of motion
   Free vibration
   Response to harmonic and periodic loading

    Unit impulse response; response to arbitrary loading; response to step and pulse excitations; shock spectra
    Numerical methods
    Response to seismic excitation; earthquake response spectra, design spectra

3. Generalized SDOF systems
    - Systems with distributed mass and elasticity; lumped-mass systems
    - Rayleigh's method for estimating natural frequencies
    - Selection of shape functions
4. Multi-degree-of-freedom (MDOF) systems
    - Equations of motion
    - Free vibration; natural vibration frequencies and mode shapes
    - Modal analysis of linear systems
    - Damping in structures
    - Response of linear systems to seismic excitation
    - Response spectrum analysis
5. Special topics (if time permits and there is interest in any of these; interspersed throughout the semester): Frequency domain methods; structural control of vibrations; wind engineering; offshore structures; floor vibration; blast/explosion loading.

Electronic Reserves web site (for handouts)

http://reserves.lib.utexas.edu/eres/coursepage.aspx?cid=3261

Grading

Homework	20%	
Exam 1	25%	(tentative date: March 2)
Exam 2	25%	(tentative date: April 13)
Final Exam	30%	(Saturday, May 10, 9am-noon)

There will be 10-12 homework assignments in this course. Homework will be collected at the start of class on the due date. Generally, no credit will be given for late homework. The dates for the two exams during the semester are subject to change.
A missed exam will be excused only in special circumstances (e.g., medical reasons). In such cases, please make every attempt to notify the instructor of absence <u>before</u> the scheduled exam time.

Course evaluation

Students will be asked to evaluate the course and the instructor using the approved forms from the Measurement and Evaluation Center.

Disabilities

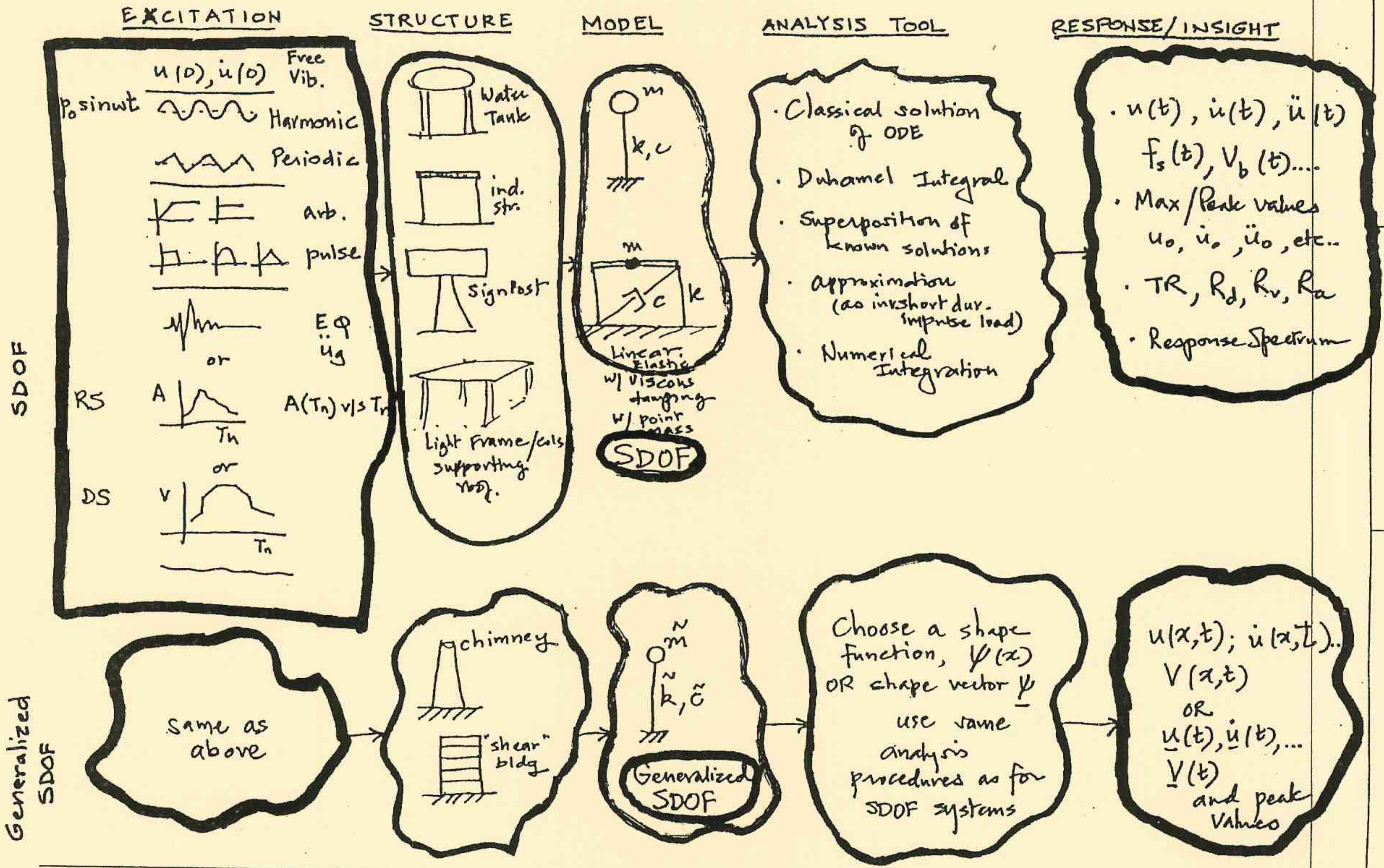
The University of Texas at Austin provides, upon request, appropriate academic adjustments for qualified students with disabilities. For more information, contact the Office of the Dean of Students at 471-6259, 471-4241 TDD or the College of Engineering Director of Students with Disabilities at 471-4321.

Important Dates

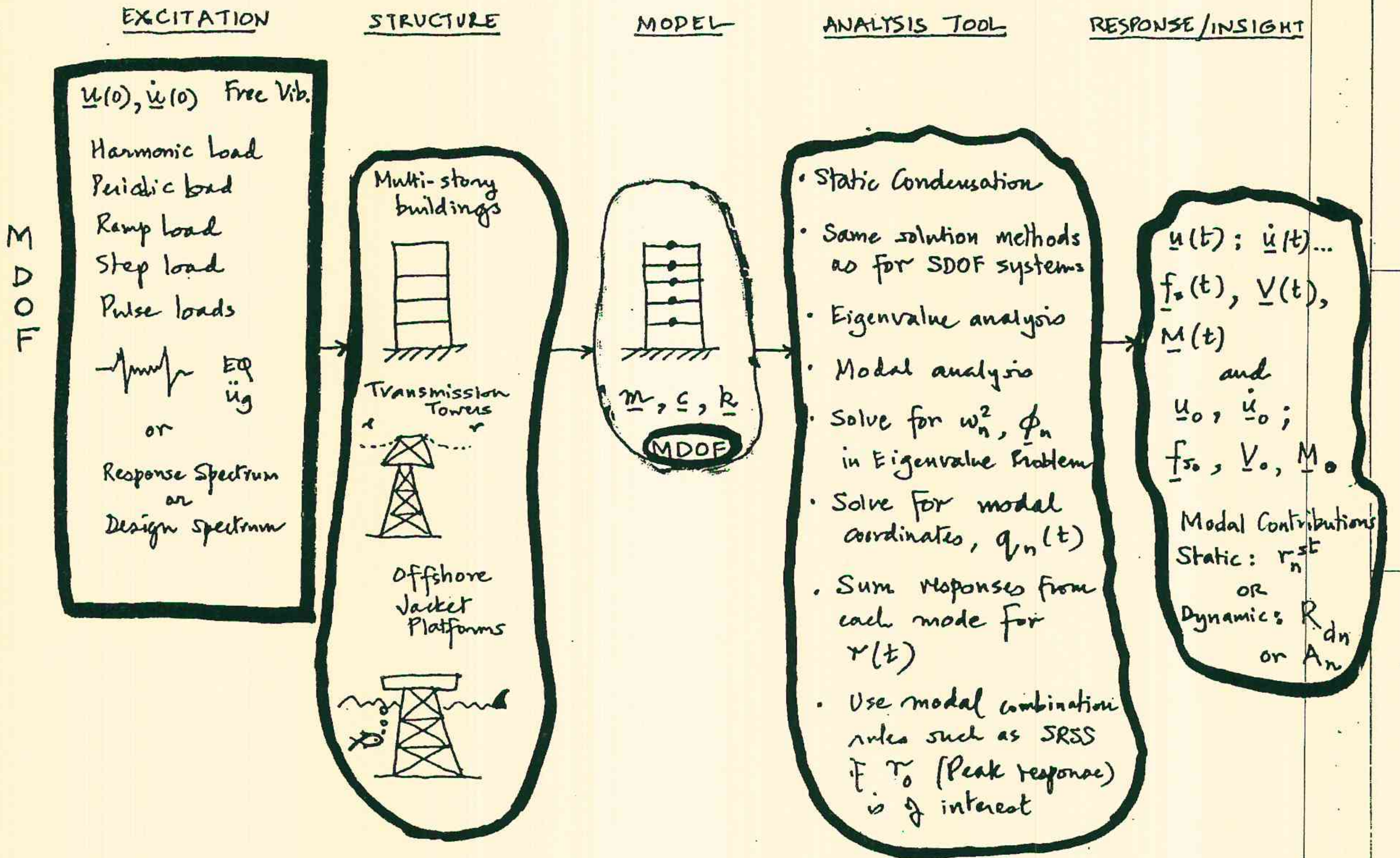
Feb 1	Last day to add/drop
March 13-17	Spring break
May 8-9	No-class days
May 10	Final exam.

National Brand

STRUCTURAL DYNAMICS - AN OVERVIEW



(OVERVIEW - Continued)



$\underline{u}(0), \dot{\underline{u}}(0)$ Free Vib.

Harmonic Load

Periodic load

Ramp load

Step load

Pulse loads

EQ $\ddot{u}_g$

or

Response Spectrum or Design spectrum

Multi-story buildings

Transmission Towers

Offshore Jacket Platforms

$\underline{m}, \underline{c}, \underline{k}$

MDOF

- Static Condensation
- Same solution methods as for SDOF systems
- Eigenvalue analysis
- Modal analysis
- Solve for $\omega_n^2, \underline{\phi}_n$ in Eigenvalue Problem
- Solve for modal coordinates, $q_n(t)$
- Sum responses from each mode for $r(t)$
- Use modal combination rules such as SRSS if r_0 (Peak response) is of interest

$\underline{u}(t)$; $\dot{\underline{u}}(t)$...

$\underline{f}_s(t)$, $\underline{V}(t)$, $\underline{M}(t)$

and

$\underline{u}_o$, $\dot{\underline{u}}_o$;

$\underline{f}_{so}$, $\underline{V}_o$, $\underline{M}_o$

Modal Contributions

Static: r_n^{st}

OR

Dynamic: R_{dn} or A_n

Chapters 1 & 2: Free Vibration

Frame Properties

$$k = \frac{24EI_c}{h^3}\frac{12\rho+1}{12\rho+4}$$

Generally, k=12EI/L^3

$$T_n = \frac{2\pi}{\omega_n}$$

$$f_n = \frac{1}{T_n}$$

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}} = \sqrt{\frac{k}{m}}$$

Damping, c, in force-time/length

$$\zeta = \frac{c}{2m\omega_n} = \frac{c}{c_{cr}}$$

Less than 1.0, underdamped

Chapter 3: Forced Motion

Steady state

$$u(t) = \frac{p_o}{k}\frac{1}{1-\left(\omega/\omega_n\right)^2}\sin(\omega t)$$

Deformation response factor, R_d

$$R_d = \frac{u_o}{(u_{st})_o} = \frac{1}{\sqrt{\left(1-\left(\omega/\omega_n\right)^2\right)^2 + \left(2\zeta\,\omega/\omega_n\right)^2}}$$

$$\frac{R_a}{\omega/\omega_n} = R_v = \frac{\omega}{\omega_n}R_d$$

Transmissibility ratio, TR

$$TR = \sqrt{\frac{1+\left(2\varsigma\,\omega/\omega_n\right)^2}{\left(1-\left(\omega/\omega_n\right)^2\right)^2 + \left(2\varsigma\,\omega/\omega_n\right)^2}}$$

Fourier series – page 114

Chapter 4: Arbitrary loadings

Duhamel integral – page 129

Summary of force responses – page 151

$$\mathrm{Im} = \int_0^{t_d} p(t)dt$$

$$u_o = \frac{\mathrm{Im}}{m\omega_n} = \frac{\mathrm{Im}}{k}\frac{2\pi}{T_n}$$

Chapter 5: Numerical Evaluations

Interpolation of Excitation – pg 169

Initial calculations

$$e^{-\xi\omega_n\Delta t}$$

$$\omega_D = \omega_n\sqrt{1-\xi^2}$$

$$\sin(\omega_D\Delta t)$$

$$\cos(\omega_D\Delta t)$$

$$u_{i+1} = Au_i + Bu_i' + Cp_i + Dp_{i+1}$$

$$u_{i+1}' = A'u_i + B'u_i' + C'p_i + D'p_{i+1}$$

Central Difference Method – pg 173

Initial calculations

$$u_o'' = \frac{p_o - cu_o' - ku_o}{m}$$

$$u_{-1} = u_o - \Delta t u_o' + \frac{\Delta t^2}{2}u_o''$$

$$khat = \frac{m}{\Delta t^2} + \frac{c}{2\Delta t}$$

$$a = \frac{m}{\Delta t^2} - \frac{c}{2\Delta t}$$

$$b = k - \frac{2m}{\Delta t^2}$$

Newmark's Method – pg 177

γ = 0.5

Average acceleration, β = 0.25

Linear acceleration, β = 1/6

Initial calculations

$$u_o'' = \frac{p_o - cu_o' - ku_o}{m}$$

$$khat = k + \frac{\gamma}{\beta\Delta t}c + \frac{1}{\beta\Delta t^2}m$$

$$a = \frac{1}{\beta\Delta t}m + \frac{\gamma}{\beta}c$$

$$b = \frac{1}{2\beta}m + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)c$$

Stability

Central Difference – pg 172

$$\frac{\Delta t}{T_n} \le \frac{1}{\pi}$$

Newmark's – pg 177

$$\frac{\Delta t}{T_n} \le \frac{1}{\pi\sqrt{2}}\frac{1}{\sqrt{\gamma - 2\beta}}$$

For average acceleration, always stable (but not accurate for large Δt)

For linear acceleration,

$$\frac{\Delta t}{T_n} \le 0.551$$

Chapter 6: Earthquake Response

Pseudo-response – pg 209

$$D = u_o$$

$$V = \omega_n D = \frac{2\pi}{T_n} D$$

$$A = \omega_n{}^2 D = \left(\frac{2\pi}{T_n}\right)^2 D$$

$$f_s = m\omega_n^2 u(t) = mA(t)$$

Structural Response – pg 217

$$u_o = D = \omega_n V = \omega_n^2 A$$

$$f_{so} = kd = mA$$

U

$$V_{bo} = kD = mA, M_{bo} = hV_{bo}, V_{bcolumn} = k_{col} u_o$$

neven column example, pg 221

Elastic Design Spectrum – pg 231

Important points:

T_a = 0.035s
T_b = 0.125s
T_c = 0.5s*
T_d = 3.0s*
T_e = 10s
T_f = 15s

*vary with damping

Damped Systems – refer back to earlier chapters

$$R_d = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}}$$

$$z_o = \frac{pt_o}{kt} R_d$$

General Equations

$$mtz'' + ctz' + ktz = -Ltu_g''(t) = pt(t)$$

$$u_{go}'' = A$$

for small values of T_n

Chapter 8: Generalized SDOF Systems

Lots of equations – pg 312+

Small ω_n, more accurate ψ

$$mt = \int_0^L m(x)(\psi(x))^2 \, dx$$

$$kt = \int_0^L EI(x)(\psi''(x))^2 \, dx$$

$$Lt = \int_0^L m(x)\psi(x) dx$$

$$\Gamma t = \frac{Lt}{mt}, \omega_n{}^2 = \frac{kt}{mt}$$

$$f_s(x,t) = \omega_n{}^2 m(x)\psi(x)z(t)$$

$$f_o(x) = \Gamma t m(x)\psi(x)A$$

$$u_o(x) = \Gamma t D\psi(x) = \psi(x) z_o$$

Shear and Bending

$$V_o(x) = \Gamma t A \int_x^L m(\chi)\psi(\chi) d\chi$$

$$M_o(x) = \Gamma t A \int_x^L (\chi - x) m(\chi)\psi(\chi) d\chi$$

Lumped Mass Systems – pg 324

$$mt = \sum m_j \psi_j{}^2$$

$$kt = \sum k_j (\psi_j - \psi_{j-1})^2$$

$$Lt = \sum m_j \psi_j$$

$pt = \Sigma p_i \psi_i$

$\mathcal{I}t = \Sigma \mathcal{I}_i \psi_i$

Shear and Bending

$$f_{jo} = \Gamma t m_j \psi_j A$$

$$V_b = \sum k\psi z_o = \sum \omega_n^2 m\psi z_o$$

$$V_{jo} = k_j(u_{jo} - u_{j-1}) = k_j z_o (\psi_j - \psi_{j-1})$$

$$V_{io} = \sum f_{jo}, M_{io} = \sum (h_j - h_i) f_{jo}$$

$$V_{bo} = Lt\Gamma tA, M_{bo} = \Gamma tA \sum h_j m_j \psi_j$$

Reduce things by ψ

Peak Earthquake Response – pg 313

$$z_o = \Gamma t D = \frac{\Gamma t}{\omega_n^2} A$$

$$f_o(x) = \Gamma t m(x)\psi(x)A$$

Γ matrix only used in earthquake response

small T_n, $A = \ddot{u}_{go}$

large T_n, $D = u_{go}$

Chapter 9: MDOF General Equations

Similar equations, but use matricies

$$[m][u''] + [c][u'] + [k][u] = [p(t)]$$

Chapter 10: MDOF Free Vibration

Eigenvalue calculation

$$\left[[k] - \omega_n^2[m]\right]\phi_n = 0, \det[[k] - \omega_n^2[m]] = 0$$

Normalization of modes

$$M_n = \phi_n^T [m]\phi_n$$

$$K_n = \phi_n^T [k]\phi_n$$

Free vibration response

$$q_n(0) = \frac{\phi_n[m][u(0)]}{M_n}$$

$$u_n(t) = \phi_n q_n(t)$$

$$[u(t)] = \sum_{n=1}^{N} \phi_n \left[q_n(0)\cos(\omega_n t) + \frac{q_n^{'}(0)}{\omega_n}\sin(\omega_n t) \right]$$

Classically damped – page 72 (forced). Free: pg 427

$$r = \omega / \omega_n$$

$$C = \frac{p_o}{k}\frac{1-r^2}{(1-r^2)^2 + (2r\zeta)^2}$$

$$D = \frac{p_o}{k}\frac{-2r\zeta}{(1-r^2)^2 + (2r\zeta)^2}$$

General procedure – page 477

Modify forcing function by mode shape

$$P_{no} = \phi_n^T p_n(t)$$

Replace u_{st} (p_o/k) with P_{no}/K_n in C, D equations

Solve for $q_n(t)$

$$q_n(t) = C\sin(\omega t) + D\cos(\omega t)$$

Multiply $q_n(t)$ by mode shape to get $u_n(t)$

Shears and Moments

$$f_{sn}(t) = \omega_n^2[m][\phi_n]q_n(t)$$

Shear on floor equals sum of shears on that floor and floors above

Chapters 12 & 13: Dynamic and Earthquake Analysis

$$r(t) = \sum_{n=1}^{N} r_n(t)$$, where r = any calculation

Spatial vector

$$s = [m][1]$$, where [1] is a column of ones

$$s_n = \Gamma_n[m]\phi_n, s = \sum s_n$$

$$P(t) = [s]p(t)$$

$$\Gamma_n = \frac{\phi_n^T[s]}{M_n}$$

Earthquake loads

Find A from data, calculate D using ω_n

$$q_n(t) = \Gamma_n D_n(t)$$

$$f_n(t) = s_n[\omega_n^2 D_n(t)]$$

Generalized calculations, r(t)

$$r_n(t) = r_n^{st} A_n(t)$$

$$r_{no} = p_o r_n^{st} R_{dn}$$ WRONG!

[redacted] $$u_n^{st} = \frac{\Gamma_n}{\omega_n^2}\phi_n$$

$$peak = \sqrt{\sum r_{no}^2}$$

TABLE 13.2.1 MODAL STATIC RESPONSES

Response, r	Modal Static Response, r_n^{st}
V_i	$V_{in}^{st} = \sum_{j=i}^{N} s_{jn}$
M_i	$M_{in}^{st} = \sum_{j=i}^{N} (h_j - h_i)s_{jn}$
V_b	$V_{bn}^{st} = \sum_{j=1}^{N} s_{jn} = \Gamma_n L_n^h \equiv M_n^*$
M_b	$M_{bn}^{st} = \sum_{j=1}^{N} h_j s_{jn} = \Gamma_n L_n^\theta \equiv h_n^* M_n^*$
u_j	$u_{jn}^{st} = (\Gamma_n/\omega_n^2)\phi_{jn}$
Δ_j	$\Delta_{jn}^{st} = (\Gamma_n/\omega_n^2)(\phi_{jn} - \phi_{j-1,n})$

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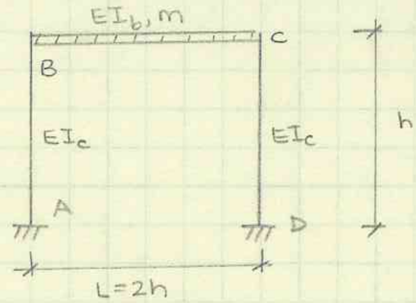
HOMEWORK #1

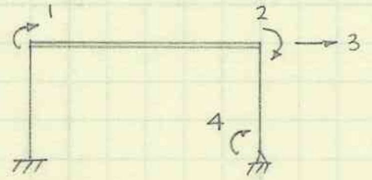
15/15

1.



If $I_c = I_b$, $k = \frac{96}{7}\frac{EI}{h^3}$ ✓

i.



$F_3 = f_s$

$$K = EI\begin{bmatrix} \frac{4}{h}+\frac{4}{2h} & \frac{2}{2h} & \frac{-6}{h^2} & 0 \\ & \frac{4}{h}+\frac{4}{2h} & -\frac{6}{h^2} & \frac{2}{h} \\ & & 2\cdot\frac{12}{h^3} & -\frac{6}{h^2} \\ & & & \frac{4}{h} \end{bmatrix}$$

$$\frac{EI}{h^3}\begin{bmatrix} 6h^2 & h^2 & -6h & 0 \\ h^2 & 6h^2 & -6h & 2h^2 \\ -6h & -6h & 24 & -6h \\ 0 & 2h^2 & -6h & 4h^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f_s \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} 6h^2 & h^2 & 0 \\ h^2 & 6h^2 & 2h^2 \\ 0 & 2h^2 & 4h^2 \end{bmatrix}^{-1}(-1)\cdot\begin{bmatrix} -6h \\ -6h \\ -6h \end{bmatrix} u_3$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} 27/29 \\ 12/29 \\ 75/58 \end{bmatrix} u_3$$

$$f_s = \begin{bmatrix} -6h & -6h & -6h \end{bmatrix}\frac{EI}{h^3}\cdot\begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} + \begin{bmatrix} 24 \end{bmatrix}\cdot u_3\cdot\frac{EI}{h^3}$$

$$\frac{f_s}{u_3} = K \qquad \boxed{\text{i. } K = \frac{237}{29}\frac{EI}{h^3}}$$

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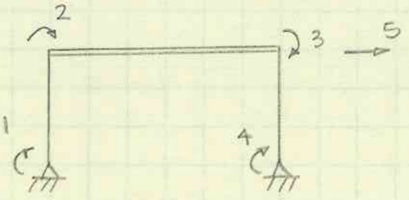
HOMEWORK #1

1. ii.



$$k = EI \begin{bmatrix} \frac{4}{h} & \frac{2}{h} & 0 & 0 & -\frac{6}{h^2} \\ & \frac{4}{h}+\frac{2}{h} & \frac{1}{h} & 0 & -\frac{6}{h^2} \\ & & \frac{4}{h}+\frac{2}{h} & \frac{2}{h} & -\frac{6}{h^2} \\ & & & \frac{4}{h} & -\frac{6}{h^2} \\ & & & & 2\cdot\frac{12}{h^3} \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = - \begin{bmatrix} 4h^2 & 2h^2 & 0 & 0 \\ 2h^2 & 6h^2 & h^2 & 0 \\ 0 & h^2 & 6h^2 & 2h^2 \\ 0 & 0 & 2h^2 & 4h^2 \end{bmatrix}^{-1} \begin{bmatrix} -6h \\ -6h \\ -6h \\ -6h \end{bmatrix} \cdot u_5 = \begin{bmatrix} 5/4 \\ 1/2 \\ 1/2 \\ 5/4 \end{bmatrix} u_5$$

$$f_s = \begin{bmatrix} -6h & -6h & -6h & -6h \end{bmatrix} \frac{EI}{h^3} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 24 \end{bmatrix} u_5 \frac{EI}{h^3}$$

$$\boxed{ii. \quad k = 3\frac{EI}{h^3}}$$

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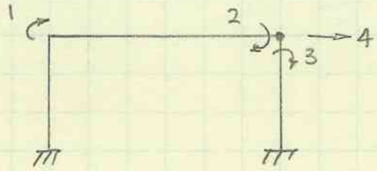
HOMEWORK #1

1. iii.



$$K = \begin{bmatrix} \frac{4}{h} + \frac{2}{h} & \frac{2}{2h} & 0 & \frac{-6}{h^2} \\ & \frac{4}{2h} & 0 & 0 \\ & & \frac{4}{h} & \frac{-6}{h^2} \\ & & & \frac{24}{h^3} \end{bmatrix}$$

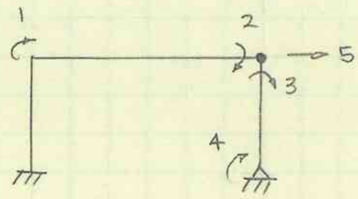
$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = - \begin{bmatrix} 6h^2 & h^2 & 0 \\ h^2 & 2h^2 & 0 \\ 0 & 0 & 4h^2 \end{bmatrix}^{-1} \begin{bmatrix} -6h \\ 0 \\ -6h \end{bmatrix} u_4 = \begin{bmatrix} 12/11 \\ -6/11 \\ 3/2 \end{bmatrix} u_4$$

$$f_s = \begin{bmatrix} -6h & 0 & -6h \end{bmatrix} \frac{EI}{h^3} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 24 \end{bmatrix} u_5 \frac{EI}{h^3}$$

$$\boxed{\text{iii. } K = \frac{93}{11} \frac{EI}{h^3}}$$

HOMEWORK #1

1. IV.



$$K = \begin{bmatrix} \frac{4}{h} + \frac{2}{h} & \frac{2}{2h} & 0 & 0 & \frac{-6}{h^2} \\ & \frac{4}{2h} & 0 & 0 & 0 \\ & & \frac{4}{h} & \frac{2}{h} & \frac{-6}{h^2} \\ & & & \frac{4}{h} & \frac{-6}{h^2} \\ & & & & \frac{24}{h^3} \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = - \begin{bmatrix} 6h^2 & h^2 & 0 & 0 \\ h^2 & 2h^2 & 0 & 0 \\ 0 & 0 & 4h^2 & 2h^2 \\ 0 & 0 & 2h^2 & 4h^2 \end{bmatrix}^{-1} \begin{bmatrix} -6h \\ 0 \\ -6h \\ -6h \end{bmatrix} u_5 = \begin{bmatrix} 12/11 \\ -6/11 \\ 1 \\ 1 \end{bmatrix} u_5$$

$$f_s = \begin{bmatrix} -6h & 0 & -6h & -6h \end{bmatrix} \frac{EI}{h^3} \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \begin{bmatrix} 24 \end{bmatrix} u_5 \frac{EI}{h^3}$$

$$\boxed{\text{IV. } k = 60/11 \frac{EI}{h^3}}$$

Summary & Comments

$K_i = 8.17$ $K_{ii} = 3.0$ $K_{iii} = 8.45$ $K_{iv} = 5.46$ fixed-fixed = 13.7

As hinges are added, stiffness decreases, being the least when both supports are pinned. A pin in the corner stops the transfer of moment through the joint, allowing the frame to sway more easily.

Additionally, as the length of members increases, the stiffness will decrease. As the moment of inertia of the system increases, the stiffness will increase.

5/5

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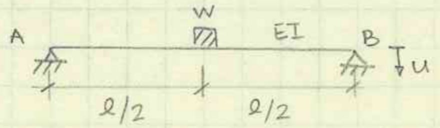
HOMEWORK #1

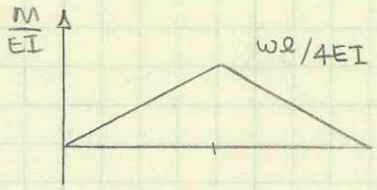
2. a)



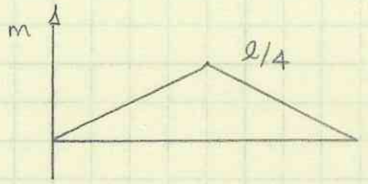
$m\ddot{u} + ku = 0$

$k = \frac{48EI}{L^3}$, $m = \frac{W}{g}$

due to symmetry,
$F_{Ay} = F_{By} = 1/2$



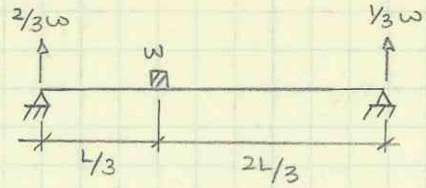
$\frac{wl}{4EI} \cdot \frac{l}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} \frac{l}{4} \cdot 2 = \frac{wl^3}{48EI} = \Delta$

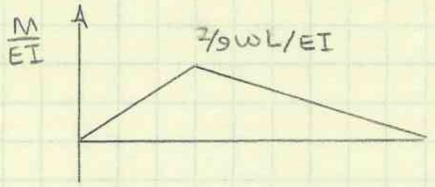


$\Delta = \frac{P}{K}$, $K = \frac{48EI}{L^3}$

$$\boxed{2.a)\ \frac{W}{g}\ddot{u} + \frac{48EI}{L^3}u = 0}$$

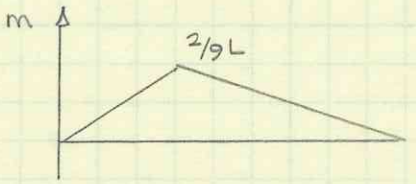
2. b)





$\Delta = \frac{2}{9}\frac{wL}{EI} \cdot \frac{L}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{9}L + \frac{2}{9}\frac{wL}{EI} \cdot \frac{2L}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{9}L$

$\frac{4wL^3}{729EI} + \frac{8wL^3}{729EI} = \frac{4wL^3}{243EI}$



$k = \frac{P}{\Delta}$, $k = \frac{243EI}{4L^3}$

$$\boxed{2.b)\ \frac{W}{g}\ddot{u} + \frac{243EI}{4L^3}u = 0}$$

Homework No. 1

Assigned: January 24, 2006
Due: January 31, 2006

1. Consider the frame shown in Figure 1. You have seen that the lateral stiffness of this frame is $96/7EI/h^3$ if $I_c = I_b$.

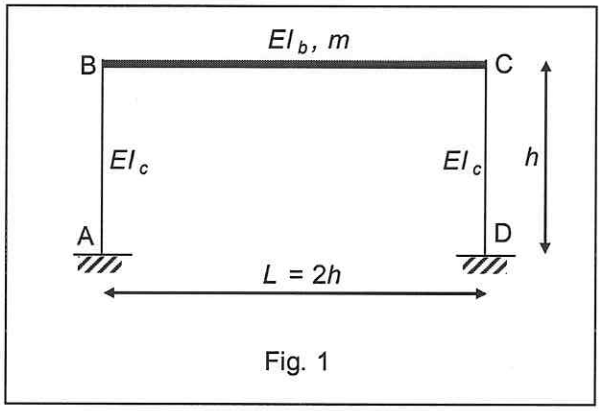


Fig. 1

Now, we will study the same frame but for different support and joint conditions.
(i) Same as Fig. 1 but with a hinged support at D
(ii) Same as Fig. 1 but with hinges at both A and D
(iii)Same as Fig. 1 but with a hinged joint at corner C
(iv)Same as Fig. 1 but with hinges at both C and D.
Determine the lateral stiffness of the frame for Cases (i) to (iv).

When you are done, comment on the influence of variation in support and joint conditions on the overall lateral stiffness of this simple frame.

2. a) Solve Problem 1.9 from the textbook.
   b) Redo Problem 1.9 for a case where the displacement of interest, u, and the weight, w, are located a distance $L/3$ from the left support instead of at the center of the beam.

CE384P-DYNAMICS

CATHERINE HOVELL
07 February 2006

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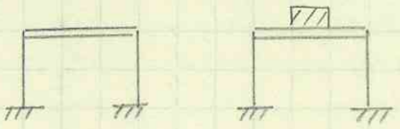
HOMEWORK #2

9/10

✓

1. Prob. 2.1



$w = 50 \text{ lb}$

$T_n = 2\pi\sqrt{\frac{m}{k}}$, $k = \left(\frac{2\pi}{T_n}\right)^2 \cdot m$

$T_{n_1} = 0.5S$ $\quad$ $T_{n_2} = 0.75S$

m $\quad$ $m + {}^{w}/_{g}$

k $\quad$ k

$$\frac{m_1}{T_{n_1}^2} = \frac{m_1 + M}{T_{n_2}^2}, \quad m_1 = \frac{T_{n_1}^2}{T_{n_2}^2}(m_1 + M)$$

$$m_1 = \frac{T_{n_1}^2/T_{n_2}^2 \, M}{1 - T_{n_1}^2/T_{n_2}^2}$$

$$m_1 = \frac{(0.5S)^2/(0.75S)^2 \, (50 \text{ lb})}{1 - (0.5S)^2/(0.75S)^2} = 40 \text{ lb}$$

$$k = \left(\frac{2\pi}{0.5S}\right)^2 \frac{40 \text{ lb}}{32.2 \text{ ft}/S^2} \cdot \frac{1 \text{ ft}}{12 \text{ in}} = 16.35 \text{ lb/in}$$

$$\boxed{\begin{array}{l} W = 40 \text{ lb} \\ k = 16.4 \text{ lb/in} \end{array}}$$

✓

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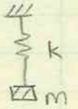
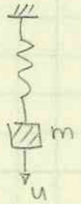
HOMEWORK #2

2. Prob 2.2





$\cancel{m}$ Mg = 400 lb

k = 100 lb/in

M = 200 lb

$$u_0 = \frac{F}{k} = \frac{200\,\text{lb}}{100\,\text{lb/in}} = 2\,\text{in}$$

$m\ddot{u} + ku = 0$ $\qquad \dot{u}_0 = 0$

$u(t) = A\cos\omega_n t + B\sin\omega_n t$

$\dot{u}(t) = -A\omega_n\sin\omega_n t + B\omega_n\cos\omega_n t$ $\qquad B = 0$, as $\dot{u}_0 = 0$

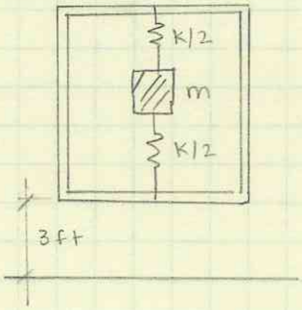
$A = 2\,\text{in}$ $\qquad \omega_n = \sqrt{\frac{k}{m}} = (1/4)^{1/2} = 0.5\,\text{rad/s}$ ✗

$$\boxed{u(t) = (2\,\text{in})\cos(0.5t)}$$

✗

$\dot{u}_{max} = ?$ (−1)

3. Prob 2.6



$m = 10\,\text{lb}/g$

$k = 50\,\text{lb/in}$

$PE = mgh = (10\,\text{lb}/g)(36\,\text{in}) = KE = \frac{1}{2}mv^2$

$v = \left[2gh\right]^{1/2} = 166.8\,\text{in/s}$ ↓

$\dot{u}(0) = 166.8\,\text{in/s}$

$u_{st} = \frac{mg}{k} = \frac{10\,\text{lb}/g}{50\,\text{lb/in}} = 0.20$

$\omega_n = \left[\frac{386\,\text{in/s}^2}{0.20\,\text{in}}\right]^{1/2} = 43.93\,\text{rad/s}$

$A = u(0) = 0$ w.r.t. box

$B = \frac{\dot{u}(0)}{\omega_n} = \frac{166.8\,\text{in/s}}{43.93\,1/\text{s}} = 3.80\,\text{in}$

$u(t) = 3.80\sin(43.9t)$

$\dot{u}(t) = 166.8\cos(43.9t)$

$\ddot{u}(t) = -7327\sin(43.9t)$

↑

max values

$$\boxed{\begin{array}{l} u_{max} = 3.80\,\text{in} \\ \ddot{u}_{max} = \pm 19.0g\,\text{in/s}^2 \end{array}}$$

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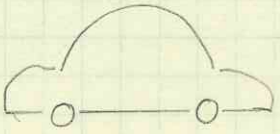
HOMEWORK #2

4. Prob 2.14



$w = 3000\,\text{lb}$

$u = 2\,\text{in}$

$k = 1500\,\text{lb/in}$ (over two springs)

✓

$$m\ddot{u} + c\dot{u} + ku = 0$$

$$c = c_{cr} = 2\sqrt{km} = 2\left[(1500\,\text{lb/in})(3000\,\text{lb})/g\right]^{1/2} = 215.8\ \text{lb}\cdot\text{s/in}$$

$$\boxed{\begin{array}{l} \text{a. } k = 1500\,\text{lb/in} \\ \quad c = 215.8\,\text{lb}\cdot\text{s/in} \end{array}}$$

✓

$+4(160\,\text{lb}) = 640\,\text{lb}$

$$\zeta = \frac{c}{c_{cr}} = \frac{2\sqrt{km_0}}{2\sqrt{km}} = \left(\frac{m_0}{m}\right)^{1/2}$$

$$\zeta = \left[\frac{3000\,\text{lb}}{3640\,\text{lb}}\right]^{1/2} = 0.908$$

$$\boxed{\text{b. } \zeta = 0.908}$$

✓

$$\omega_n = \sqrt{\frac{k}{m}} = \left[\frac{1500\,\text{lb/in}}{3640\,\text{lb}/g}\right]^{1/2} = 12.62\ \text{rad/s}$$

$$\omega_D = \omega_n\sqrt{1-\zeta^2} = (12.62\,\text{rad/s})\left[1-(0.908)^2\right]^{1/2}$$

$$\omega_D = 5.29\,\text{rad/s}$$

$$\boxed{\text{c. } \omega_D = 5.29\,\text{rad/s}}$$

✓

4/4

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HOMEWORK #2

5. Prob 2.15

$m = 0.1 \text{ lb}\cdot\text{s}^2/\text{in}$

$u_0 = 1 \text{ in}$

$t = 3\text{s}, \; j = 20 \quad u = 0.2 \text{ in}$

$$T_n = \frac{3.0\text{s}}{20} = 0.15 \text{ s}$$

$$\omega_n = \frac{2\pi}{T_n} = 41.89 \text{ rad/s}$$

$$k = \omega_n^2 \cdot m = 175.5 \text{ lb/in}$$

$$c_{cr} = 2\sqrt{km} = 8.38 \text{ lb}\cdot\text{s/in}$$

$$\zeta = \frac{1}{2\pi j} \ln \frac{u_i}{u_{i+j}} = \frac{1}{2\pi(20)} \ln \left[\frac{1.0 \text{ in}}{0.2 \text{ in}}\right] = 0.0128$$

$$c = \zeta \cdot c_{cr} = (0.0128)(8.38 \text{ lb}\cdot\text{s/in})$$

$$c = 0.107 \text{ lb}\cdot\text{s/in}$$

$$\boxed{\begin{array}{l} k = 175.5 \text{ lb/in} \\ c = 0.107 \text{ lb}\cdot\text{s/in} \end{array}}$$

Homework No. 2

Assigned: January 31, 2006
Due: February 7, 2006

1. *Use of free vibration experiments to determine mass and stiffness of a structure.* Solve Problem 2.1 from the textbook.

2. Solve Problem 2.2 from the textbook.
What is the maximum velocity of the electromagnet?
What is the displacement of the mass at that instant? Why?

3. Solve Problem 2.6 from the textbook.

4. Solve Problem 2.14 from the textbook.

5. Solve Problem 2.15 from the textbook. You may assume light damping but you need to verify if this assumption may be justified from the test results.

*6. *Optional Question*
Consider a SDOF system with any mass and stiffness of your choice.
Sketch phase plane plots (displacement vs. velocity) for free vibration if your system is
(a) undamped;
(b) underdamped;
(c) critically damped;
(d) overdamped.
Use any reasonable but general initial conditions that help to illustrate the time-varying response in each case.

You do not need to solve the differential equations for critically damped and overdamped free vibration. Displacement solutions for cases (b) and (c) are given in the problem statements of Problems 2.8 and 2.9 (pg. 63), respectively.

Comment on your results.

It would be interesting to see if you can animate your results and compare the behavior of these four systems, starting each one from the same initial conditions. Send me by e-mail your ".avi" or other animation file (zipped if it's very large) and I'll show it to the rest of the class. The use of Matlab or MathCad might help.

CE 384P: DYNAMICS

CATHERINE HOVELL

14 February 06

1/5

HOMEWORK #3

19/20

1. Problem 3.1

m, k, ω_n

$f = 4\,Hz = f_n$

$\Delta W = 5\,lb$

$f_n = 3\,hz$

$f_n = \frac{\omega_n}{2\pi}$ $\quad \omega_{n_0} = f_{n_0}(2\pi) = (4hz)(2\pi)$ ✓

$\omega_{n_0} = 8\pi\ rad/s$

$\omega_n = \sqrt{\frac{k}{m}}$

$\omega_n' = (3hz)(2\pi) = 6\pi$

$k_0 = k'$

$\frac{k}{m_0} = \omega_{n_0}^2, \quad \frac{k}{m'} = \omega_n'^2$

$\omega_{n_0}^2 m_0 = \omega_n'^2 m'$

$(8\pi)^2 m_0 = (6\pi)^2 (m_0 + {}^{5lb}/g)$

$m_0 = \frac{36\pi^2(5/g)}{64\pi^2 - 36\pi^2} = 6.43\ lb/g$

$$\boxed{\begin{array}{c} m_0 = 6.43\ lb/g \\ k = 10.51\ lb/in \end{array}}$$

2. Problem 3.4

$f_n = 200\ cyc/min, \quad \omega_n = 20.94\ rad/s$

$p(t) = p_0 \sin \omega t$	$u_0 = 0.2\,in$	$1.042\,in$	$0.0248\,in$
	$f = 20\,rev/min$	$180\,rpm$	$600\,rpm$
	$\omega = 2.09\ rad/s$	$18.85\,rad/s$	$62.83\ rad/s$

add $\zeta = 0.25$

$u(t) = u_{st}\left[\frac{1}{1-(\omega/\omega_n)^2}\right]\sin\omega t$

$0.2\,in\left[1 - ({}^{20}/_{200})^2\right] = 0.198\,in = u_{st}$

$u(t) = u_{st}\left[(1-(\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2\right]^{-1/2}$

$u(t) = 0.198\left[(1-({}^{20}/_{200})^2)^2 + (2 \cdot 0.25 \cdot {}^{20}/_{200})^2\right]^{-1/2} = 0.1997$

$u(t)_{\omega=180} = 0.405\ in$

$u(t)_{\omega=600} = 0.0243\,in$

Speed rev/min	Umax (in) $\zeta=0$	$\zeta=0.25$
20	0.2	0.1997
180	1.042	0.405
600	0.0248	0.0243

* Damping helps greatly near resonance ($\omega = 180$ vs $\omega_n = 200$),

HOMEWORK #3

3. case (I) – Problem 3.5

$W = 1200\ \text{lb}$

$I = 10\ \text{in}^4$

$\omega = 300\ \text{rpm}$

$p_o = 60\ \text{lb}$

$\zeta = 0.01$

$E = 30000\ \text{ksi}$

$L = 8\ \text{ft}$

$$k_{beam} = \frac{48EI}{L^3}$$ for each beam – two springs

$$k_b = \frac{48}{(96\text{in})^3}(30000\text{ksi})(10\text{in}^4) = 16.28\ \text{k/in}$$

$$u\overset{o}{\cancel{(t)}} = \frac{p_o}{k}\left[\left(1-(\omega/\omega_n)^2\right)^2 + (2\zeta\omega/\omega_n)^2\right]^{-1/2}$$

$$u(t) = \frac{60\text{lb}\cdot\frac{1}{2}}{16.28\ \text{k/in}}\left[\left[1-\left(31.42/102.34\right)^2\right]^2 + \left[2(0.01)\left(31.42/102.34\right)\right]^2\right]^{-\frac{1}{2}}$$

$\omega = 300\text{rpm} = 31.42\ \text{rad/s}$

$$\omega_n = \left[\frac{16.28\ \text{k/in}}{1200\ \text{lb}}(2000)(386\ \text{in/s}^2)\right]^{\frac{1}{2}} = 102.34\ \text{rad/s}$$

$u(t) = 0.00203\ \text{in}$

$$\boxed{u\overset{u_o}{\cancel{(t)}} = 2.03\times10^{-3}\ \text{in}}$$ ✓

$$\ddot{u}(t) = \frac{-p_o}{m}\left(\frac{\omega}{\omega_n}\right)^2 R_d = \frac{-60\text{lb}/(1200\text{lb}/386\ \text{in/s}^2)\cdot\left(31.42/102.34\right)^2}{\sqrt{\left[\left(1-\left(31.42/102.3\right)^2\right)^2 + \left(0.02\cdot 31.42/102.3\right)^2\right.}}$$

$$\boxed{\begin{array}{c}\ddot{u}\overset{\ddot{u}_o}{\cancel{(t)}} = 0.0052g\ \text{in/s}^2\\ \text{(slowing)}\end{array}}$$ ✓

case (II)

$$K = \frac{48}{(288\text{in})^3}(30000\ \text{ksi})(10\ \text{in}^4) = 0.603\ \text{k/in}$$

$$k_{tot} = 2000(0.603\ \text{k/in}) = 1205.6\ \text{lb/in}$$

$\omega_n = 19.69$

$r = \omega/\omega_n = 1.595$

$$u(t) = \frac{60\ \text{lb}}{1206\ \text{lb/in}}\left[(1-r^2)^2 + (0.02r)^2\right]^{-1/2} = 0.0321\ \text{in}$$

$$\ddot{u}(t) = \frac{-60\text{lb}}{1200\text{lb}}(386\text{in/s}^2)\cdot r^2 \downarrow \quad = 0.082g\ \text{in/s}^2$$

$$\boxed{\begin{array}{l}u_o\ \cancel{u(t)} = 0.032\text{in}\\ \ddot{u}_o\ \cancel{\ddot{u}(t)} = 0.082g\ \text{in/s}^2\end{array}}$$

✓

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HOMEWORK #3

3. (cont'd) case (iii)

L = 16 ft
fixed-roller

$$k = \frac{768^*}{7}(30000\text{ ksi})(10\text{ in}^4)(192\text{ in})^{-3} = 4.65\text{ k/in}$$

$$K_{tot} = 9301 \text{ lb/in}$$

* stiffness value $k = \frac{768EI}{7L^3}$ found in the steel manual

$\omega_n = 54.70$ rad/s ✓
$r = 0.574$

$$u(t) = \frac{60\text{ lb}}{9301\text{ lb/in}} \quad R_d = 0.0079\text{ in}$$ ✗ −0.5

$$\ddot{u}(t) = \frac{-60\text{ lb}}{1200\text{ lb/g}} r^2 \cdot R_d = -0.0246g \text{ in/s}^2$$ ✓

$$\boxed{\begin{aligned} u(t) &= 0.0079\text{ in} \\ \ddot{u}(t) &= 0.025g \text{ in/s}^2 \end{aligned}}$$

Notes:
- increasing the span decreases the stiffness, increasing $u, \ddot{u}$
- changing the supports can allow for longer members
- forced vibration near the natural frequency with low damping will cause large deflections.
- small amounts of damping can drastically reduce the amplitude of deflections at resonance.

✓

4/5

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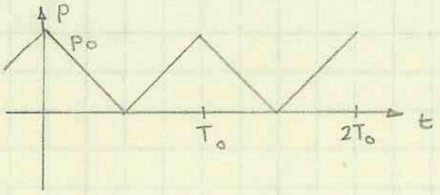
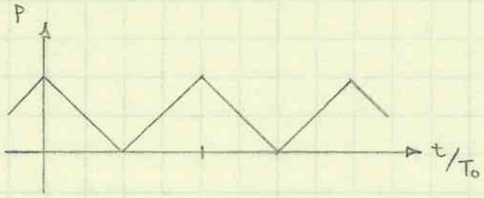
HOMEWORK #3

4. Problem 3.26



T_n, ζ, p_o, T_o

$$p(t) = \begin{cases} -2p_o/T_o\, t + p_o & 0 \le t \le T_o/2 \\ 2p_o/T_o\, t - p_o & T_o/2 \le t \le T_o \end{cases}$$



$$p(t) = a_o + \sum_{j=1}^{\infty} a_j \cos(j\omega_o t) + \sum_{j=1}^{\infty} b_j \sin(j\omega_o t)$$

$$a_o = \frac{1}{T_o}\int_0^{T_o} p(t)\,dt$$

– average value

$$= \frac{1}{T_o}\cdot\frac{1}{2}T_o\cdot p_o = \frac{p_o}{2}$$

$$a_j = \frac{2}{T_o}\int_0^{T_o} p(t)\cos\left(j\frac{2\pi}{T_o}t\right)dt$$

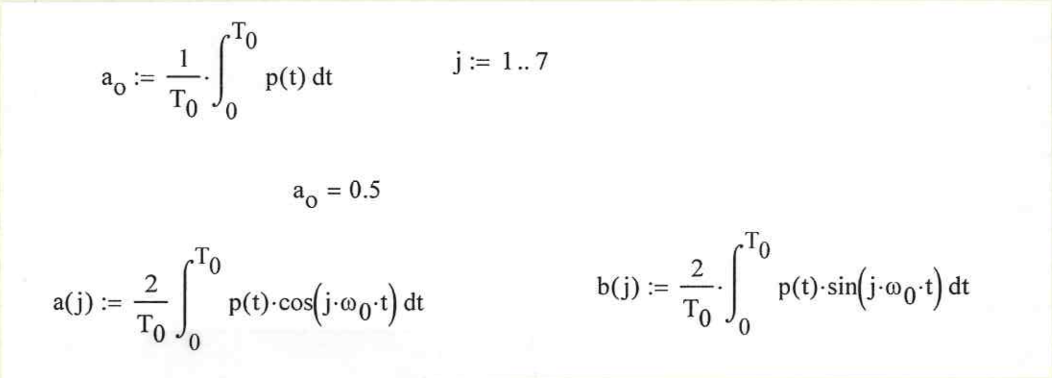
$= 0$ if j is even

$= \frac{4p_o}{\pi^2 j^2}$ if j is odd

$$b_j = \frac{2}{T_o}\int_0^{T_o} p(t)\sin\left(j\frac{2\pi}{T_o}t\right)dt$$

because fxn mirrors over y-axis, sin coefficients = 0

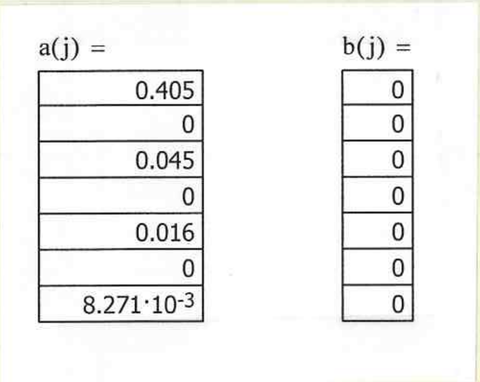
$$\boxed{a.\ p(t) = \frac{p_o}{2} + \frac{4p_o}{\pi^2}\sum_{j=1,3,5..}^{\infty}\frac{1}{j^2}\overset{\cos}{\cancel{\sin}}(j\cdot\omega_o t)}$$



$$a_o := \frac{1}{T_0}\cdot\int_0^{T_0} p(t)\,dt \qquad j := 1..7$$

$$a_o = 0.5$$

$$a(j) := \frac{2}{T_0}\int_0^{T_0} p(t)\cdot\cos(j\cdot\omega_0\cdot t)\,dt \qquad b(j) := \frac{2}{T_0}\cdot\int_0^{T_0} p(t)\cdot\sin(j\cdot\omega_0\cdot t)\,dt$$



a(j) =	b(j) =
0.405	0
0	0
0.045	0
0	0
0.016	0
0	0
$8.271\cdot10^{-3}$	0

$0.405 = \frac{4}{\pi^2}$

$0.045 = 4/9 \cdot 1/\pi^2$

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HOMEWORK #3

4. (cont'd)

steady state, $\xi = 0$

$$u_{jc}(t) = \frac{a_j}{k} \frac{2\xi B_j \sin(j\omega_0 t) + (1-\beta_j^2)\cos(j\omega_0 t)}{(1-\beta_j^2)^2 + (2\xi\beta_j)^2}$$

, $u_{js}(t) \propto b_j = 0$, so ignore

$$\beta_j = \frac{j\omega_0}{\omega_n}$$

$$u_0(t) = \frac{a_0}{k} = \frac{p_0}{2k}$$

$$u_{jc}(t) = \frac{a_j}{k} \cdot \frac{(1-\beta_j^2)\cos(j\omega_0 t)}{(1-\beta_j^2)^2} = \frac{a_j}{k} \frac{\cos(j\omega_0 t)}{1-\beta_j^2}$$

$$a_j = \frac{4p_0}{\pi^2 j^2} \text{ for odd } j \text{ values}$$

What values of T_0 ?

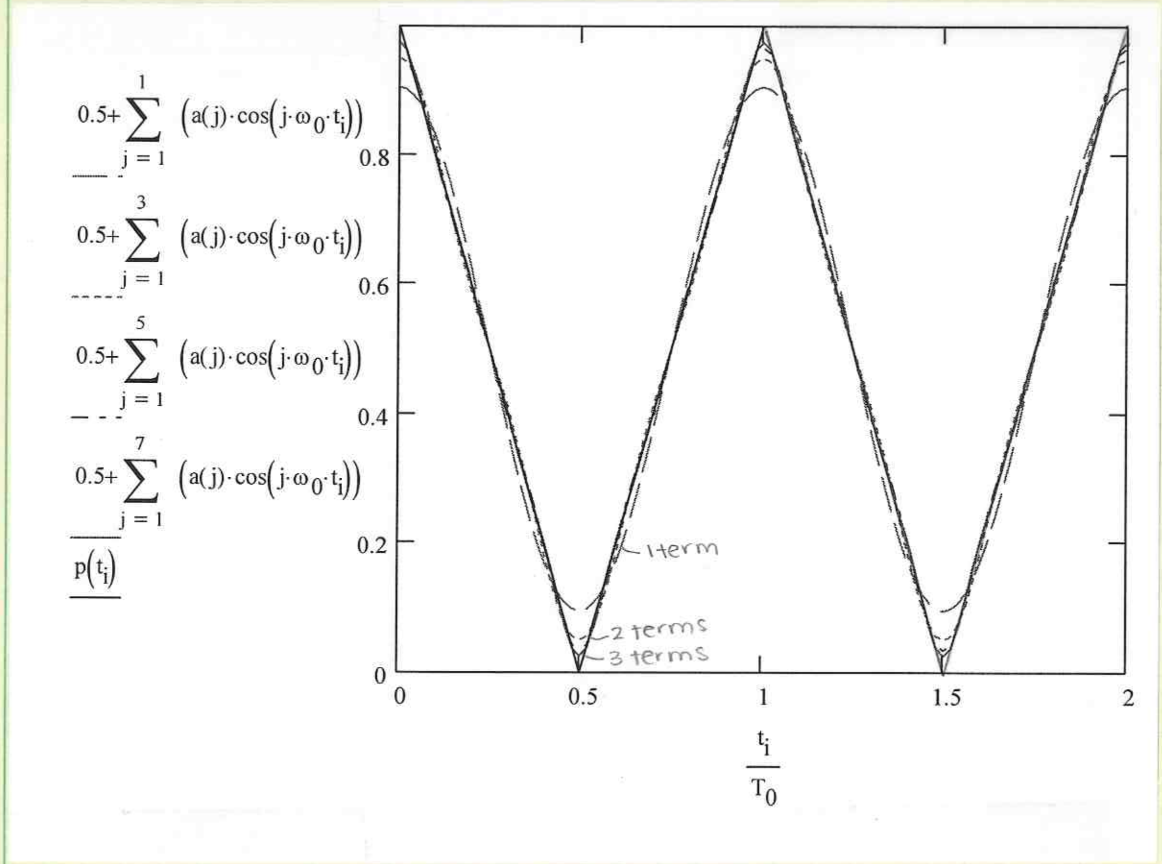
(−0.5)

$$u(t) = u_0(t) + \sum u_{jc} + \sum u_{js}$$

$$\boxed{\text{b. } u(t) = \frac{p_0}{k}\left[\frac{1}{2} + \frac{4}{\pi^2}\sum_{j=1,3,5\ldots}^{\infty} \frac{1}{j^2} \frac{\cos(j\omega_0 t)}{(1-\beta_j^2)}\right] \quad \beta_j = \frac{j\omega_0}{\omega_n},\ \beta_j \neq 1}$$

✓

c)



Two or three terms can accurately match p(t).

✓

Homework No. 3

Assigned: February 7, 2006
Due: February 14, 2006

1. Solve Problem 3.1 from the textbook.

2. Solve Problem 3.4 from the textbook.
In a tabular form, summarize your results for the three operating speeds (for the undamped and 25% damping cases).

3. Air-conditioner supported on two beams.
Case (I) Solve Problem 3.5 as given in the textbook.
Case (II) The steady-state displacements and accelerations in Case (I) seem rather small. How much would they change if the two beams have spans of 24 feet instead of 8 ft?
Case (III) Suppose Case (II) is unacceptable.
Instead of simply-supported beams of 8-ft spans, you try fixed supports at the left end of each beam and rollers on the right. You also change each beam span to 16 feet. How does the steady-state response compare with Cases (I) and (II)?
Note: In Cases (II) and (II), assume that the air-conditioning unit is placed at mid-span.

Make brief comments on what you learned here.

4. Solve Problem 3.26 form the textbook.

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HOMEWORK #4

25/25

Nice work!

1. Test 1

$\omega_1 = 16$ rad/s
$u_{meas} = 0.0072$ in
$\phi_{meas} = 15° = \pi/12$

Test 2

$\omega_2 = 25$ rad/s
$u_{meas} = 0.0145$ in
$\phi_{meas} = 55° = 11\pi/36$

(1) Test two is more likely closer to resonance, given only the phase angles 15° and 55°. Figure 3.2.6 shows that, below 90°, a higher ϕ is a result of being closer to resonance.

(2) Test two is more likely closer to resonance because the maximum amplitude of vibration is larger, indicating ω/ω_n closer to 1.0

$P_o = 500$ lb, need m, c, K

$$\phi = \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right)$$

with $\omega = 16$ rad/s, $\phi = \pi/12$
$\omega = 25$ rad/s, $\phi = 11\pi/36$
solve for ζ and ω_n

$$\tfrac{1}{2}\tan\phi \cdot \left[1-(\omega/\omega_n)^2\right] = \zeta\,\omega/\omega_n$$

$$\left[\tfrac{1}{2}\tan\phi - \tfrac{1}{2}\tan\phi\,(\omega/\omega_n)^2\right]\frac{\omega_n}{\omega} = \zeta = \frac{\omega_n}{\omega_2}\left[\tfrac{1}{2}\tan\phi_2 - \tfrac{1}{2}\tan\phi_2\,(\omega_2/\omega_n)^2\right]$$

$\omega_n = 27.89$ rad/s , $\omega_n = \sqrt{K/m}$
$\zeta = 0.157$, $\zeta = \dfrac{c}{2\sqrt{Km}}$

$u_{st} = \dfrac{P_o}{K}$, $u_o = u_{st} R_d$

$$K = \frac{P_o}{u_o} R_d = \frac{500\text{ lb}}{u_o}\left[\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta(\omega/\omega_n)\right]^2\right]^{-\frac{1}{2}}$$

$$K = \frac{500\text{ lb}}{0.00721\text{ in}}\left[\left[1-(16/27.89)^2\right]^2 + \left[2(0.157)(16/27.89)\right]^2\right]^{-\frac{1}{2}}$$

$$\boxed{\begin{array}{l} K = 100\text{ k/in} \\ m = 129\text{ lb/g} \\ c = 1.13\text{ k}\cdot\text{in/s} \end{array}}$$

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HOMEWORK #4

2. $w = 4000\ \text{lb}$

$+w' = 100\ \text{lb}$

$u' = 0.08\ \text{in}$

$F = 1.2 \sin(\omega t)$

$\dot{u}(0) = 45\ \text{mph} \to 66\ \text{ft/s}$

$\xi = 0.40$

$F = kx$

$4000 = ku, \quad 4100 = k(u + 0.08\ \text{in})$

$$4100 = k\left[\frac{4000}{k} + 0.08\right]$$

$100\ \text{lb} = k(0.08\ \text{in})$

$k = 1.25\ \text{k/in}$

$$\omega = \frac{2\pi(66\ \text{ft/s})}{40\ \text{ft}} = 3.3\pi\ \text{rad/s}$$

$$\omega_n = \left[\frac{1250\ \text{lb/in}}{4000\ \text{lb}} \cdot 386\right]^{1/2} = 10.98\ \text{rad/s}$$

$$TR = \frac{u_o^t}{u_{go}} = \left[\frac{1 + (2\xi\omega/\omega_n)^2}{\left[1 - (\omega/\omega_n)^2\right]^2 + (2\xi\omega/\omega_n)^2}\right]^{1/2}, \quad \frac{\omega}{\omega_n} = 0.944$$

(1) considering $\omega/\omega_n = 0.944$, which is less than $\sqrt{2} = 1.414$, Figure 3.5.1 shows graphically the response of TR to changing damping. As damping increases, the transmissibility ratio goes down. Correspondingly, the amplitude of the car's motion also goes down.

$u_o^t = u_{go}\, TR, \quad u_{go} = 1.2\ \text{in}$

$$TR = \left[\frac{1 + \left[2(0.40)(0.944)\right]^2}{\left[1 - (0.944)^2\right]^2 + \left[2(0.40)(0.944)\right]^2}\right]^{1/2} = 1.64$$

(2) $u_o^t = 1.97\ \text{in}$

$u_o^t = 1.86\ \text{in}, \xi = 0.44$

A 10% increase in damping resulted in vibrations 5% smaller than initially.

if $\xi = 0.44$, $TR = 1.55$

span = 25 ft, $\omega = 5.28\pi\ \text{rad/s}$

$$\frac{\omega}{\omega_n} = \frac{5.28\pi\ \text{rad/s}}{10.98\ \text{rad/s}} = 1.511$$

$TR = 0.89, \quad u_{go} = 1.2\ \text{in}$

If $k = 1125\ \text{lb/in}$, $\omega_n = 10.42$

$\omega/\omega_n = 1.59$, $TR = 0.814$

if $\xi = 0.44$, $TR = 0.901$

(3) $u_o^t = 1.07\ \text{in}$

$u_o^t = 0.97\ \text{in}, k = 1125\ \text{k/in}$ (lower!)

$u_o^t = 1.08\ \text{in}, \xi = 0.44$ (higher!)

* Until ω/ω_n increases above $\sqrt{2}$, increasing damping will decrease the amplitude of motion. Past that point, increasing damping actually increases vibrations.

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HOMEWORK #4

3. $u(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{p_o}{k}\frac{1}{1-(\omega/\omega_n)^2}\sin\omega t$

if $u(0)=0$ and $\dot{u}(0)=0$,

$$u(t) = \frac{p_o}{k}\frac{1}{1-(\omega/\omega_n)^2}\left(\sin\omega t - \omega/\omega_n \sin\omega_n t\right)$$

what if $\omega \rightarrow \omega_n$?

assume $\omega/\omega_n = 1.0$ on top

$$u(t) = u_{st}\frac{1}{1-(\omega/\omega_n)^2}\left(\sin\omega t - \sin\omega_n t\right)$$

$$\sin At - \sin Bt = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} = 2\cos\frac{2\omega}{2}t\sin\frac{\omega-\omega_n}{2}t$$

or, reversing the order initially,

$$-2\cos\omega t \cdot \sin\frac{\omega_n-\omega}{2}t$$

define $\varepsilon = \frac{\omega_n-\omega}{2}$

$$u(t) = u_{st}\frac{-2\omega_n^2}{\omega_n^2-\omega^2}\cos\omega t\sin\varepsilon t$$

but $\omega_n^2-\omega^2 = (\omega_n-\omega)(\omega_n+\omega) = 2\varepsilon\cdot 2\omega$

$$u(t) = u_{st}\frac{-2\omega^2}{2\varepsilon\cdot 2\omega}\cos\omega t\sin\varepsilon t$$

$$\boxed{u(t) = \frac{-(u_{st})_o\omega}{2\varepsilon}\cos\omega t\sin\varepsilon t}$$

with $\varepsilon = \frac{\omega_n-\omega}{2}$

$T_b = \frac{2\pi}{\varepsilon}$, No. osc $= \frac{\omega}{\varepsilon}$

for $\omega_n = 1800$ cpm

$\omega = 1785$ cpm , $\varepsilon = \frac{1800\text{cpm}-1785\text{cpm}}{2} = 7.5$ cpm

7.5cpm = 15π rad/min

$$T_b = \frac{2\pi}{15\pi} = 0.133 \text{ min}$$

$$\text{No. osc} = \frac{1785\text{cpm}}{7.5\text{cpm}} = 238$$

$$\boxed{\begin{array}{l} T_b = 0.133 \text{ min} \\ \text{No. osc} = 238 \end{array}}$$

HOMEWORK #4

3. (cont'd)

Eq. 3.1.6(b)

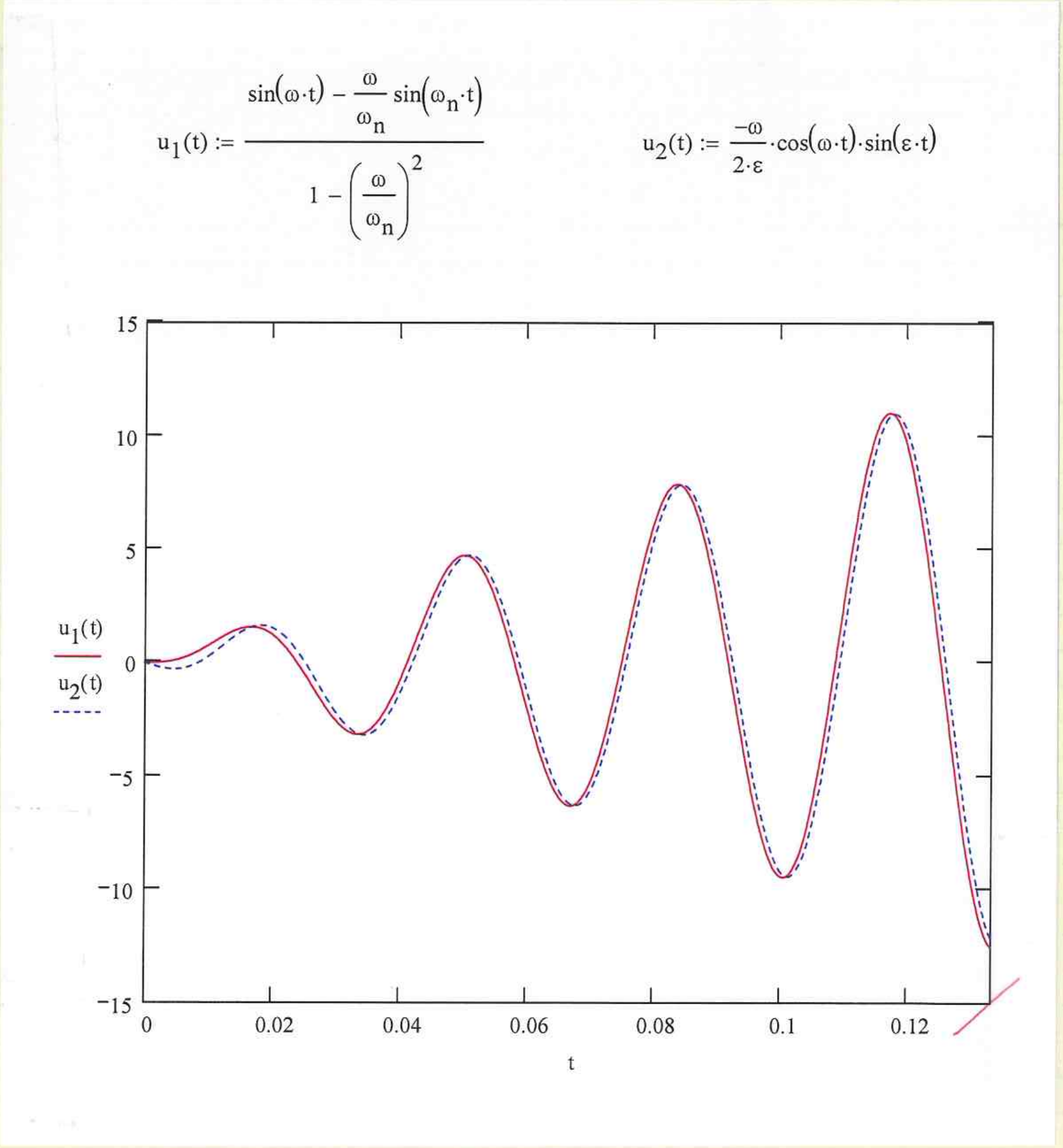
$$\frac{u(t)}{u_{st}} = \frac{\sin \omega t - \omega/\omega_n \sin \omega_n t}{1 - (\omega/\omega_n)^2}$$

Simplification

$$\frac{u(t)}{u_{st}} = \frac{-\omega \cos \omega t \sin\left(\frac{\omega_n - \omega}{2} t\right)}{\omega_n - \omega}$$

$w = 1785$ cpm $= 186.9$ rad/s

$\omega_n = 1800$ cpm $= 188.4$ rad/s



$$u_1(t) := \frac{\sin(\omega \cdot t) - \frac{\omega}{\omega_n}\sin(\omega_n \cdot t)}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \qquad u_2(t) := \frac{-\omega}{2 \cdot \varepsilon} \cdot \cos(\omega \cdot t) \cdot \sin(\varepsilon \cdot t)$$

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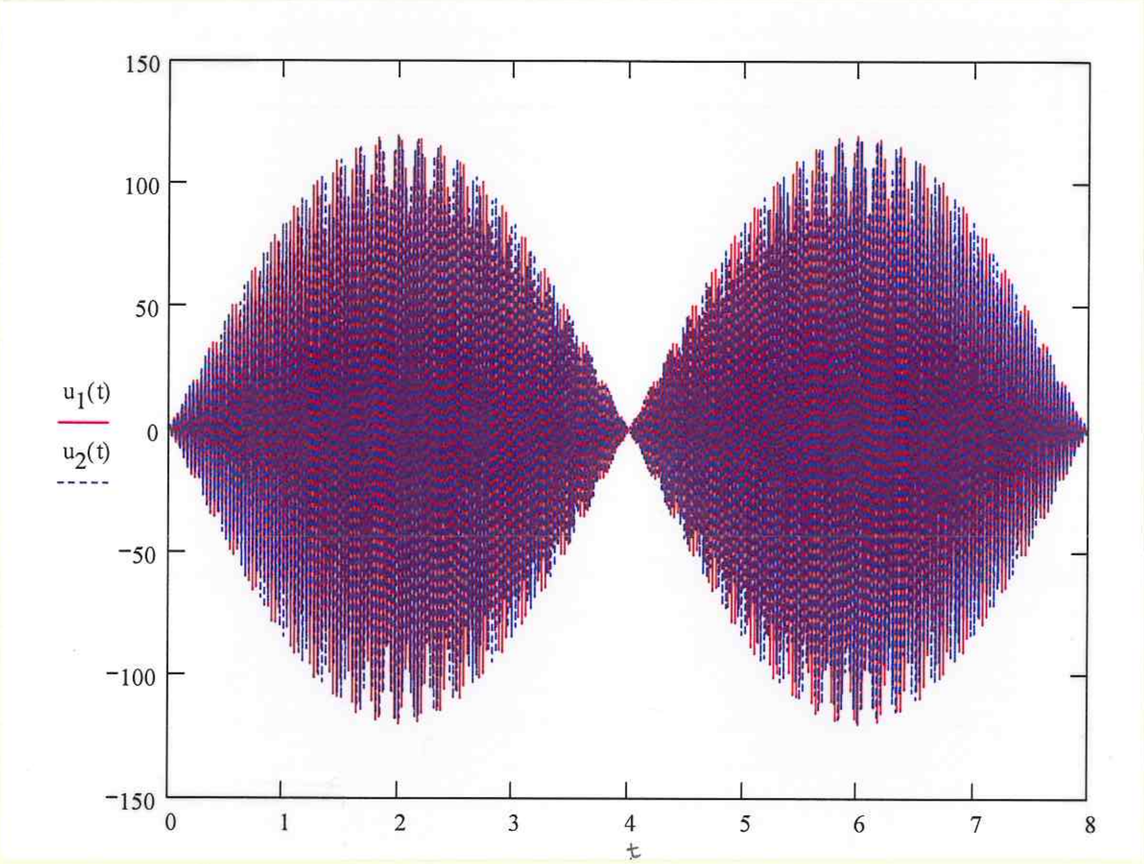
HOMEWORK #4

3. (cont'd)

Expanding the plot to see through T_b:



Although it's not clear, there are oscillations within the large oscillation. The approximate and exact solutions stop lining up perfectly, but do ultimately show the same shape.

Homework No. 4

Assigned: February 14, 2006
Due: February 21, 2006

Problem 1

A portable harmonic-loading machine provides an effective means for evaluating the dynamic properties of structures in the field. By operating the machine at different frequencies and measuring the resulting structural response amplitude and phase relationships in each case, it is possible to determine the mass, damping, and stiffness of a single-degree-of-freedom structure.

Two tests were performed each with a force amplitude of 500 lb.

Test 1: Operating frequency, $\omega_1 = 16$ rad/sec
Measured response amplitude = 0.0072 in.
Measured phase angle = 15°

Test 2: Operating frequency, $\omega_2 = 25$ rad/sec
Measured response amplitude = 0.0145 in.
Measured phase angle = 55°

1. If you only knew the phase angles (i.e., you did not know the amplitude), which test would you say was conducted at a frequency closer to resonance? Why?
2. Would you expect that the natural frequency of the structure is closer to 16 or to 25 rad/sec? Why?
3. Evaluate the mass, damping coefficient, and stiffness of the structure.

Problem 2

Deflections sometimes develop in concrete bridge girders due to creep, and if the bridge consists of a long series of identical spans, these deformations will cause a harmonic excitation in a vehicle traveling over the bridge at constant speed. One can usually assume that the springs and shock absorbers of the car will provide a vibration isolation system that will limit the vertical motions transmitted from the road to the occupants.

Consider an idealized model of a vehicle with a weight of 4000 lb. Assume that the spring stiffness is measured by a test that showed that adding 100 lb to the vehicle caused a deflection of 0.08 in. The bridge profile is approximated by a sine curve having a wavelength (girder span) of 40 ft and an amplitude of 1.2 in. Assume that the car is traveling at a speed of 45 mph and that damping is 40 percent of critical.

1. Based solely on this information, would you expect that an increase in the damping would lead to larger or smaller vertical motions, u_0^t, in the vehicle?
   (*Hint*: Note that u_0^t is related to the effective ground "displacement" amplitude, u_{g0} such that $u_0^t = TR \times u_{g0}$. Does TR increase or decrease with a damping increase for this frequency ratio?)
2. Estimate the steady state vertical motions, u_0^t, in the car.
   What would be the effect of a 10% increase in the damping ratio?
   Comment on the effectiveness of an increased damping.
3. If the girder span were changed to 25 ft with the same amplitude of 1.2 in and the vehicle were still to traveling at 45 mph, estimate the steady state vertical motions, u_0^t.
   What would be the effect of a 10% reduction in the spring stiffness now?
   What would be the effect of a 10% increase in the damping ratio?
   Does an increase in damping reduce vertical motions?
   Comment on your observations.

Problem 3

The phenomenon of "beating"

Consider undamped forced vibrations of a SDOF system to a harmonic force.
The general solution for this problem is given by Eq. 3.1.5 in your textbook.

Now, consider the special case where the initial displacement and initial velocity are both zero. This will result in Eq. 3.1.6b.

When the difference between ω and ω_n is very small but not zero, show that the solution for the displacement, $u(t)$ may be approximately given as:

$$u(t) = -\frac{(u_{st})_0 \omega}{2\varepsilon} \cos \omega t \sin \varepsilon t \qquad (1)$$

where $\varepsilon = (\omega_n - \omega)/2$.

When the solution is written in this manner, it can be observed that your response is a product of two oscillatory terms. The term $\sin\varepsilon t$ will oscillate with a period much longer than the term $\cos\omega t$. The resulting motion is a rapid oscillation with a slowly varying amplitude and this is termed a "beat." The beat period is $2\pi/\varepsilon$ and the number of oscillations in each beat is $(2\pi/\varepsilon)/(2\pi/\omega) = \omega/\varepsilon$. Sometimes, the two oscillating terms add together and sometimes they cancel one another out.

An undamped system is excited by a harmonic force close to resonance, resulting in a beating condition. The natural frequency of the system is 1800 cpm (cycles per minute) and the frequency of the load is 1785 cpm. Determine the beat period and the number of oscillations within each beat.

Graph your displacement for at least one whole beat period using the approximate solution (Eq. 1) as well as using Eq. 3.1.6(b). You can plot $u(t)/(u_{st})_0$ so that you won't need to know the force amplitude or stiffness.

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HOMEWORK #5

$\frac{20}{20}$ ✓

1. Prob 4.5

$$\frac{u(t)}{(u_{st})_0} = \frac{1}{1+a^2/\omega^2}\left[\frac{a}{\omega_n}\sin\omega_n t - \cos\omega_n t + e^{-at}\right]$$

$$p(t) = p_0 e^{-at}$$

using MathCAD

$$p(p_o,t,a) := p_o \cdot e^{-a\cdot t}$$

$$u(t,p_o,a,m,\omega_n,\tau) := \frac{1}{m\cdot\omega_n}\left[\int_0^t p(p_o,\tau,a)\cdot\sin[\omega_n\cdot(t-\tau)]\,d\tau\right]$$

$$u(t,p_o,a,m,\omega_n,\tau) \rightarrow \frac{1}{m\cdot\omega_n}\cdot\left(\frac{\omega_n}{a^2+\omega_n^2}\cdot e^{-t\cdot a}\cdot p_o + \frac{-\omega_n\cdot\cos(\omega_n\cdot t) + a\cdot\sin(\omega_n\cdot t)}{a^2+\omega_n^2}\cdot p_o\right)$$

Knowing that $u_{st_0} = \frac{p_0}{k}$

$$m = \frac{k}{\omega_n^2},$$

$$\frac{u(t)}{u_{st_0}} = \omega_n^2\left[\frac{e^{-at} + a/\omega_n \sin(\omega_n t) - \cos(\omega_n t)}{a^2+\omega_n^2}\right]$$

or, $\boxed{\text{a)}\ \frac{u(t)}{u_{st_0}} = \frac{e^{-at} + a/\omega_n \sin(\omega_n t) - \cos(\omega_n t)}{a^2/\omega_n^2 + 1}}$

✓

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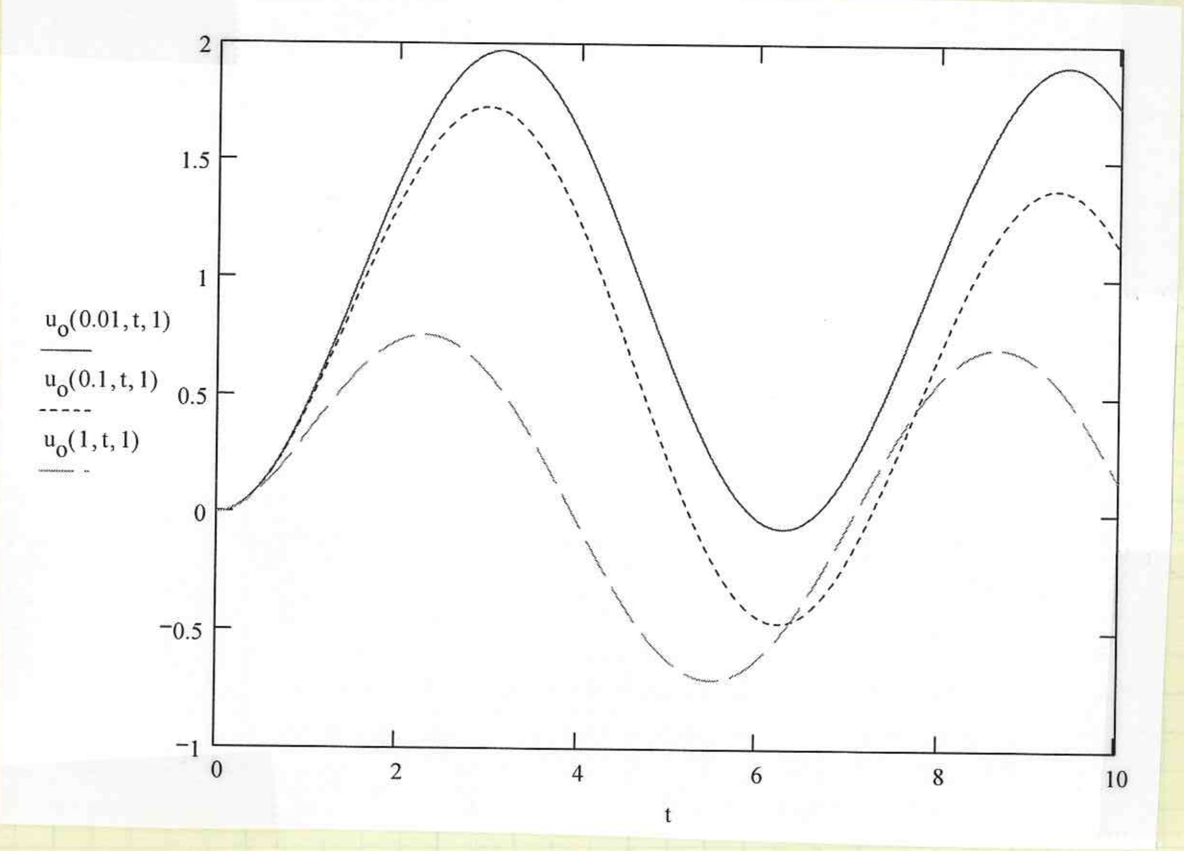
HOMEWORK #5

1. (cont'd)

b) Plot of (a) $u_o(a,t,\omega_n)$; $\omega_n=1.0$, $a=0.01, 0.1, 1.0$; $u_o=\frac{u(t)}{u_{st_o}}$



Amplitude at steady state ($t\to\infty$)

$$\frac{u(t)}{u_{st_o}}=\frac{a/\omega_n\sin(\omega_n t)-\cos(\omega_n t)}{a^2/\omega_n^2+1}\ ,\quad e^{-ta}\to 0$$

for $a\sin x-b\cos x$, $A=\sqrt{a^2+b^2}=\left[(a/\omega_n)^2+1^2\right]$

$$A=\frac{\sqrt{(a/\omega_n)^2+1}}{a^2/\omega_n^2+1}\ ,\quad \frac{\sqrt{x}}{x}=\frac{1}{\sqrt{x}}$$

$$\boxed{c)\ A=\left[a^2/\omega_n^2+1\right]^{-1/2}}$$

By $t=10s$, e^{-at} → almost zero
(steady state conditions)

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HOMEWORK #5

2. Prob. 4.14

$$p(t) = \begin{cases} p_o & 0 \le t \le t_d \\ 0 & t > t_d \end{cases} \quad \text{or} \quad \begin{array}{l} p_1(t) = p_o \text{ from } t=0 \\ p_2(t) = -p_o \text{ from } t = t_d \end{array}$$

$$u_1(t) = u_{st_o}\left(1 - \cos\frac{2\pi}{T_n} t\right) \qquad (I)$$

$$u_2(t) = u_1(t_d)\left[\cancel{1-}\cos\frac{2\pi}{T_n}(t - t_d)\right] + \frac{\dot{u}(t_d)}{\omega_n}\sin\left[\omega_n(t - t_d)\right]$$

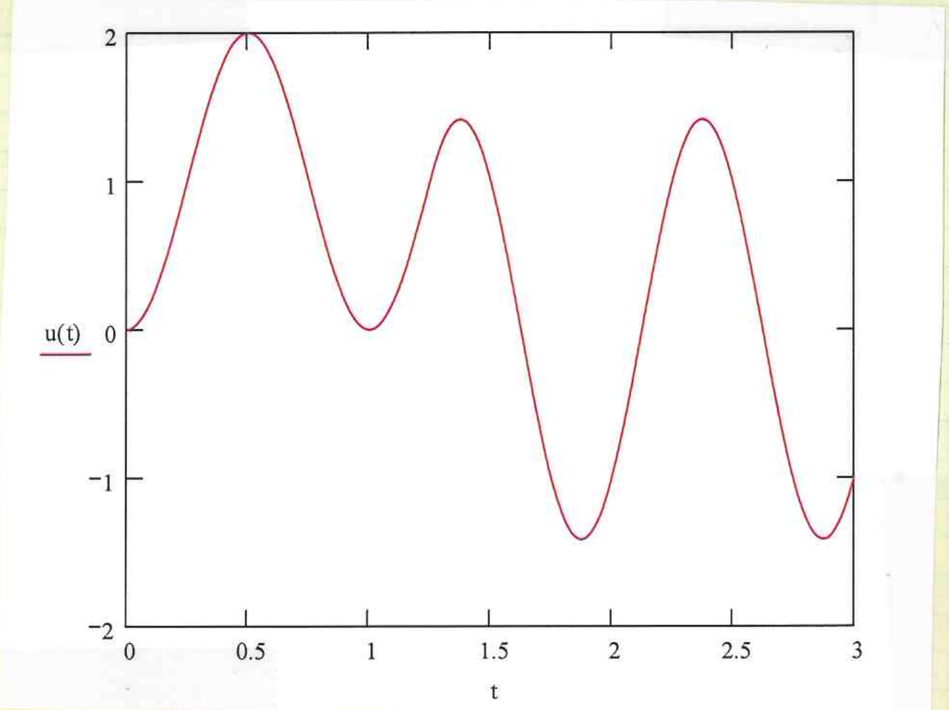
$$u(t_d) = u_{st_o}\left[1 - \cos\frac{2\pi}{T_n} t_d\right]$$

$$\dot{u}(t_d) = u_{st_o}\,\omega_n \sin \omega_n t_d$$

$$\frac{u(t)}{u_{st_o}} = 2\sin\frac{\pi t_d}{T_n}\sin\left[2\pi\left(t/T_n - 1/2\; t_d/T_n\right)\right] \qquad (II)$$

$$\boxed{\frac{u(t)}{u_{st_o}} = \begin{cases} 1 - \cos\frac{2\pi}{T_n}\cdot t & 0 \le t \le t_d \\ 2\sin\left(\frac{\pi}{T_n}t_d\right)\sin\left[2\pi\left(t/T_n - t_d/2T_n\right)\right] & t \ge t_d \end{cases}}$$

For $t_d = 1.25\text{s}$, $T_n = 1.0\text{s}$,



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HOMEWORK #5

3. Prob. 4.15

$$u(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin\left[\omega_n (t-\tau)\right] d\tau$$

$$p(t) = \left| \begin{array}{ll} p_o & 0 \le t \le t_d \\ 0 & t > t_d \end{array} \right. \qquad \frac{p_o}{m\omega_n} = u_{st_o}\omega_n$$

$$(1): \quad \omega_n \int_0^t \sin\left[\omega_n(t-\tau)\right] d\tau = \frac{u(t)}{u_{st_o}} \qquad 0 \le t \le t_d$$

$$(11): \quad u(t) = \frac{1}{m\omega_n} \int_{t_d}^{t} p(\tau) \sin(\omega_n(t-\tau))\, d\tau + u(t_d) \cos\omega_n \underset{\overset{\uparrow}{t-t_d}}{t} + \frac{\dot{u}(t_d)}{\omega_n} \sin\omega_n \underset{\overset{\uparrow}{t-t_d}}{t}$$

$$p(\tau) = 0$$

$$u(t) = u(t_d) \cos\frac{2\pi}{T_n}\tau + \frac{\dot{u}(t_d)}{2\pi} T_n \sin\omega_n\tau$$

using same process as for Prob. 4.14,

$$\frac{u(t)}{u_{st_o}} = 2\sin\left(\frac{\pi t_d}{T_n}\right) \sin\left[2\pi\left(t/T_n - t_d/2T_n\right)\right] \quad , \; t > t_d$$

Simplifying (1):

$$\omega_n \left[1 - \cos\left(\frac{2\pi}{T_n}t\right)\right] \cdot \frac{1}{\omega_n} = 1 - \cos\frac{2\pi}{T_n}t$$

$$\boxed{\frac{u(t)}{u_{st_o}} = \left| \begin{array}{l} 1 - \cos\frac{2\pi}{T_n}t \\ 2\sin\left(\pi\, t_d/T_n\right) \sin\left[2\pi\left(t/T_n - t_d/2T_n\right)\right] \end{array} \right.}$$

example on Prob. 2 page.

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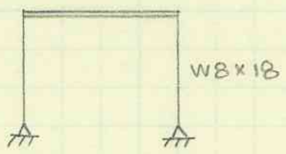
HOMEWORK #5

4. Prob. 4.18

$P_o = 5\,k$

$t_d = 0.25\,s$



$L = 12\,ft$

$I_x = 61.9\,in^4$

$S = 15.2\,in^3$

$E = 30{,}000\,ksi$

$T_n = 0.5\,s$

$$\frac{t_d}{T_n} = \frac{0.25S}{0.5S} = 0.5$$

$R_d = 1.57$ from 4.8.3(c)

$K_{col} = 1.865\,k/in$, $K_{Tot} = 3.73\,k/in$ (from Ex. 4.1)

$$u_{st_o} = \frac{P_o}{K} = \frac{5\,KIP}{3.73\,K/in} = 1.34\,in$$

$$u_o = R_d\,(u_{st})_o = (1.34\,in)(1.57) = 2.105\,in$$

$$M = \frac{3EI}{L^2}\,u_o = \frac{3(30000\,ksi)(61.9\,in^4)}{(144\,in)^2}\,(2.105\,in)$$

$$M = 565.4\,kip\cdot in$$

$$\sigma = \frac{M}{S} = \frac{565.4\,k\cdot in}{15.2\,in^3} = 37.2\,ksi$$

$$\boxed{\begin{aligned} u_o &= 2.11\,in \\ \sigma &= 37.2\,ksi \end{aligned}}$$

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28 Feb 06

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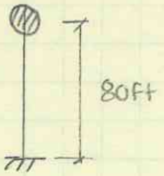
HOMEWORK #5

5. Prob. 4.26



$W_0 = 100.03\,K$

$w_1 = 20.03\,K$

$k = 8.2\,K/in$

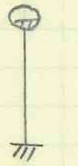
$T_n = 1.12\,s$

$\zeta = 1.23\%$

$t_d/T_n < 0.25$, so

$$F = \int_0^{0.08} p(t)dt = 1.2\,k\cdot s \text{, as given in E4.2}$$

vs.



but, $m = 0.0519$ slug, not 0.259 slug

$$T_n = 2\pi\sqrt{\frac{m}{k}} = 0.4998\,s \sim 0.5\,s$$

$$t_d/T_n = \frac{0.08\,s}{0.5\,s} = 0.16 < 0.25$$

$$u_0 = \frac{F(2\pi)}{kT_n} = 1.839\,in$$

$$f_{s0} = ku_0 = 15.08\,K, \quad M = f_{s0}\cdot h$$

a) $V_{max} = 15.08\,K$
$M_{max} = 1206\,K\cdot ft$

b) A lower mass means a shorter period, higher impulsive (V, M) forces, and larger displacements.

	FULL	EMPTY
Displacement	0.821 in	1.839 in
Base shear	6.73 K	15.08 K
overturn. M	538 K·ft	1206 K·ft

Because this force likes impulse force

$$\dot{u}(0) = \frac{I}{m}$$

Homework No. 5

Assigned: February 21, 2006
Due: February 28, 2006

1. Solve Problem 4.5 from the textbook.

2. Solve Problem 4.14 from the textbook.

3. Solve Problem 4.15 from the textbook.

4. Solve Problem 4.18 from the textbook.
   Note: Use of Figure 4.8.3(c) is all you need to obtain the deformation response factor.

5. Solve Problem 4.26 from the textbook.
   In part (b), first summarize all your results for the empty and full tank by including in a single table, the maximum displacement, maximum base shear, and maximum overturning moment in each case. Then, comment on what effect the mass has on these various response measures.

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HOMEWORK #6

$\frac{24.5}{25}$

1. $\Delta t = 1/3\ s$

$m = 0.2533\ k \cdot s^2/in$ $\qquad u_0 = 0$

$k = 10\ k/in$ $\qquad \dot{u}_0 = 0$

$T_n = 1s \quad (\omega_n = 6.283\ rad/s)$

$\xi = 0.05$ $\qquad c = 0.1592$

Initial calcs:

$$\ddot{u}_0 = \frac{p_0 - \dot{u}_0 c - k u_0}{m} = 0$$

$$u_{-1} = u_0 - \Delta t \dot{u}_0 + \frac{(\Delta t)^2}{2} \ddot{u}_0 = 0$$

$$\hat{k} = \frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t} = \frac{0.2533 k \cdot s^2/in}{(1/3\ s)^2} + \frac{0.1592\ k \cdot s/in}{2(1/3 s)} = 2.519\ k/in$$

$$a = \frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t} = 2.041\ k/in$$

$$b = k - \frac{2m}{(\Delta t)^2} = 10 k/in - \frac{2(0.2533\ k \cdot s^2/in)}{(1/3 s)^2} = 5.441\ k/in$$

Timestep calcs:

$$\hat{p}_i = p_i - a u_{i-1} - b u_i = p_i - 2.041 u_{i-1} - 5.441 u_i$$

$$u_{i+1} = \frac{\hat{p}_i}{\hat{k}} = \frac{\hat{p}_i}{2.519}$$

t_i	p_i	u_{i-1}	u_i	p_i	u_{i+1}
0.000	0.0	0.0	0.0	0.0	0.0
0.333	9.8481	0.0	0.0	9.8481	3.9103
0.667	0.0	0.0	3.9103	-21.2744	-8.4472
1.000	0.0	3.9103	-8.4472	37.9775	15.0794
1.333	0.0	-8.4472	15.0794	-64.8011	-25.7300
1.667	0.0	15.0794	-25.7300	109.2112	43.3636
2.000	0.0	-25.7300	43.3636	-183.4115	-72.8257

These values are so far off the theoretical (shown →, with Δt = 0.1 s, up to t = 1.0 s) because of the much larger value of Δt. The larger the Δt used, the less accurate the approximation.

"Stability"

$$\frac{\Delta t}{T} \overset{?}{<} \frac{1}{\pi}$$

t	Theoretical
0	0
0.1	0.0328
0.2	0.2332
0.3	0.6487
0.4	1.1605
0.5	1.5241
0.6	1.4814
0.7	0.9245
0.8	0.0593
0.9	-0.7751
1.0	-1.2718

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HOMEWORK #6

2. $\Delta t = 0.1\,s$

$\ddot{u}_0 = 0$

$u_{-1} = 0$

$\hat{k} = 26.126$ K/in

$a = 24.534$ K/in

$b = -40.66$ K/in

$m = 0.2533$ K·s²/in

$k = 10$ K/in

$c = 0.1592$ K·s/in

t_i	p_i	u_{i-1}	u_i	p_i	u_{i+1}
0.0	0.0	0.0	0.0	0.0	0.0
0.1	5.0	0.0	0.0	5.0	0.1913802
0.2	8.660	0.0	0.1914	16.4418	0.6293
0.3	10.0	0.1914	0.6293	30.8931	1.1825
0.4	8.660	0.6293	1.1825	41.2994	1.5808
0.5	5.0	1.1825	1.5808	40.2638	1.5411
0.6	0.0	1.5808	1.5411	23.8799	0.9140
0.7	0.0	1.5411	0.9140	-0.6459	-0.0247
0.8	0.0	0.9140	-0.0247	-23.4300	-0.8968
0.9	0.0	-0.0247	-0.8968	-35.8576	-1.3725
1.0	0.0	-0.8968	-1.3725	-33.8032	-1.2939
1.1	0.0	-1.3725	-1.2939	-18.9353	-0.7248
1.2	0.0	-1.2939	-0.7248	2.2742	0.0870
1.3	0.0	-0.7248	0.0870	21.3209	0.8161
1.4	0.0	0.0870	0.8161	31.0461	1.1883
1.5	0.0	0.8161	1.1883	28.2955	1.0830
1.6	0.0	1.1883	1.0830	14.8821	0.5696
1.7	0.0	1.0830	0.5696	-3.4102	-0.1305
1.8	0.0	0.5696	-0.1305	-19.2826	-0.7381
1.9	0.0	-0.1305	-0.7381	-26.8072	-1.0261
2.0	0.0	-0.7381	-1.0261	-23.6125	-0.9038

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HOMEWORK #6

3. $\Delta t = 0.1\,s$

$\gamma = 1/2, \beta = 1/4$

$\ddot{u}_0 = 0$

$$\hat{k} = k + \frac{\gamma}{\beta \Delta t} c + \frac{m}{\beta (\Delta t)^2} = 10 + \frac{1/2}{1/4(0.1)}(0.1592) + \frac{0.2533}{1/4\,(0.1)^2} = 114.50$$

$$a = \frac{m}{\beta \Delta t} + \frac{\gamma}{\beta} c = \frac{0.2533}{1/4(0.1)} + \frac{1/2}{1/4}(0.1592) = 10.45$$

$$b = \frac{m}{2\beta} + \Delta t \left(\frac{\gamma}{2\beta} - 1\right) c = \frac{0.2533}{2(1/4)} + (0.1)(0.1592)\left[\frac{0.5}{0.5} - 1\right] = 0.5066$$

4. $\gamma = 1/2, \beta = 1/6$

$\hat{k} = 166.76$

$a = 15.68$

$b = 0.768$

Average Acceleration (3)

Δt	p_i	$\ddot{u}_i$	Δp_i	$\Delta \hat{p}_i$	Δu_i	$\Delta \mathring{u}_i$	$\Delta \ddot{u}_i$	$\mathring{u}_i$	u_i
0.0	0.0	0.0	5.000	5.000	0.043667	0.873332	17.46664	0.0	0.0
0.1	5.000	17.467	3.660	21.636	0.189	2.032	5.713	0.873	0.044
0.2	8.660	23.180	1.340	43.448	0.379	1.778	-10.808	2.906	0.233
0.3	10.000	12.372	-1.340	53.870	0.470	0.043	-23.889	4.683	0.612
0.4	8.660	-11.517	-3.660	39.893	0.348	-2.484	-26.644	4.726	1.083
0.5	5.000	-38.161	-5.000	-0.902	-0.008	-4.642	-16.511	2.242	1.431
0.6	0.0	-54.672	0.0	-52.774	-0.461	-4.419	20.973	-2.400	1.423
0.7	0.0	-33.700	0.0	-88.325	-0.771	-1.791	31.579	-6.818	0.962
0.8	0.0	-2.121	0.0	-91.045	-0.795	1.316	30.563	-8.609	0.191
0.9	0.0	28.442	0.0	-61.808	-0.540	3.791	18.928	-7.293	-0.604
1.0	0.0	47.370						-3.503	-1.144

Linear Acceleration (4)

Δt	p_i	$\ddot{u}_i$	Δp_i	$\Delta \hat{p}_i$	Δu_i	$\Delta \mathring{u}_i$	$\Delta \ddot{u}_i$	$\mathring{u}_i$	u_i
0.0	0.0	0.0	5.000	5.000	0.029984	0.899518	17.99036	0.0	0.0
0.1	5.000	17.990	3.660	31.575	0.189347	2.08235	5.666285	0.900	0.030
0.2	8.660	23.657	1.340	66.247	0.397271	1.789691	-11.5195	2.982	0.219
0.3	10.000	12.137	-1.340	82.777	0.496396	-0.02967	-24.8677	4.772	0.617
0.4	8.660	-12.730	-3.660	60.897	0.365183	-2.63365	-27.212	4.742	1.113
0.5	5.000	-39.943	-5.000	-2.622	-0.01573	-4.79936	-16.1022	2.108	1.478
0.6	0.0	-56.045	0.0	-85.219	-0.51104	-4.45568	22.97581	-2.691	1.462
0.7	0.0	-33.069	0.0	-137.423	-0.82409	-1.62898	33.55816	-7.147	0.951
0.8	0.0	0.489	0.0	-137.190	-0.8227	1.621918	31.45987	-8.776	0.127
0.9	0.0	31.949	0.0	-87.609	-0.52537	4.103026	18.16229	-7.154	-0.695
1.0	0.0	50.111						-3.051	-1.221

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HOMEWORK #6

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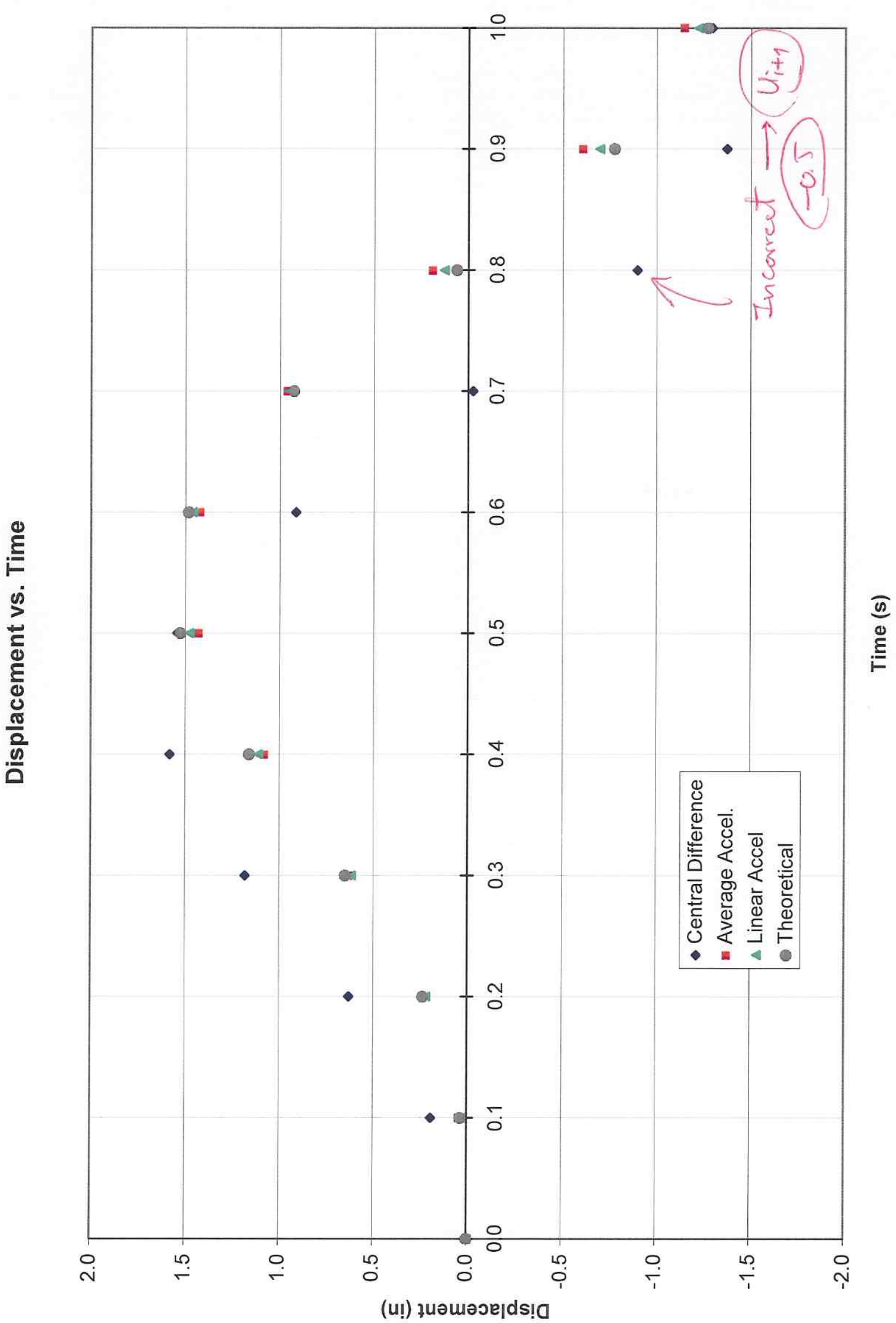
5.



Incorrect → u_{i+1}

-0.5

Homework No. 6

Assigned: February 28, 2006
Due: March 21, 2006

1. For the half-cycle sine pulse shown in Figure E5.1 of the textbook, use the central difference method with $\Delta t = 1/3$ sec and compute the displacement up to 2 sec. Comment on what happens to your solution. Why?

2. Repeat Problem 1 using with $\Delta t = 0.1$ sec.

3. For the same half-cycle sine pulse force, use the average acceleration method with $\Delta t = 0.1$ sec to compute the displacement up to 1 sec.

4. Repeat Problem 3 using the linear acceleration method instead of the average acceleration method.

5. Create a plot showing displacement versus time up to $t = 1$ sec based on your results from Problems 2, 3, and 4. Also, include the theoretical (exact) displacement values on your plot. Thus, on one plot, you should show theoretical values as well as results from three methods: central difference, average acceleration, and linear acceleration (all with $\Delta t = 0.1$ sec).

   Note: You may use the theoretical values from Table E5.4 if you wish but for extra credit, derive expressions for the theoretical displacement and use your values instead. Include any work associated with computing theoretical values.

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HOMEWORK #7

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04 April 2006

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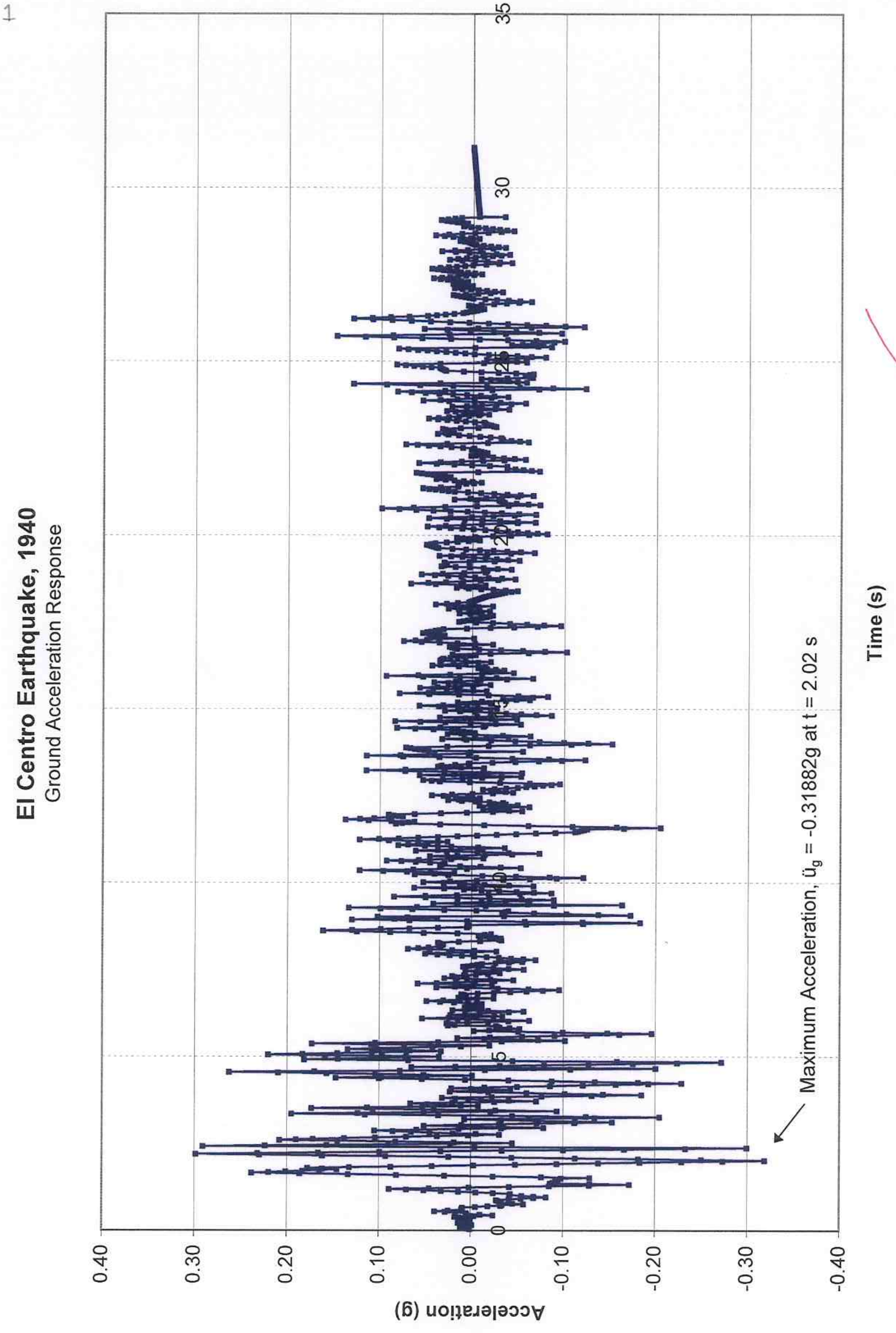
$\frac{30}{30} + \frac{14}{20} = \frac{44}{50}$

1. Plot 1



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HOMEWORK #7

1. (cont'd)

Plot 2 – interpolation of excitation

calculations:

$\omega_n, K, e^{-\xi\omega_n\Delta t}, \omega_D$
$A, B, C, D, A', B', C', D'$
} summarized on pg 169, table 5.2.1

find maximum u_i value

$D = u_{io}$
$V = \omega_n u_{io}$
$A = \omega_n^2 u_{io}$

normalize to g

	ξ = 0	ξ = 0.05	ξ = 0.20
T_n	A/g	A/g	A/g
0.02	0.32512	0.318149	0.315948
0.035	0.531516	0.315314	0.313788
0.125	1.229945	0.714503	0.480896
0.5	1.314004	0.914939	0.462185
1	0.758426	0.451431	0.180629
1.5	0.340737	0.188712	0.111393
3	0.260323	0.122274	0.066755

Plot 2 on next page

Plot 3 – Newmark's method, linear acceleration

$\ddot{u}_0, \hat{k}, a, b$ – calculate with $\gamma = 1/2, \beta = 1/6$

$\Delta\hat{p}_i, \Delta u_i, \Delta\dot{u}_i, \Delta\ddot{u}_i$ from equations in table 5.4.2

find A in same way, normalize

	ξ = 0	ξ = 0.05	ξ = 0.20
T_n	A/g	A/g	A/g
0.02			
0.035			
0.125	1.869696	0.682444	0.493227
0.5	1.650122	0.912848	0.478489
1	0.753167	0.448008	0.192945
1.5	0.342574	0.193534	0.121721
3	0.260157	0.127212	0.077342

} approx. is unstable

Plot 3 after plot 2

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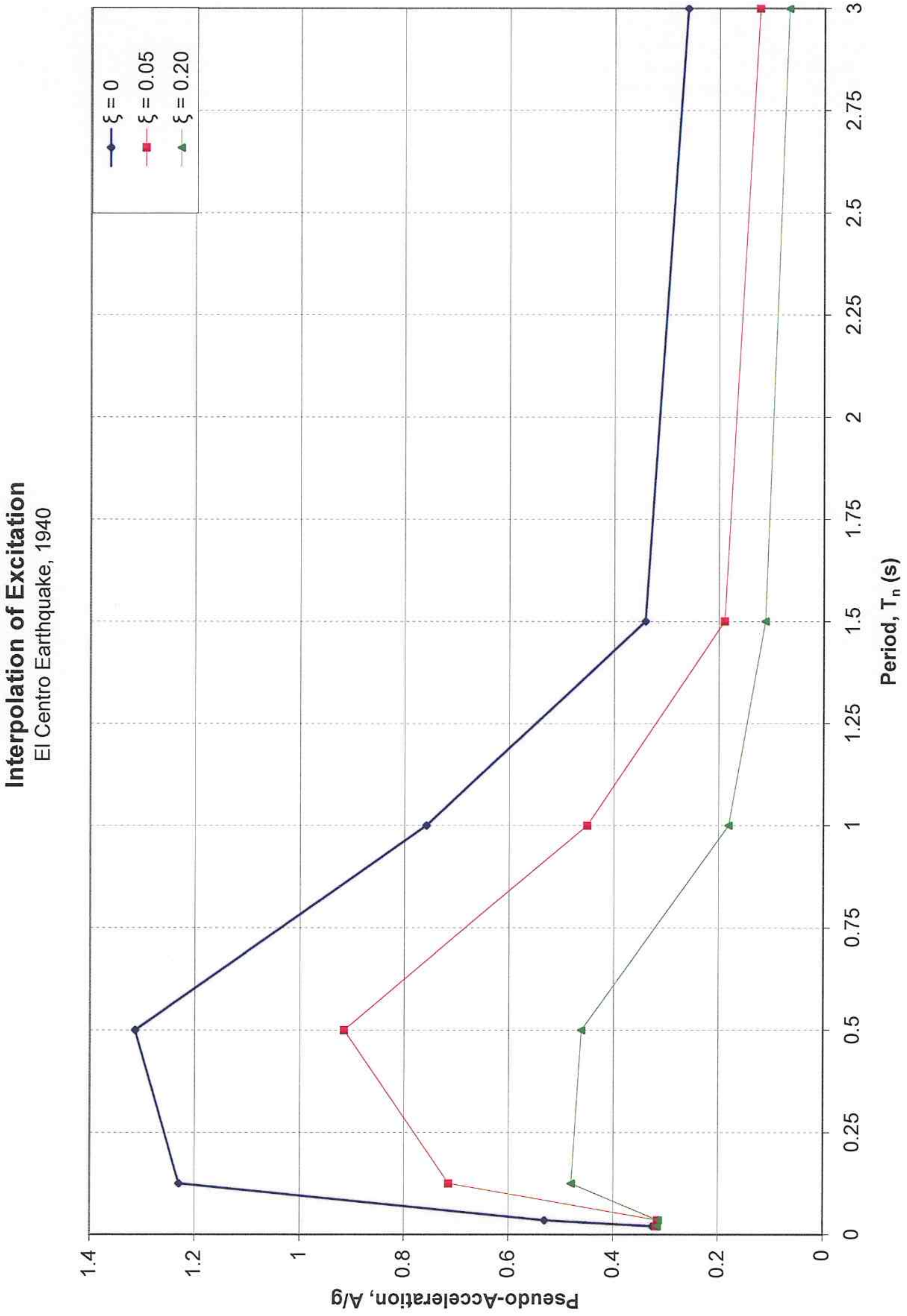
1. Plot 2



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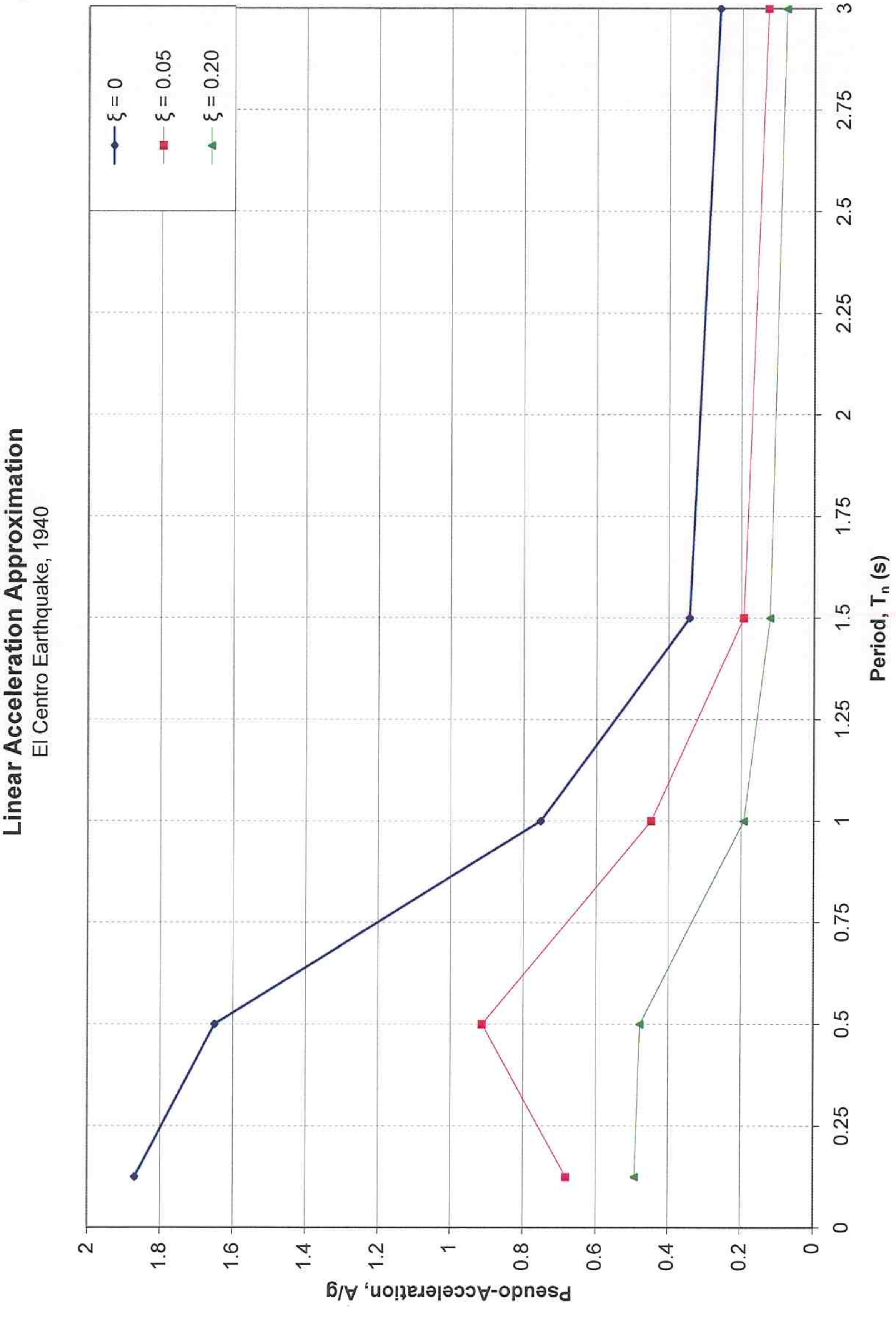
HOMEWORK #7

1. Plot 3



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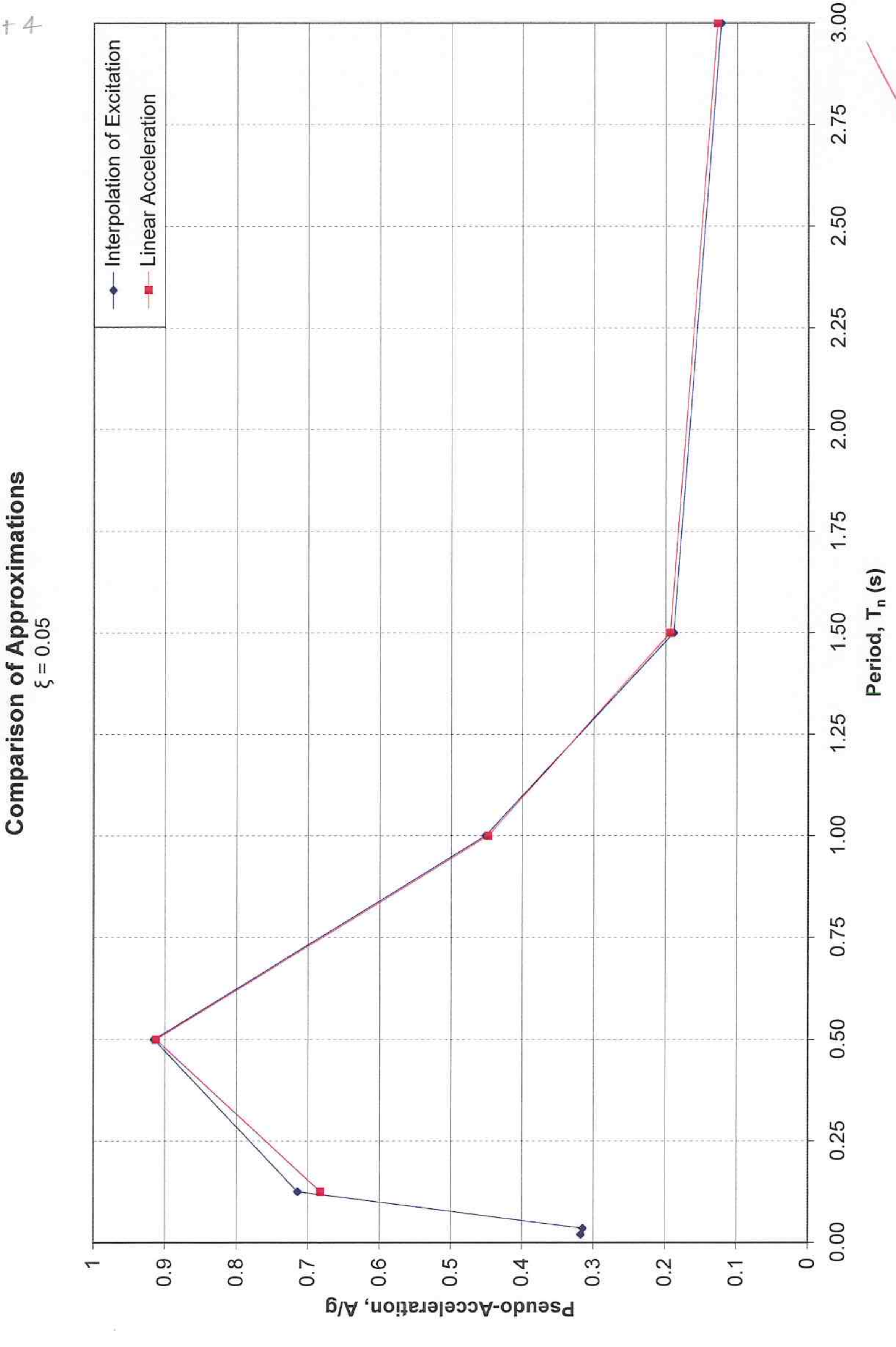
HOMEWORK #7

1. Plot 4



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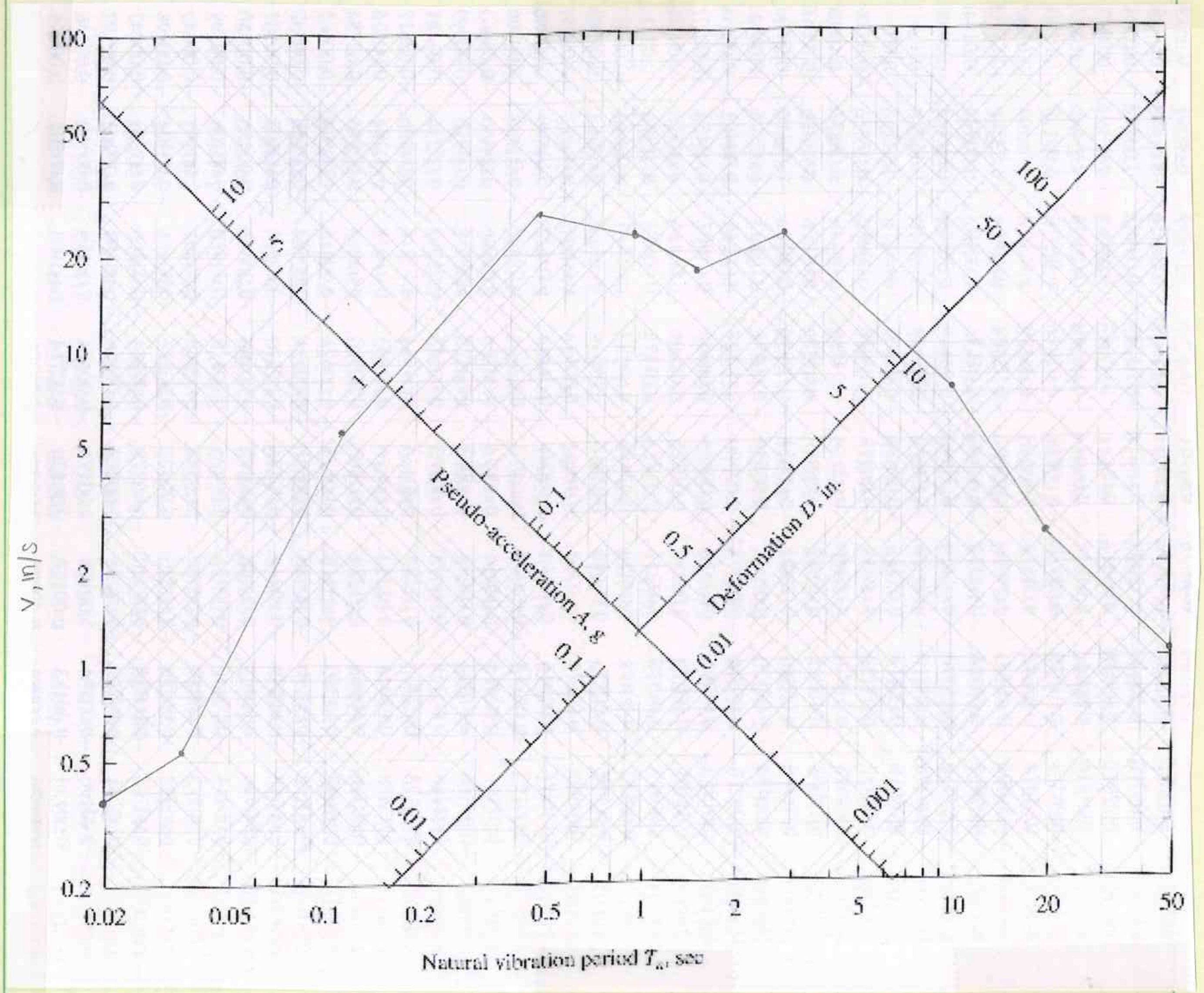
HOMEWORK #7

1. (cont'd)

Plot 5

5% damping, interpolation data points
Tn, D, V, A/g



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HOMEWORK #7

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1. Excel details

ξ	ξ
T_n	1.0

Interpolation of Excitation

t	p_i*g	p_i	Cp_i	Dp_{i+1}	Bu'_i	u'_i	Au_i	u_i
4.7	25.03486	0.06479	0.003327	0.000429	0.424948	21.43753	3.04106	3.065176
4.72	6.456744	0.01671	0.000858	-0.0008	0.375038	18.91971	3.442463	3.469763
4.74	-12.1214	-0.03137	-0.00161	-0.00204	0.312599	15.76982	3.787518	3.817554
4.76	-30.6995	-0.07945	-0.00408	-0.00327	0.238763	12.045	4.064236	4.096467
4.78	-49.2776	-0.12753	-0.00655	-0.00451	0.154828	7.810701	4.261849	4.295647
4.8	-67.8557	-0.17561	-0.00902	-0.00574	0.062233	3.139495	4.37096	4.405623
4.82	-86.4338	-0.22369	-0.01149	-0.00697	-0.03747	-1.89012	4.383671	4.418436
4.84	**-105.012**	**-0.27177**	**-0.01396**	**-0.00407**	**-0.14262**	**-7.19502**	**4.293694**	**4.327745**
4.86	-61.2483	-0.15851	-0.00814	-0.00116	-0.23921	-12.0676	4.100529	4.133048
4.88	-17.4846	-0.04525	-0.00232	0.001745	-0.31367	-15.824	3.82171	3.852018
4.9	26.28293	0.06802	0.003493	0.004651	-0.36512	-18.4194	3.479863	3.50746
4.92	70.04659	0.18128	0.009309	0.003711	-0.39304	-19.8278	3.098317	3.122887
4.94	55.8889	0.14464	0.007427	0.002771	-0.40873	-20.6192	2.696911	2.718298
4.96	41.7312	0.108	0.005546	0.001831	-0.42339	-21.3588	2.280299	2.298382
4.98	27.57737	0.07137	0.003665	0.000891	-0.4368	-22.0354	1.849622	1.86429
5	13.41967	0.03473	0.001783	0.00248	-0.44877	-22.6393	1.406228	1.417379

peak

Linear Acceleration

t	p_i*g	p_i	$\ddot{u}_i$	Δp_i	Δp_i-hat	Δu_i	$\Delta u'_i$	$\Delta \ddot{u}_i$	u'_i	u_i
4.7	25.03486	0.06479	-110.328	-18.5781	6152.429	0.406538	-2.53689	-33.0336	21.54027	3.024286
4.72	6.456744	0.01671	-143.361	-18.5781	5287.272	0.34937	-3.17101	-30.3783	19.00338	3.430824
4.74	-12.1214	-0.03137	-173.74	-18.5781	4238.666	0.280081	-3.7476	-27.2806	15.83237	3.780194
4.76	-30.6995	-0.07945	-201.02	-18.5781	3025.31	0.199905	-4.25835	-23.7945	12.08477	4.060275
4.78	-49.2776	-0.12753	-224.815	-18.5781	1668.246	0.110234	-4.69608	-19.9793	7.826428	4.26018
4.8	-67.8557	-0.17561	-244.794	-18.5781	190.5056	0.012588	-5.05487	-15.899	3.130343	4.370414
4.82	**-86.4338**	**-0.22369**	**-260.693**	**-18.5781**	**-1383.28**	**-0.0914**	**-5.33006**	**-11.6207**	**-1.92452**	**4.383002**
4.84	-105.012	-0.27177	-272.314	43.76366	-2964.94	-0.19592	-4.9005	54.57719	-7.25459	4.291598
4.86	-61.2483	-0.15851	-217.736	43.76366	-4280.25	-0.28283	-3.78167	57.30539	-12.1551	4.095682
4.88	-17.4846	-0.04525	-160.431	43.76753	-5249.6	-0.34688	-2.61755	59.1065	-15.9368	3.812853
4.9	26.28293	0.06802	-101.324	43.76366	-5862.12	-0.38735	-1.42697	59.95239	-18.5543	3.465972
4.92	70.04659	0.18128	-41.3721	-14.1577	-6170.58	-0.40774	-0.80301	2.443665	-19.9813	3.078618
4.94	55.8889	0.14464	-38.9284	-14.1577	-6405.65	-0.42327	-0.74834	3.022529	-20.7843	2.670881
4.96	41.7312	0.108	-35.9059	-14.1538	-6622.47	-0.4376	-0.68261	3.550705	-21.5326	2.247611
4.98	27.57737	0.07137	-32.3552	-14.1577	-6817.87	-0.45051	-0.60701	4.009068	-22.2152	1.810014
5	13.41967	0.03473	-28.3461	23.92975	-6950.98	-0.4593	-0.14539	42.1537	-22.8223	1.359505

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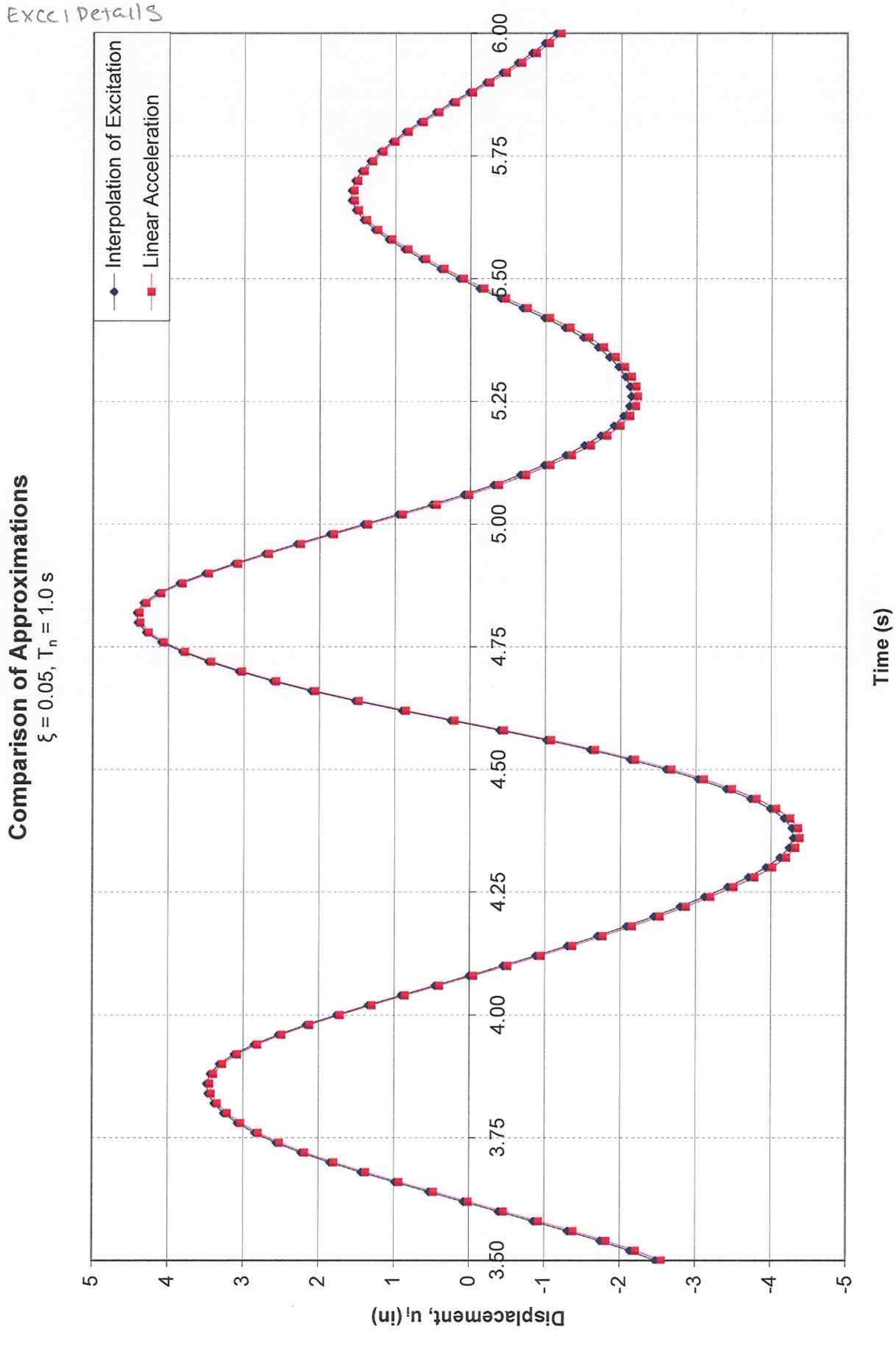
HOMEWORK #7

1. EXCEL Details



✓

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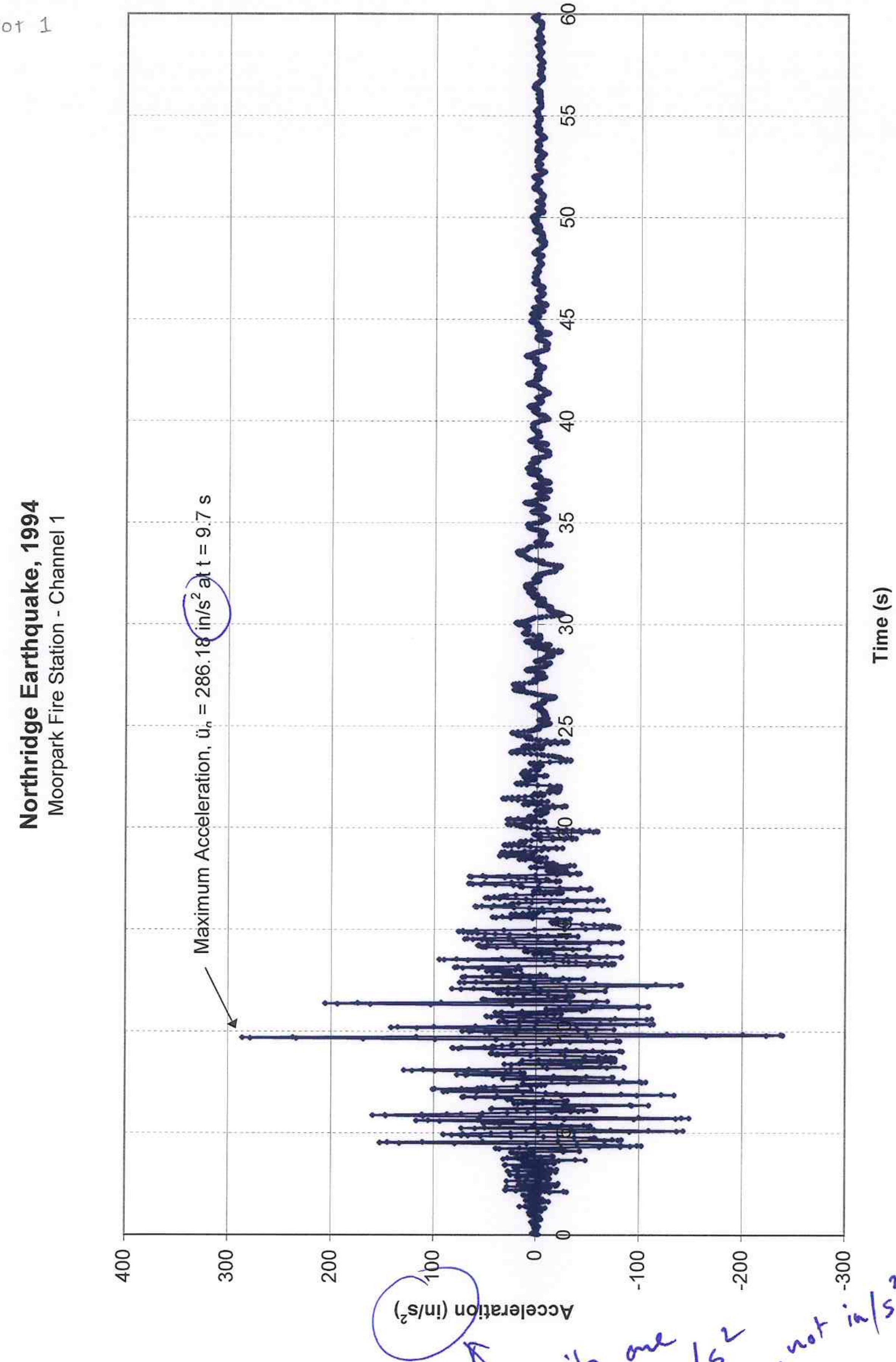
HOMEWORK #1

2, Plot 1



units are cm/s^2, not in/s^2

(-2)

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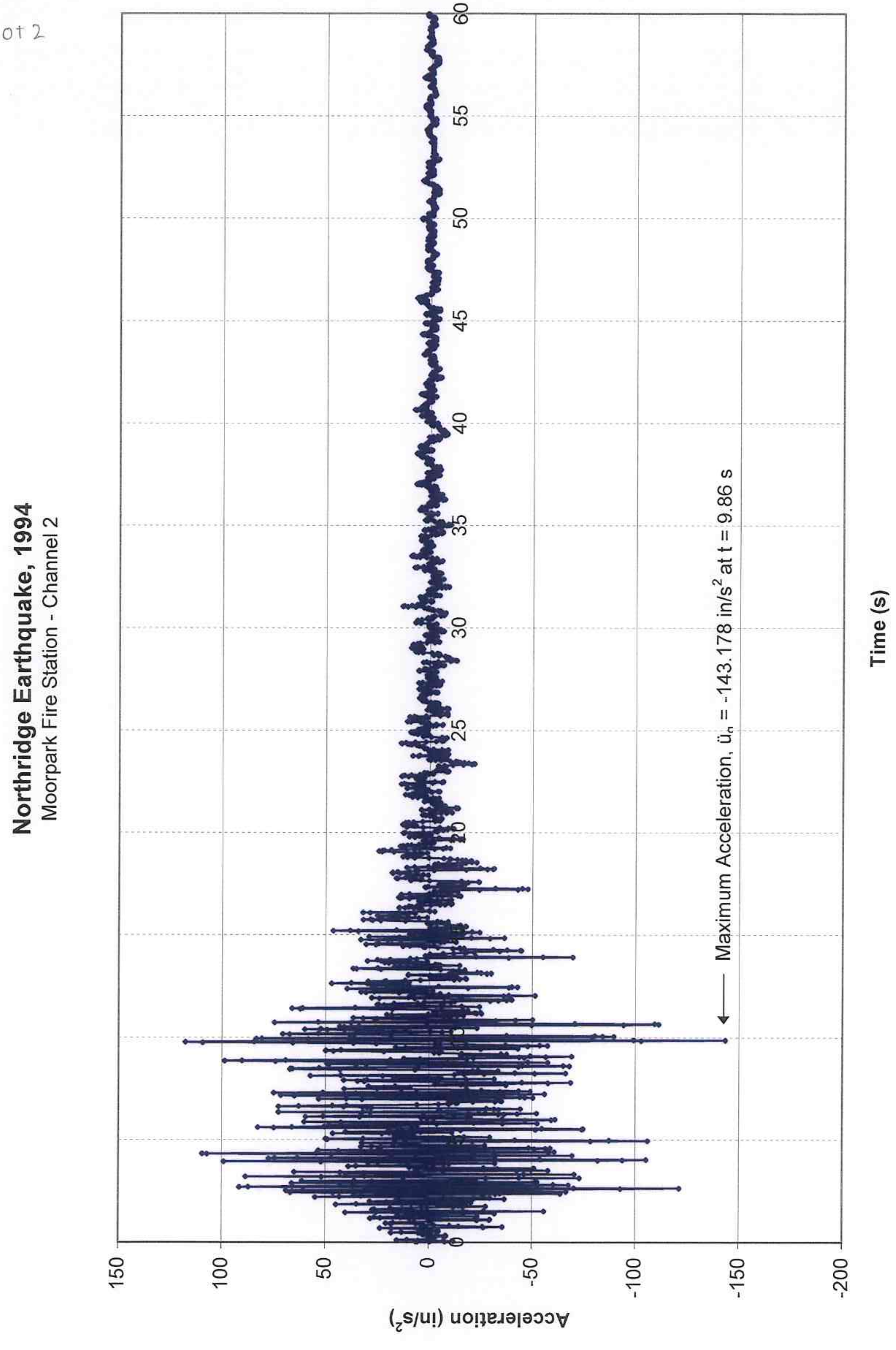
HOMEWORK #7

2. Plot 2



✓

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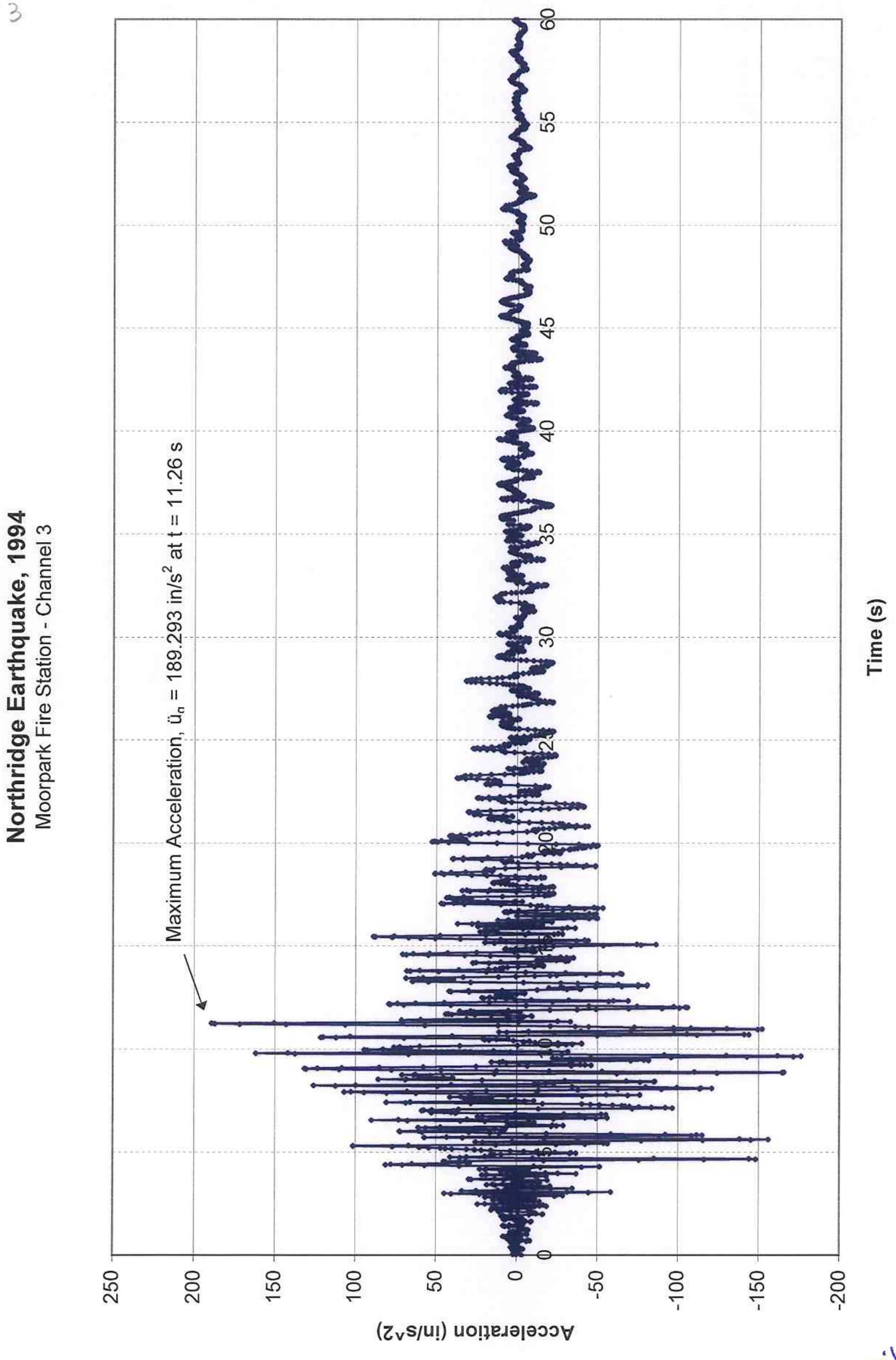
HOMEWORK #7

2. Plot 3



units.

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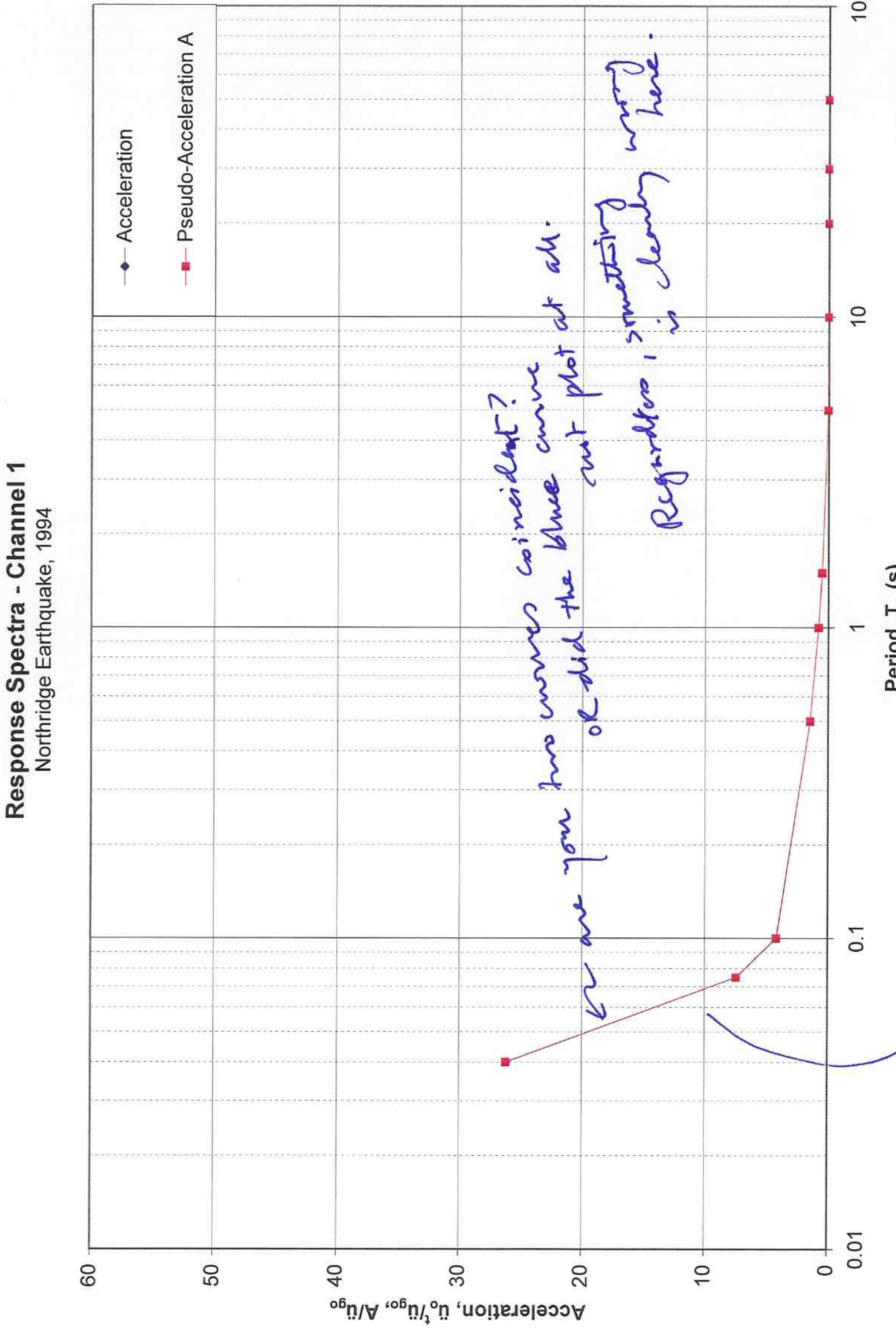
HOMEWORK #7

2. Plot 4



$\boxed{-4}$

Are your two curves coincident? or did the blue curve not plot at all? Regardless, something is clearly wrong here.

Something is wrong with all three plots.

$\lim_{T_n \to 0} \frac{A}{\ddot{u}_{go}} \to 1$ & $\frac{\ddot{u}_o^t}{\ddot{u}_{go}} \to 1$

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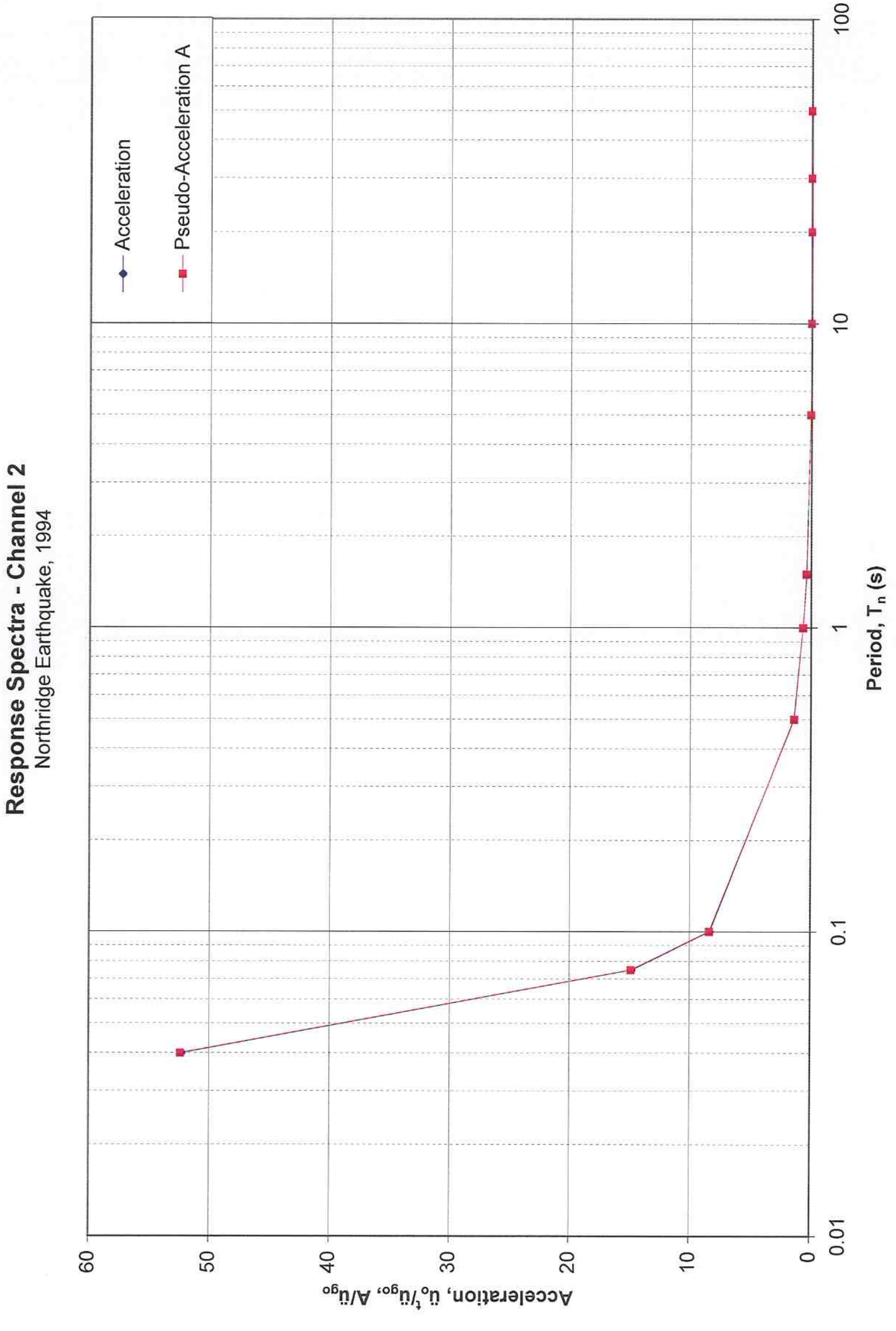
HOMEWORK #7

2. Plot 5



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HOMEWORK #7

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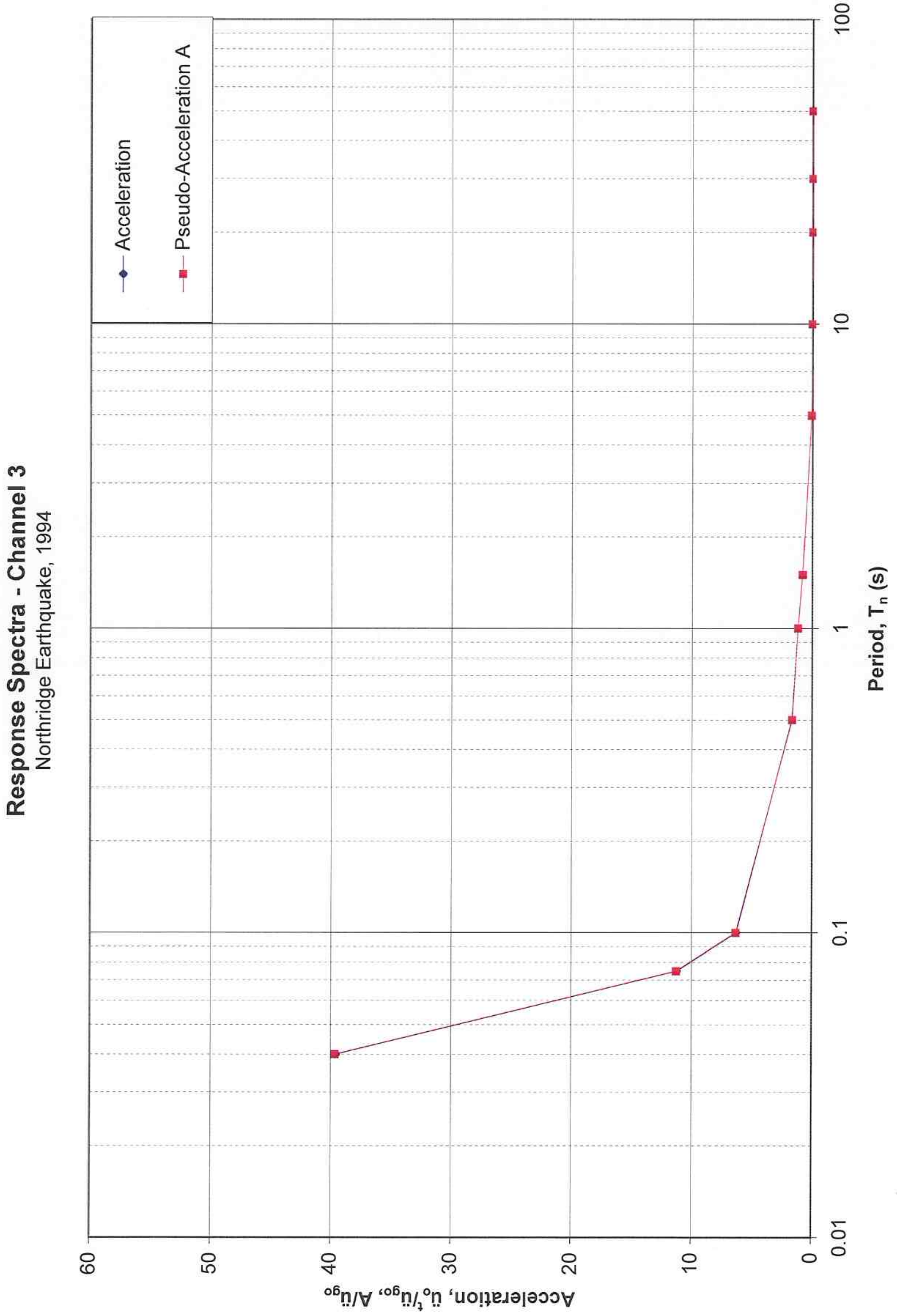
2. Plot 6



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HOMEWORK #7

2. DISCUSSION

1) For my station (Moorpark Fire Station), the peak vertical acceleration was larger and occured sooner than that of channel 2. The max acceleration occured just after the max for channel 1. The peak values are summarized below:

Channel	t	$\ddot{u}_{go}$
1	9.75	286.18 in/s^2
2	11.26 s	143.178 in/s^2
3	9.86 s	189.293 in/s^2

units.

The change in acceleration for channel 3 seems much smaller than the other two, and the tail end of the results have a higher amplitude.

2) In all three directions, buildings with short natural periods would have experienced the highest acceleration response (and thus, forces in the building). The Newmark's approximation was unstable for periods below $T_n = 0.04$ s. Structures with $T_n > 0.1$ s would have experienced accelerations $A/\ddot{u}_{go} \leq 10$.

↖ 10 is too large! something is wrong.

3) As shown on plots 4 through 6, there was almost no difference between the plots of $\ddot{u}^t_o/\ddot{u}_{go}$ and $A/\ddot{u}_{go}$, indicating that the pseudo acceleration value A predicted $\ddot{u}^t_o$ very accurately. On the plots, the two lines are essentially indistinguishable.

×

↳ they shouldn't be.

Homework No. 7

Assigned: March 23, 2006
Due: April 4, 2006

Problem 1

The N-S component of ground acceleration time history recorded at the Imperial Valley Irrigation District substation during the El Centro 1940 earthquake is available at:
http://www.ce.utexas.edu/prof/Manuel/Spring2006_CE384P/homework.htm

The data, recorded at time intervals of 0.02 seconds, are available in two formats:

EC.XLS	Excel format
EC.TXT	Plain text format

Use either of these files to complete your homework.

Required Plot 1
Plot the time history of the ground acceleration given.
What is the peak ground acceleration (in units of g)? Indicate this value on your plot.

Next, you are required to create response spectra for this earthquake record.

Use the following TWO methods:
Interpolation of excitation
Newmark's method (linear acceleration)

For THREE damping ratios equal to 0%, 5%, and 20% each taken one at a time, obtain a response spectrum showing peak pseudo-acceleration response, A, versus period.

Use the following TEN natural periods for your plot(s).
0.02, 0.035, 0.125, 0.5, 1.0, 1.5, 3.0, 10.0, 20.0, 50.0 (seconds)

Required Plot 2
Method: Interpolation of Excitation
Plot A versus T_n for periods up to 3.0 seconds.
This plot should look like Figure 6.6.5 except that you are using a lot fewer periods (seven) and, so, will have a smoother spectrum. Include spectra for 0%, 5%, and 20% damping ratios on the same plot. Use units of g for A.

Required Plot 3
Method: Linear Acceleration
Plot A versus T_n for periods up to 3.0 seconds.
This plot should look like Figure 6.6.5 but will again be smoother. Include spectra for 0%, 5%, and 20% damping ratios on the same plot. Use units of g for A.

Required Plot 4
Damping ratio: 5%
Plot *A* versus T_n for periods up to 3.0 seconds. Include spectra based on both methods on the same plot. This plot should look like Figure 6.6.5. Use units of g for *A*.

Required Plot 5 (to be done by hand)
Make a copy of the four-way log scale paper in Figure A6.1 of the text and plot the 5%-damped spectrum using all the ten natural periods (ranging from 0.02 to 50.0 seconds). Plot only the spectrum resulting from the use of the method based on interpolation of excitation. Your plot should look like the one in Figure 6.8.1 (with a lot fewer points for plotting).

Important Notes for Problem 1
If you use Excel, turn in only any one response analysis worksheet (i.e., one damping ratio and one natural period). Keep the output submitted to a minimum (save some trees!) by printing only the most interesting part of the history where the response peaks (say +/- 0.2 seconds around the time where the peak response is observed). Print these once for each method.
Summary: Choose one period (say, 0.5 seconds) and one damping ratio (say, 5%).
Then, for each of the two methods, print a small portion of the time history around where the peak response was recorded.

If you use a program other than Excel, please include your program listing as well as a part of the output for one response analysis.
Summary: Choose one period (say, 0.5 seconds) and one damping ratio (say, 5%).
Then, for each of the two methods, print a small portion of the time history around where the peak response was recorded.

Problem 2

From the January 17, 1994 Northridge earthquake that occurred in the Los Angeles area, there are a large number of ground acceleration records available at the following Web site: **http://www.quake.ca.gov/cisn-edc/login/register.asp** (register at the site first). After registering, if you start at **http://www.quake.ca.gov/cisn-edc/**, you can search in different ways for what you need. For instance, you can search by Earthquake or by Station name.

Using the attached sheet, identify the station assigned to you and download the data for all three components of ground acceleration (two orthogonal horizontal components and one vertical (designated UP)) at your station. If you have difficulties with the records assigned to you, pick one of the extra stations listed at the bottom of the attached sheet. Please identify your station in your report.

Note that you will use the V2 file that contains three channels of data for the three components of ground acceleration. The time interval for the data will be specified in the file. You will need to find a way to separate out the data of the three channels from the rest of the stuff in the V2 file before you can use the records. Also, the data might be in a format that shows 8 acceleration values per line; this might not be an ideal format for your analysis. To facilitate analysis, you may want to have one value per line. Please ask me or someone else for help with formatting if you need it.

Required Plots 1-3
Plot the time histories for the three components of the ground acceleration at your station. On your three plots, indicate the peak ground acceleration values.

Required Plots 4-6
For the three components, use Newmark's linear acceleration method to obtain 5%-damped spectra for (i) pseudo-acceleration response, and (ii) total acceleration. Include both spectra on the same plot. Choose axes for your plots such as the ones used in Figure 6.12.2 (i.e., use a logarithmic scale for natural period on the horizontal axis, and use acceleration values, normalized with respect to the corresponding peak ground acceleration, on the vertical scale).

Using your results, include comments on:

1. the vertical peak ground acceleration value versus the two horizontal peak ground acceleration values for the station that you analyzed;
2. the types of structures (i.e., what natural period range) that might have experienced the largest base shear values at your station site;
3. the differences between the pseudo-acceleration response spectra and the total acceleration response spectra.

Include any noteworthy comments that you learned about the records you used and from doing this exercise.

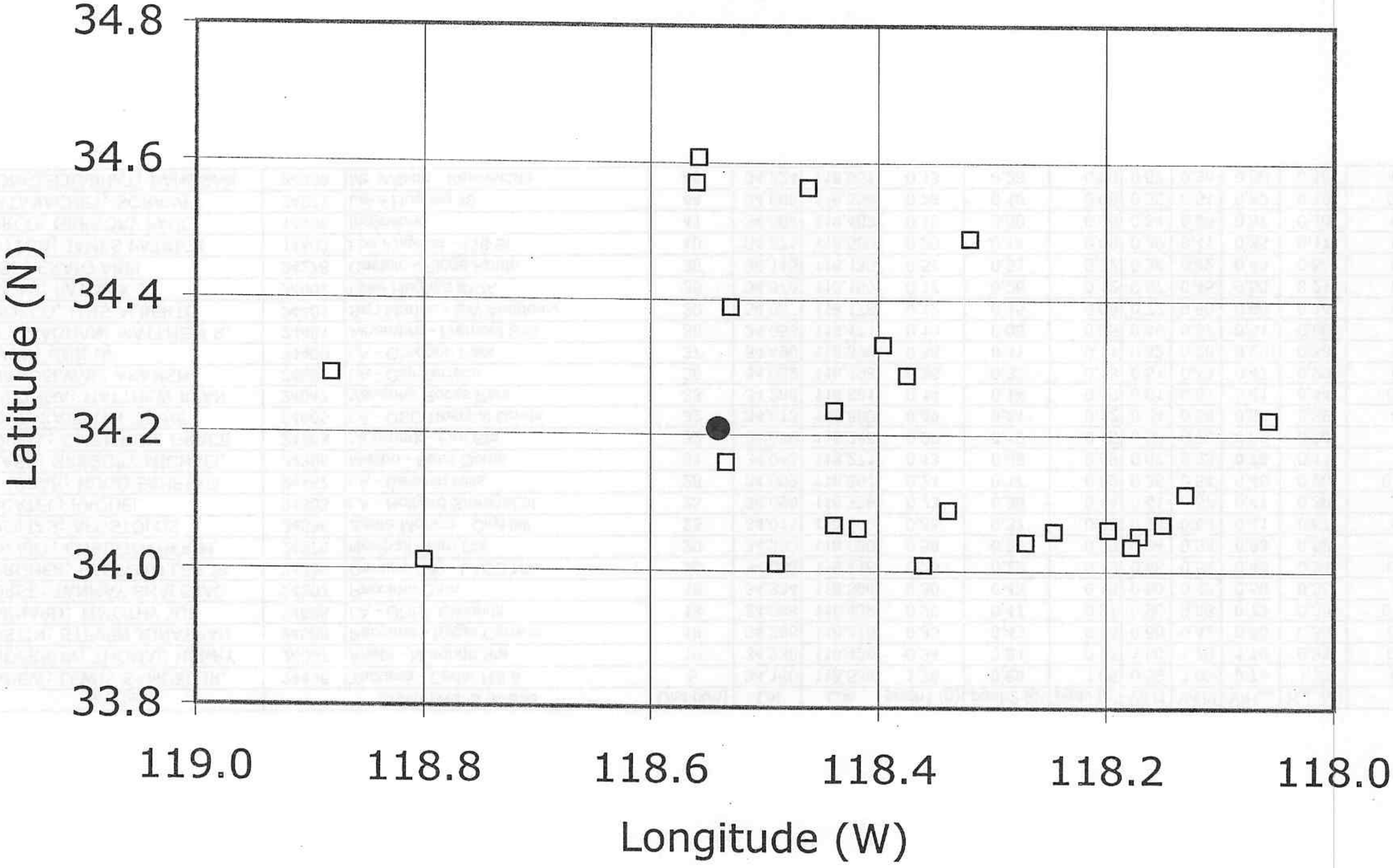
Student	Station No. & Name		Dist (km)
AGNEW; LEWIS SAMUEL JR.	24436	Tarzana - Cedar Hill A	5
ANDERSON; THOMAS HENRY	24087	Arleta - Nordhoff Sta	10
AUSTIN; STEVEN JONATHAN	24088	Pacoima - Kagel Canyon	18
BARNARD; TIMOTHY JOE	24688	LA - UCLA Grounds	18
BORSE; TANMAY BHALERAO	24207	Pacoima Dam	19
BURGHER; BEDFORD LEE JR.	24389	Century City - LACC Nth	20
COZART; CHRISTOPHER M.	24279	Newhall - Fire Sta	20
DROLIAS; APOSTOLOS	24538	Santa Monica - CityHall	23
ESCATEL; RAQUEL	24303	LA - Hollywd StorageLot	23
ESCOBAR; HUGO ERNESTO	24157	LA - Baldwin Hills	28
GLASS; GREGORY MICHAEL	24396	Malibu - Point Dume	32
HOVELL; CATHERINE GRACE	24283	Moorpark - Fire Sta	33
HOYT; KATHRYN DIANE	24605	LA - USC Hospital Grnds	36
HUIZINGA; MATTHEW RYAN	24047	Vasquez Rocks Park	37
KHANDELWAL; AKANSHA	24592	LA - City Terrace	38
KWON; GUN UP	24400	LA - Obregon Park	39
O'CALLAGHAN; MATTHEW R.	24461	Alhambra - Fremont Sch	39
OROZCO; LUIS ALBERTO	24401	San Marino - SW Academy	39
RAGAN; PATRICK J.	24607	Lake Hughes #12A	40
RIOS; CRAIG ABEL	24278	Castaic - Ridge Route	41
SUTTON; JAMES PATRICK	14403	Los Angeles - 116 St	41
TURCO; GREGORY PAUL	14196	Inglewood	42
WATANACHET; SORAWIT	24272	Lake Hughes #9	44
WONGJEERAPHAT; RANGSAN	24399	Mt. Wilson - SeismicSta	45

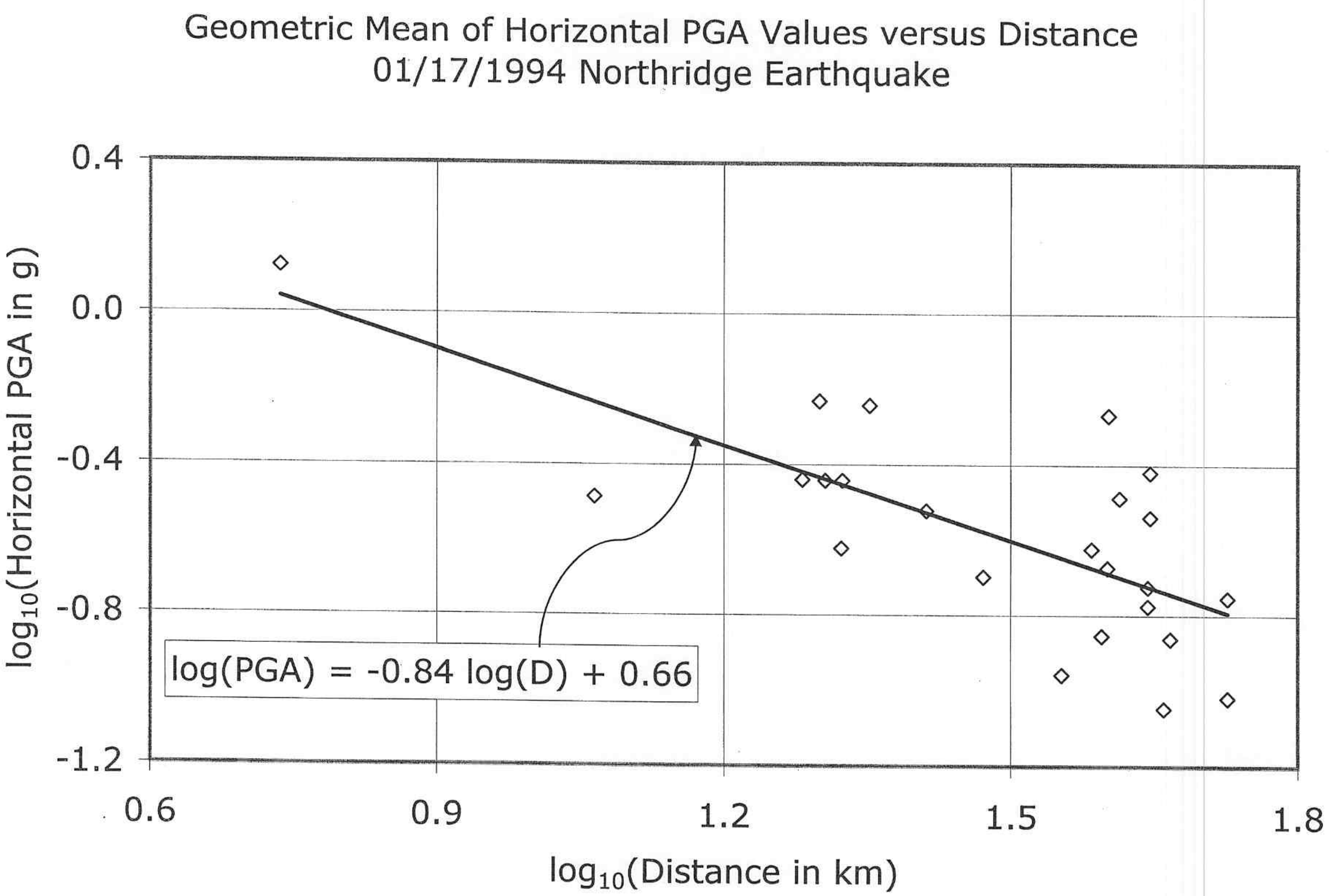
extra records (if you have any problem with yours)......	14368	Downey	47
	24469	Lake Hughes #4	49
	24523	Lake Hughes #4A	49

CSMIP Strong-Motion Records from the Northridge, California Earthquake of 1/17/1994

Web Site --- **http://www.quake.ca.gov/cisn-edc/**

1994 Northridge Earthquake Ground Motion Recording Stations analyzed by Spring 2006 CE384P Students





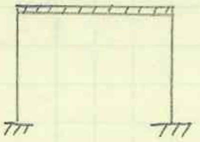
S_a = spectral acceleration

34.5/35

nice work!

HOMEWORK #8

1. Prob. 6.18



$h = 12\ ft$ $\quad L = 24\ ft$

$I_b = 160\ in^4$ $\quad I_c = 320\ in^4$

$E = 30{,}000\ ksi$ $\quad w = 100\ kip$

$\zeta = 0.05$

$$k = \frac{24EI_c}{h^3}\ \frac{12\rho+1}{12\rho+4}, \quad \rho = \frac{I_b}{4I_c} = \frac{160\ in^4}{4(320\ in^4)} = 0.125$$

$$k = \frac{24(30{,}000\ ksi)(320\ in^4)}{(144\ in)^3} \cdot \frac{12(0.125)+1}{12(0.125)+4} = 35.07\ k/in$$

$$\omega_n = \left[\frac{k}{m}\right]^{1/2} = \left[\frac{35.07\ k/in}{100k/386\ in/s^2}\right]^{1/2} = 11.64\ rad/s$$

$T_n = 0.54\ s$

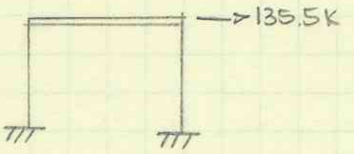
From 6.9.5, $A = 2.71g$ at $T_n = 0.54S$

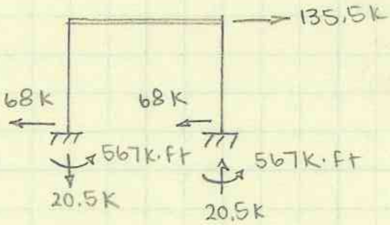
scaled, $A = 1.355g$

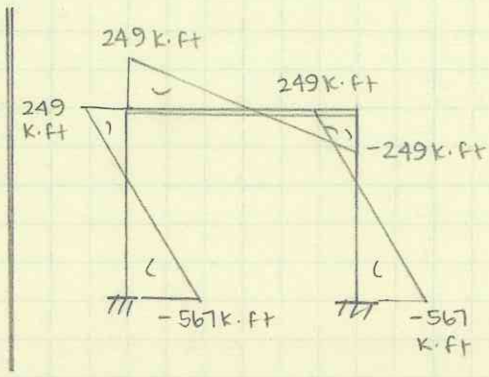
$$D = u_0 = \frac{A}{\omega_n^2} = \frac{1.355(386\ in/s^2)}{(11.64\ rad/s)^2} = 3.86\ in$$

$$\boxed{u_0 = 3.86\ in}$$

$$f_{so} = (A/g)w = 1.355(100\ kip) = 135.5K$$







Analysis run in Arcade. ?

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HOMEWORK #8

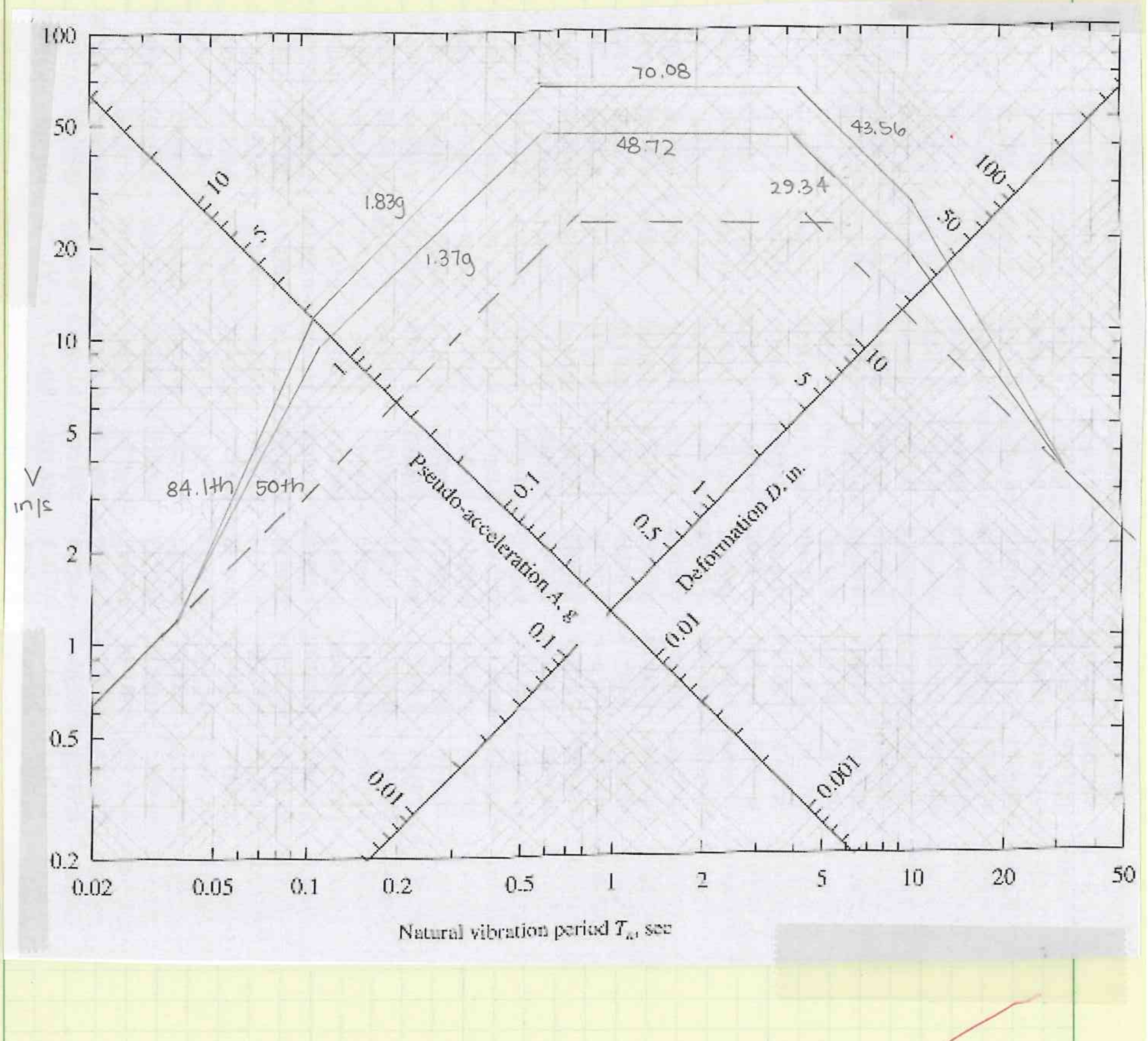
2. Prob. 6.23

$\ddot{u}_{go} = 0.5g$ $\xi = 0.02$

$\dot{u}_{go} = 24 \text{ in/s}$ 50th, 84.1th spectra

$u_{go} = 18 \text{ in}$

	50th	84.1th
$\alpha_A =$	2.74	3.66
$\alpha_V =$	2.03	2.92
$\alpha_D =$	1.63	2.42
$\ddot{u}_{go}\alpha_A =$	1.37g	1.83g
$\dot{u}_{go}\alpha_V =$	48.72	70.08
$u_{go}\alpha_D =$	29.34	43.56

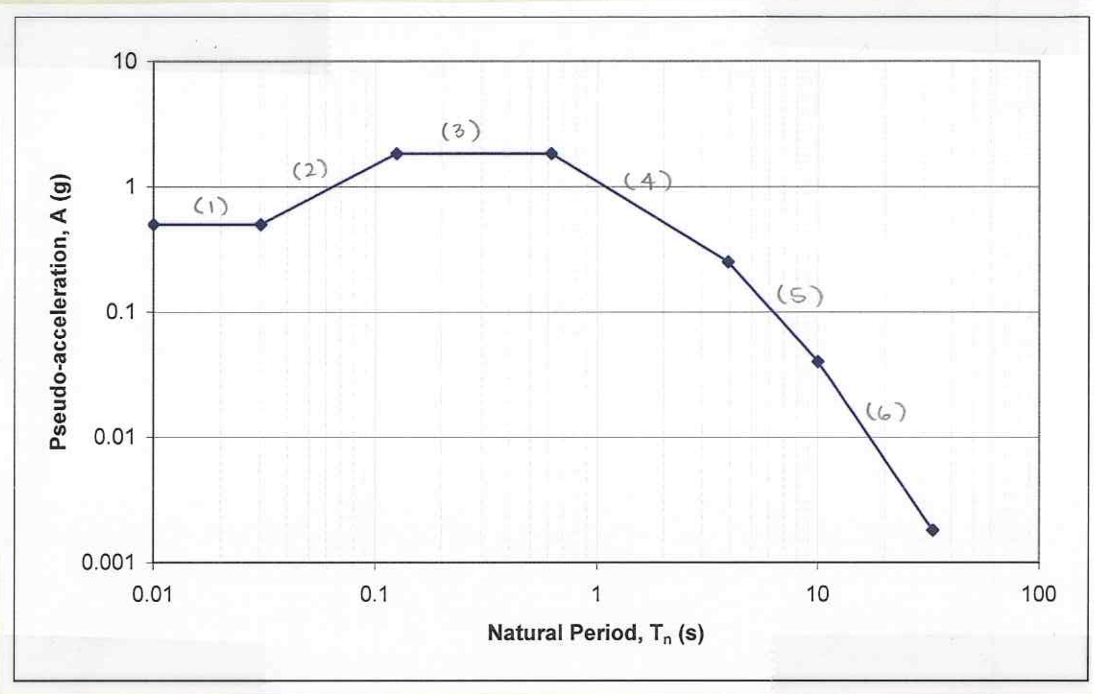


HOMEWORK #8

2. (cont'd)

b) From my plot (and my classmates', as accuracy was difficult with hand drawn curves).

Tn	A/g
1/33	0.5
1/8	1.83
0.62	1.83
3.91	0.25
10	0.04
33	0.0018



$A_1 = 0.5$

$A_2 \sim 11.7 T_n^{0.9}$

$A_3 = 1.83$

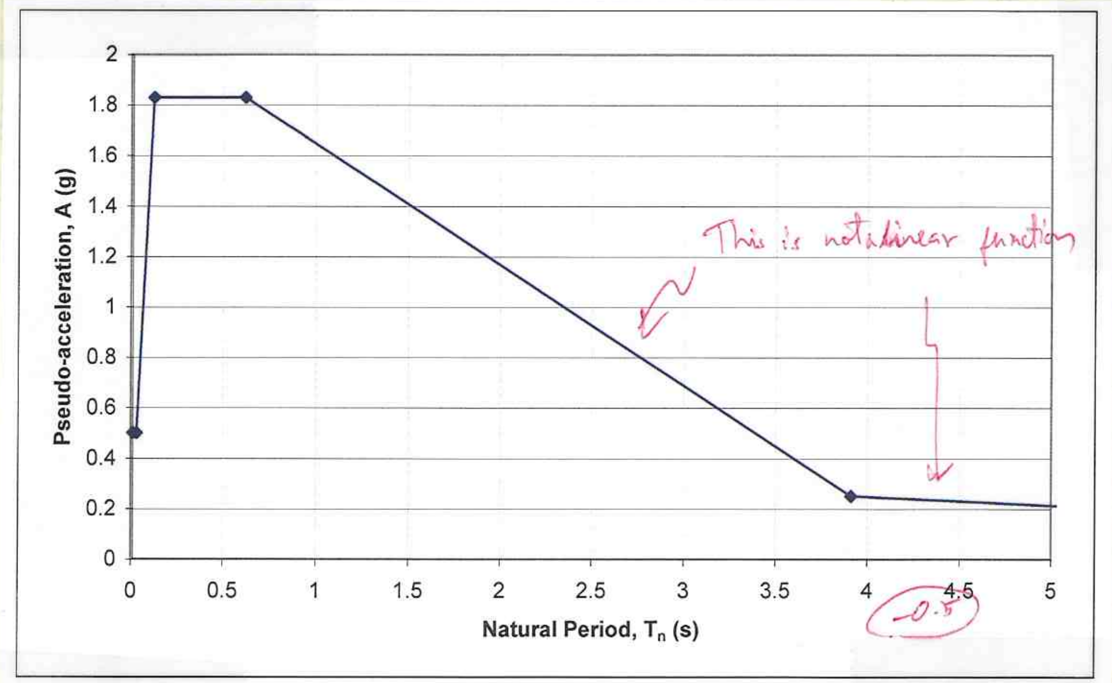
$A_4 \sim 1.06 T_n^{-1}$

$A_5 \sim 4.0 T_n^{-2}$

$A_6 \sim 21 T_n^{-2.7}$

Exponents guessed from ! Fig. 6.9.5, coefficients determined and averaged with two points.

c)

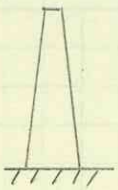


This is not a linear function

-0.5

HOMEWORK #8

3. Prob 8.7



$\psi(x) = 1 - \cos\frac{\pi x}{2L}$

$\zeta = 5\%$
$E_c = 3600 \text{ ksi} = 518400 \text{ ksf}$

$L = 600 \text{ ft}$
$d_o = 50 \text{ ft}$
$d_L = 25 \text{ ft}$ $\Big\rangle$ $R(x) = 25 - x/48$

$A(x) = \pi\left[R(x)^2 - (R(x) - 2.5)^2\right]$

$m(x) = \frac{(0.150 \text{ k/ft}^3)\,A(x)}{32.2 \text{ ft/s}^2}$

$I(x) = \frac{\pi}{4}\left[R(x)^4 - (R(x) - 2.5)^4\right]$

$mt := \int_0^L m(x)\cdot(\psi(x))^2\,dx$ $\qquad$ $kt := \int_0^L E_c\cdot I(x)\cdot(\psi_2(x))^2\,dx$ $\qquad$ $Lt := \int_0^L m(x)\cdot\psi(x)\,dx$

$mt = 134.367$ $\qquad$ $kt = 485.293$ $\qquad$ $Lt = 231.462$

t signifies ~ on value
$mt = \tilde{m}$

$w_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = 1.9 \text{ rad/s}$

$T_n = 3.31 \text{ s}$ → in sloped section of Fig. 6.9.5

$A(T_n) = 1.80 T_n^{-1} \cdot \frac{1}{4} \cdot 32.2 \text{ ft/s}^2 = 4.383$

$\tilde{\Gamma} = \frac{\tilde{L}}{\tilde{m}} = 1.723$

$u_o(x) = \frac{\tilde{\Gamma}}{w_n^2} A(T_n)\,\psi(x) \cdot 12 \text{ in/ft}$

$u_o(600) = 25.1 \text{ in}$

$\boxed{\text{b. } u_o(L) = 25.1 \text{ in}}$

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HOMEWORK #8

3. (cont'd)

using mathcad to solve for shears and moments:

$f_o(x) := \Gamma t \cdot m(x) \cdot \psi(x) \cdot A(T_n)$

$V_o(x) := \int_x^L f_o(\xi)\, d\xi$

$M_o(x) := \int_x^L (\xi - x) \cdot f_o(\xi)\, d\xi$

$V_o(300) = 1.432 \times 10^3$

$M_o(300) = 2.41 \times 10^5$

$V_o(0) = 1.747 \times 10^3$

$M_o(0) = 7.401 \times 10^5$

a) Bottom:	Mid-height:
$V_o = 1747$ k	$V_o = 1432$ k
$M_o = 740{,}100$ k·ft	$M_o = 241{,}000$ k·ft

CE 384P: DYNAMICS

C.HOVELL
11 April 06

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HOMEWORK #8

4. Prob. 8.9

Similar in set up to 8.7

$$\mathcal{I} = \text{impulse load} = \int_0^L \text{area} \cdot \psi(x) \cdot \tfrac{x}{L}\, dx$$

$$\hookrightarrow = 4\left(0.1/2 + 0.15/2\right) = 0.5$$

$$\mathcal{I} = 80.599$$

$$u_o(x) := \frac{\text{Impulse}}{mt \cdot \omega_n} \cdot \psi(x) \cdot 12$$

$$u_o(600) = 3.788$$

$$A_{max} := \frac{u_o(L) \cdot \omega_n^{\ 2}}{\Gamma t \cdot 12}$$

$\boxed{\text{b) } u_o = 3.79 \text{ in}}$

$$f_o(x) = \tilde{\Gamma} \cdot m(x) \cdot \psi(x) \cdot A_{max}$$

$$V_o(x) := \int_x^L f_o(\xi)\, d\xi$$

$$M_o(x) := \int_x^L (\xi - x) \cdot f_o(\xi)\, d\xi$$

$$V_o(300) = 216.268$$

$$M_o(300) = 3.638 \times 10^4$$

$$V_o(0) = 263.861$$

$$M_o(0) = 1.118 \times 10^5$$

a) Bottom:	Mid-height:
$V_o = 263.9$ K	$V_o = 216.3$ K
$M_o = 111{,}800$ K·ft	$M_o = 363{,}800$ K·ft

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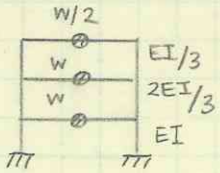
HOMEWORK #8

5. Prob. 8.11



$E = 29000 \text{ ksi}$

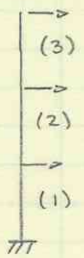
$w = 100 \text{ k}$

$I = 1400 \text{ in}^4$

$\zeta_n = 5\%$

$$k = \frac{12EI}{L^3} \cdot 2 = 24\frac{EI}{L^3}$$

$$m = \frac{w}{g} = 0.259 \text{ k} \cdot \text{s}^2/\text{in}$$



$$k_1 = \begin{bmatrix} 1 \end{bmatrix}$$ scale from this

$$k_2 = \begin{bmatrix} 2/3 & -2/3 \\ -2/3 & 2/3 \end{bmatrix}$$

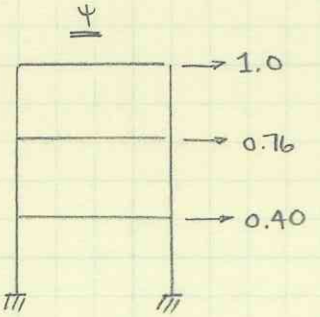
$$k_3 = \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix}$$

$$\underline{k} = \begin{bmatrix} 5/3 & -2/3 & 0 \\ -2/3 & 1 & -1/3 \\ 0 & -1/3 & 1/3 \end{bmatrix}$$

$$\underline{p} = \begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}$$

$$\underline{u} = k^{-1} \cdot p = \begin{bmatrix} 2.5 \\ 4.75 \\ 6.25 \end{bmatrix}$$

normalize to 6.25



$$\tilde{m} = \sum m\psi^2 = \frac{m}{2}(1.0)^2 + m(0.76)^2 + m(0.40)^2 = 1.2376m$$

$$\tilde{k} = \sum k(\psi - \psi_{j-1})^2 = k(0.40)^2 + 2/3k(0.76 - 0.40)^2 + 1/3k(1.0 - 0.76)^2 = 0.2656k$$

$$\tilde{L} = \sum m\psi = \frac{m}{2}(1.0) + m(0.76) + m(0.40) = 1.66m$$

$$\omega_n^2 = \frac{\tilde{k}}{\tilde{m}} = \frac{0.2656(24EI/L^3)}{1.2376m}, \quad L = 12 \text{ ft}$$

$\omega_n = 16.44 \text{ rad/s}$

$T_n = 0.382 \text{ s}$

$$A_{max} = \frac{2.71g}{4} = 261.5$$ (from 6.9.5)

$$D = \frac{A}{\omega_n^2} = 0.968 \text{ in}$$

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HOMEWORK #8

5. (cont'd)

$$\tilde{\Gamma} = \frac{\tilde{L}}{\tilde{m}} = \frac{1.66m}{1.2376m} = 1.34$$

$$z_o = \tilde{\Gamma} \cdot D = (1.34)(0.968\,in) = 1.298\,in$$

Story Displacements: $u = \psi \cdot z_o$

$u_1 = 0.519\,in$
$u_2 = 0.986\,in$
$u_3 = 1.298\,in$

Equivalent Static Forces

$$f_{10} = \tilde{\Gamma} \cdot m \psi A = (1.34)(m)(0.4)(261.5) = 36.3k$$

$$f_{20} = (1.34)(0.259\,k \cdot s^2/in)(0.76)(261.5) = 69.0k$$

$$f_{30} = (1.34)\frac{m}{2}(1.0)(261.5) = 45.4k$$

Shears & Moments

$V_o = \Sigma f_{jo}$, $M_o = \Sigma hf$ for all floors above

Story	V	M
Base	150.7k	3726 k·ft
1	150.7k	1918 k·ft
2	114.4k	545 k·ft
3	45.4k	0

Homework No. 8

Assigned: April 4, 2006
Due: April 11, 2006

1. Solve Problem 6.18 from the textbook.
2. Solve Problem 6.23 from the textbook.
3. Solve Problem 8.7 from the textbook.
4. Solve Problem 8.9 from the textbook.
5. Solve Problem 8.11 from the textbook.

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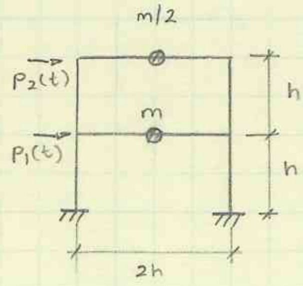
HOMEWORK #9

29.5/30

Nice work!

1. Prob. 9.5



$$K_{\substack{\text{one} \\ \text{column}}} = \frac{12EI}{h^3}$$

$$K_{\text{story}} = \frac{24EI}{h^3}$$

when 1st floor moves, two sets of columns resist motion.

$$\underline{m} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} m, \quad \underline{k} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \frac{24EI}{h^3}$$

$$\boxed{m \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \frac{24EI}{h^3} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}}$$

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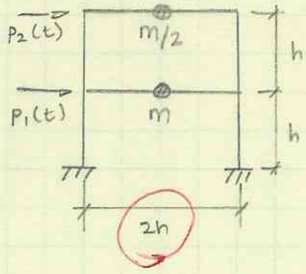
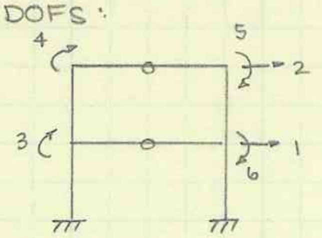
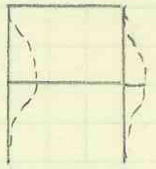
HOMEWORK #9

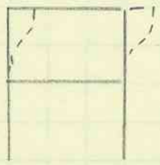
2. Prob. 9.6

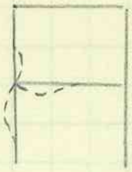


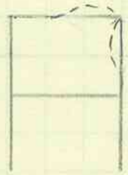
a) DOFS:











a) $K = EI \begin{bmatrix} \begin{array}{cc|cccc} 48/h^3 & -24/h^3 & 0 & 6/h^2 & 6/h^2 & 0 \\ -24/h^3 & 24/h^3 & -6/h^2 & -6/h^2 & -6/h^2 & -6/h^2 \\ \hline 0 & -6/h^2 & 12/h & 2/h & 0 & 2/h \\ 6/h^2 & -6/h^2 & 2/h & 8/h & 2/h & 0 \\ 6/h^2 & -6/h^2 & 0 & 2/h & 8/h & 2/h \\ 0 & -6/h^2 & 2/h & 0 & 2/h & 12/h \end{array} \end{bmatrix}$

-0.5

b)

$$m = \begin{bmatrix} m & & & & & \\ & m/2 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

$$p(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

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HOMEWORK #9

2. (cont'd)

Partition matricies:

$$\underline{u}_0 = \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}, \quad \underline{u}_t = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

✓

$$\underline{m}_{tt} = \begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix}, \quad \underline{m}_0 = \underline{0}$$

$$\underline{k}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 48 & -24 \\ -24 & 24 \end{bmatrix}, \quad \underline{k}_{0t} = \begin{bmatrix} 0 & -6 \\ 6 & -6 \\ 6 & -6 \\ 0 & -6 \end{bmatrix}\frac{EI}{h^2}$$

$$\underline{k}_{t0} = \begin{bmatrix} 0 & 6 & 6 & 0 \\ -6 & -6 & -6 & -6 \end{bmatrix}\frac{EI}{h^2}$$

✓

$$\underline{k}_{00} = \frac{EI}{h}\begin{bmatrix} 12 & 2 & 0 & 2 \\ 2 & 8 & 2 & 0 \\ 0 & 2 & 8 & 2 \\ 2 & 0 & 2 & 12 \end{bmatrix}$$

✗

condensed stiffness matrix

$$\hat{k}_{tt} = k_{tt} - k_{0t}^T k_{00}^{-1} k_{0t}$$

c)
$$\boxed{\begin{gathered}\begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \left[\underline{k}_{tt} - \underline{k}_{t0}\underline{k}_{00}^{-1}\underline{k}_{0t}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \\ \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = -\underline{k}_{00}^{-1}\underline{k}_{0t}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\end{gathered}}$$

✓

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HOMEWORK #9

3. Prob. 10.6

$$m = \begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix}, \quad k = \frac{24EI}{h^3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

define $\lambda = \frac{mh^3}{24EI}\omega^2$

$$k - \omega^2 m = \frac{24EI}{h^3}\begin{bmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda/2 \end{bmatrix}$$

$$\det = 0, \quad (2-\lambda)(1-\lambda/2) - (-1)(-1) = 0$$

$$2 - \lambda - \lambda + \lambda^2/2 - 1 = 0$$

$$\frac{\lambda^2}{2} - 2\lambda + 1 = 0$$

$$\lambda = 3.414, 0.586$$

subbing back in,

a)

$$\omega_1 = 9.05\left[\frac{EI}{mh^3}\right]^{1/2}, \quad \lambda = 3.414 \text{ — Mode 2}$$

$$\omega_2 = 3.75\left[\frac{EI}{mh^3}\right]^{1/2}, \quad \lambda = 0.586 \text{ — Mode 1}$$

assigning

$$k = \frac{24EI}{h^3}, \quad k\begin{bmatrix} -\sqrt{2} & -1 \\ -1 & -\sqrt{2}/2 \end{bmatrix}\begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 1.5$, assume $\phi_{21} = 1.0$

$$-\sqrt{2}\,\phi_{11} = 1.0, \quad \phi_{11} = -1/\sqrt{2}$$

$\lambda = 1.0$, $\phi_{22} = 1.0$

$$\sqrt{2}\,\phi_{12} = 1.0, \quad \phi_{12} = 1/\sqrt{2}$$

a)

$$\phi_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1.0 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1.0 \end{bmatrix}$$

CE 384P: DYNAMICS

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HOMEWORK #9

3. (cont'd)

b) check orthogonality

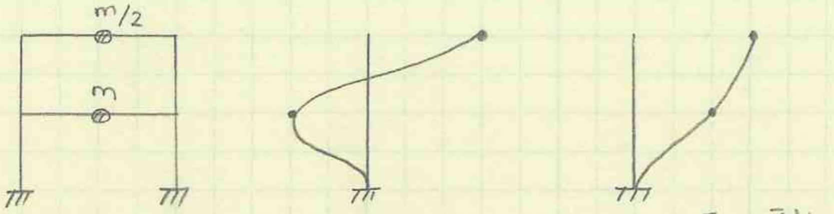
$\phi_1^T m \phi_2 = 0$

$$\begin{bmatrix} -1/\sqrt{2} & 1.0 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1.0 \end{bmatrix} = [0] \quad ✓$$

$\phi_1^T k \phi_2 = 0$

$$\begin{bmatrix} -1/\sqrt{2} & 1.0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1.0 \end{bmatrix} = [0] \quad ✓$$

c)



$$\omega_n = 9.05 \left[\frac{EI}{mh^3}\right]^{1/2} \qquad \omega_n = 3.75 \left[\frac{EI}{mh^3}\right]^{1/2}$$

Mode shapes

Normalize mass:

$M_1 = \phi_1^T m \phi_1 = m$

$M_2 = \phi_2^T m \phi_2 = m$

$$\boxed{d) \quad \phi_{\cancel{1}\,2} = \begin{bmatrix} -1/\sqrt{2m} \\ 1/\sqrt{m} \end{bmatrix}, \quad \phi_{\cancel{2}\,1} = \begin{bmatrix} 1/\sqrt{2m} \\ 1/\sqrt{m} \end{bmatrix}}$$

mode shapes are scaled by $1/\sqrt{m}$

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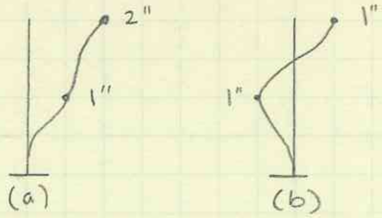
HOMEWORK #9

4. Prob. 10.8

Initial displacement



$$q_1 = \frac{\phi_1^T m u}{M_1}$$

$$q_{11} = \frac{1}{m}\begin{bmatrix} -1/\sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & \frac{m}{2} \end{bmatrix}\begin{bmatrix} 1.0 \\ 2.0 \end{bmatrix}$$

$$q_{11} = \frac{-1}{\sqrt{2}} + 1.0 = 0.293$$

$$q_{21} = 1/\sqrt{2} + 1.0 = 1.707$$

$$\dot{q}_{11} = \dot{q}_{21} = 0$$

$$q_1(t) = \left(\frac{-1}{\sqrt{2}} + 1.0\right)\cos\omega_1 t$$

$$q_2(t) = \left(1/\sqrt{2} + 1.0\right)\cos\omega_2 t$$

$$\underline{u}(t) = \Sigma\, \phi_n\, q_n(t)$$

Displacement (a)

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} -0.207 \\ 0.293 \end{bmatrix}\cos\underset{2}{\omega_{\cancel{1}}}t + \begin{bmatrix} 1.207 \\ 1.707 \end{bmatrix}\cos\underset{1}{\omega_{\cancel{2}}}t$$

with ω_1, ω_2 defined in Prob.3

$$q_{12} = \frac{1}{m}\begin{bmatrix} -1/\sqrt{2} & 1 \end{bmatrix} m \begin{bmatrix} -1.0 \\ 1.0 \end{bmatrix} = 1.207$$

$$q_{22} = -1/\sqrt{2} + 1/2 = -0.207$$

Displacement (b)

$$u_1(t) = -0.853\cos\underset{2}{\omega_{\cancel{1}}}t - 0.146\cos\underset{1}{\omega_{\cancel{2}}}t$$

$$u_2(t) = 1.207\cos\underset{2}{\omega_{\cancel{1}}}t - 0.207\cos\underset{1}{\omega_{2}}t$$

Homework No. 9

Assigned: April 11, 2006
Due: April 20, 2006

Problem 1
Equation of motion for a 2-DOF system.
Solve Problem 9.5 from the textbook.

Problem 2
Same as Problem 1 but beams are not rigid.
Solve Problem 9.6 from the textbook.
In part (c), you may write your stiffness matrix in terms of inverses and/or transverses of submatrices of the original stiffness matrix – i.e., you do not need to carry out any needed matrix multiplication or inversion. State, with equations, how you would determine the rotations.

Problem 3
Eigenvalue analysis of frame in Problem 1.
Solve Problem 10.6 from the textbook.

Problem 4
Free vibration analysis of frame in Problem 1.
Solve Problem 10.8 from the textbook.

Catherine Hovell <cghovell@gmail.com>

homework questions

Lance Manuel <lmanuel@mail.utexas.edu> **Sun, Apr 30, 2006 at 12:24 PM**
To: Catherine Hovell <cghovell@gmail.com>

Catherine,

Here are some ideas (and things to check):

1. Let's forget about the spatial pattern method first and see if we can't do this using the general approach (Handout No. 25). First, let's work out the amplitude Pn0 for each mode n (note: Pn0 = phi_n{transpose} * p(t); where p(t) is a 5X1 vector with 5000 kips at the third entry and zeroes everywhere else). Then use, the modal stiffness Kn for each mode n. You should be able to get the static value for qn - that is, you can calculate qn(st,0) = Pn0/Kn. This is the static value for the nth mode generalized coordinate.

2. Now, we need to compute the dynamic response at any time t of the generalized coordinate.
Let's go back to Ch 3. For a SDOF system where the response needed is u(t), you can calculate u(t) using Eqs. (3.2.4) and (3.2.5).
Let's use the same idea here. For the nth mode, we can do the following:
Basically, we can find C and D by replacing p0/k in Eq. (3.2.4) with the qn(st,0) we found in Step 1 above.
The remaining multiplying terms in Eq. (3.2.4) can be obtained exactly as shown. For each mode, you will have a different omega_n. Note zeta = 0.05 for all the modes. So, you can find C and D (different for each mode).
Thus, you can find qn(t).

3. Now, it's easy. To get the nth mode contribution to the physical displacement vector, u. Let's call it u_n(t).
u_n(t) = phi_n * qn(t).

Note that in the way I have explained it above, the usual Rd and phase angle are sort of embedded in the C and D you're calculating for each mode. Recall that if you want the Rd, you can get it as (C^2 + D^2)^(1/2) divided by the static response and the phase = arctan(-D/C) [See pg. 76] If you're trying to get insights into your answers, you can see what these values are.

The spatial pattern method should also work if you want to try it. It requires that you first decompose the 5X1 s vector where the third element is 1 (the rest, zero), using Eq. 12.8.3 and 12.8.4. The method might be a bit more involved than the general method I outlined above.

Good luck! Let me know how it works.

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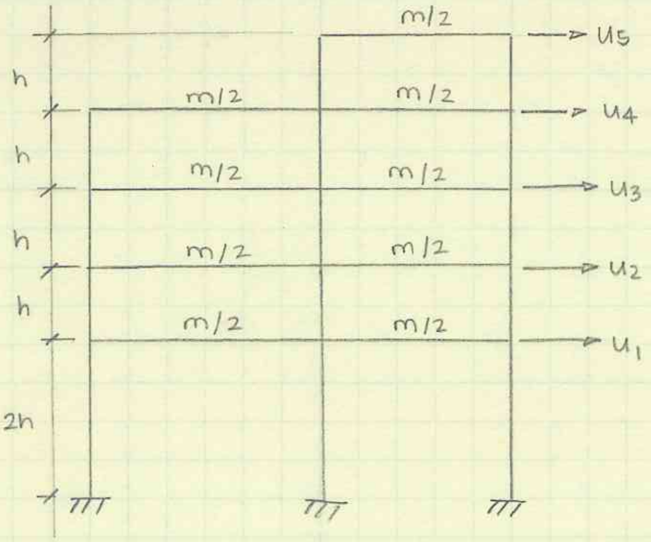
HOMEWORK #10

$$\frac{25 + 2.5}{25} = \boxed{\frac{27.5}{25}}$$

Nice work!

Some additional comments and response plots would have helped.



h = 10 ft
$I = 4320 \text{ in}^4$ (columns)
$m = 4 \text{ k}\cdot\text{s}^2/\text{in}$
E = 30000 ksi

$$m = \begin{bmatrix} m & & & & 0 \\ & m & & & \\ & & m & & \\ & & & m & \\ 0 & & & & m/2 \end{bmatrix}$$

$$k = \begin{bmatrix} 3k_1 + 3k & -3k & 0 & 0 & 0 \\ -3k & 2(3k) & -3k & 0 & 0 \\ 0 & -3k & 6k & -3k & 0 \\ 0 & 0 & -3k & 3k + 2k & -2k \\ 0 & 0 & 0 & -2k & 2k \end{bmatrix}, \quad k = \frac{12EI}{h^3}, \quad k_1 = \frac{12EI}{h_1^3}$$

✓

$$k = \frac{12(30000 \text{ ksi})(4320 \text{ in}^4)}{(10 \text{ ft})^3}$$

k = 900 k/in

k_1 = 112.5 k/in (very small, comparatively)

$$k = \begin{bmatrix} 3037.5 & -2700 & 0 & 0 & 0 \\ -2700 & 5400 & -2700 & 0 & 0 \\ 0 & -2700 & 5400 & -2700 & 0 \\ 0 & 0 & -2700 & 4500 & -1800 \\ 0 & 0 & 0 & -1800 & 1800 \end{bmatrix} \text{ k/in}$$

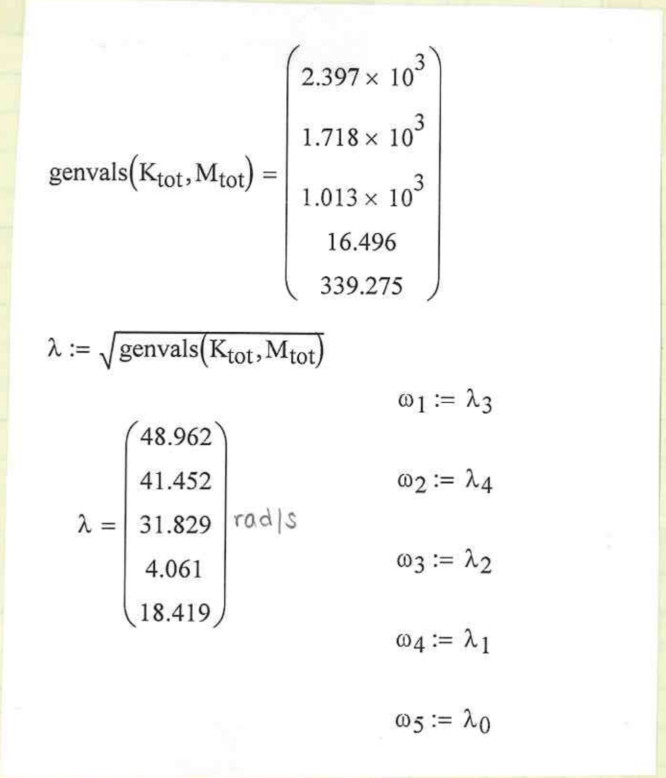
✓

HOMEWORK #10

Natural Frequencies

$$k-\omega_n^2 m = \begin{bmatrix} 3037.5-4\omega_n^2 & -2700 & 0 & 0 & 0 \\ -2700 & 5400-4\omega_n^2 & -2700 & 0 & 0 \\ 0 & -2700 & 5400-4\omega_n^2 & -2700 & 0 \\ 0 & 0 & -2700 & 4500-4\omega_n^2 & -1800 \\ 0 & 0 & 0 & -1800 & 1800-2\omega_n^2 \end{bmatrix}$$

$\det|k-\omega_n^2 m| = 0$ to solve for ω_n



$$\text{genvals}(K_{tot}, M_{tot}) = \begin{pmatrix} 2.397\times10^3 \\ 1.718\times10^3 \\ 1.013\times10^3 \\ 16.496 \\ 339.275 \end{pmatrix}$$

$$\lambda := \sqrt{\text{genvals}(K_{tot}, M_{tot})}$$

$$\lambda = \begin{pmatrix} 48.962 \\ 41.452 \\ 31.829 \\ 4.061 \\ 18.419 \end{pmatrix} \text{rad/s}$$

$\omega_1 := \lambda_3$

$\omega_2 := \lambda_4$

$\omega_3 := \lambda_2$

$\omega_4 := \lambda_1$

$\omega_5 := \lambda_0$

← reorder values to go from smallest to largest

From Mathcad

$$\boxed{\begin{aligned} \omega_{n_1} &= 4.061 \\ \omega_{n_2} &= 18.419 \\ \omega_{n_3} &= 31.829 \\ \omega_{n_4} &= 41.452 \\ \omega_{n_5} &= 48.962 \end{aligned}}$$

(or ω_0 to ω_4, from here on out)

$$T_n = \frac{2\pi}{\omega},$$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{bmatrix}, \text{ so } T_n = \begin{bmatrix} 1.547 \\ 0.341 \\ 0.197 \\ 0.152 \\ 0.128 \end{bmatrix} s$$

HOMEWORK #10

Part one (cont'd)

Mode Shapes – using mathcad

- genvecs (Ktot, Mtot)
- reorder columns
- normalize by story 5 displacement:

$$\Phi_{one} = \begin{pmatrix} 0.389 & 0.553 & 0.423 & 0.346 & -0.228 \\ 0.428 & 0.344 & -0.159 & -0.492 & 0.553 \\ 0.456 & -0.038 & -0.503 & -0.078 & -0.63 \\ 0.474 & -0.401 & -0.092 & 0.535 & 0.425 \\ 0.483 & -0.643 & 0.731 & -0.588 & -0.255 \end{pmatrix}$$

← reordered, non-normalized mode shape vectors

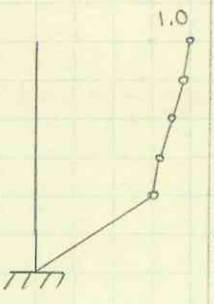
$$\Phi^{\langle 0\rangle} := \frac{\Phi_{one}^{\langle 0\rangle}}{\left(\Phi_{one}^{\langle 0\rangle}\right)_4} \quad \Phi^{\langle 1\rangle} := \frac{\Phi_{one}^{\langle 1\rangle}}{\left(\Phi_{one}^{\langle 1\rangle}\right)_4} \quad \Phi^{\langle 2\rangle} := \frac{\Phi_{one}^{\langle 2\rangle}}{\left(\Phi_{one}^{\langle 2\rangle}\right)_4} \quad \Phi^{\langle 3\rangle} := \frac{\Phi_{one}^{\langle 3\rangle}}{\left(\Phi_{one}^{\langle 3\rangle}\right)_4} \quad \Phi^{\langle 4\rangle} := \frac{\Phi_{one}^{\langle 4\rangle}}{\left(\Phi_{one}^{\langle 4\rangle}\right)_4}$$

$$\Phi^{\langle 0\rangle} = \begin{pmatrix} 0.805 \\ 0.886 \\ 0.945 \\ 0.982 \\ 1 \end{pmatrix} \quad \Phi^{\langle 1\rangle} = \begin{pmatrix} -0.86 \\ -0.535 \\ 0.059 \\ 0.623 \\ 1 \end{pmatrix} \quad \Phi^{\langle 2\rangle} = \begin{pmatrix} 0.579 \\ -0.218 \\ -0.687 \\ -0.126 \\ 1 \end{pmatrix} \quad \Phi^{\langle 3\rangle} = \begin{pmatrix} -0.589 \\ 0.837 \\ 0.132 \\ -0.909 \\ 1 \end{pmatrix} \quad \Phi^{\langle 4\rangle} = \begin{pmatrix} 0.893 \\ -2.167 \\ 2.469 \\ -1.664 \\ 1 \end{pmatrix}$$

$$\boxed{\Phi = \begin{pmatrix} 0.805 & -0.86 & 0.579 & -0.589 & 0.893 \\ 0.886 & -0.535 & -0.218 & 0.837 & -2.167 \\ 0.945 & 0.059 & -0.687 & 0.132 & 2.469 \\ 0.982 & 0.623 & -0.126 & -0.909 & -1.664 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}}$$

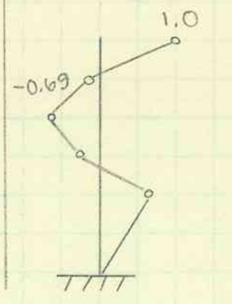
Each mode shape is more complex than the one before, with an additional inflection point. This is akin to higher modes in buckling of members. It should take more force to cause each mode to occur.

Mode 1

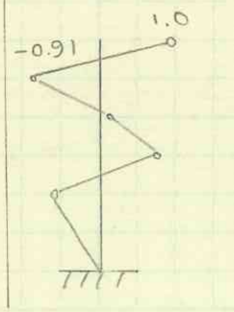


Mode 2

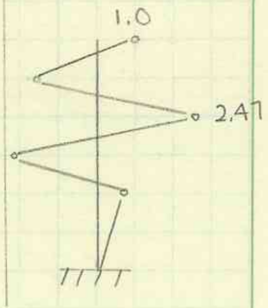
Mode 3



Mode 4



Mode 5



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HOMEWORK #10

Part One (cont'd)

Modal mass, stiffness

$$M_n = \phi_n^T m \phi_n$$
$$K_n = \phi_n^T k \phi_n$$

$$M_1 = \begin{bmatrix} 0.805 & 0.886 & 0.945 & 0.982 & 1.0 \end{bmatrix} \underline{m} \begin{bmatrix} 0.805 \\ 0.886 \\ 0.945 \\ 0.982 \\ 1.0 \end{bmatrix} = 15.16 \frac{k \cdot s^2}{in}$$

$$K_1 = [\phi_1]^T \underline{k} [\phi_1] = 250.1 \ k/in$$

$$M_n(j) := \Phi^{\langle j \rangle T} M_{tot} \cdot \Phi^{\langle j \rangle}$$

$M_n(0) = 15.165$

$M_n(1) = 7.672$

$M_n(2) = 5.484$

$M_n(3) = 9.567$

$M_n(4) = 59.426$

k · s²/in

$$K_n(j) := \Phi^{\langle j \rangle T} K_{tot} \cdot \Phi^{\langle j \rangle}$$

$K_n(0) = 250.15$

$K_n(1) = 2.603 \times 10^3$

$K_n(2) = 5.555 \times 10^3$

$K_n(3) = 1.644 \times 10^4$

$K_n(4) = 1.425 \times 10^5$

k/in

✓

Increasing stiffness values demonstrate how more force is needed to cause the same deflections in each mode.

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HOMEWORK #10

Part Two

$$u_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \end{bmatrix}, \quad \dot{u}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\zeta = 0.05$$

$$\omega_{Dj} = \omega_j \sqrt{1-\zeta^2}$$

$$q_{0j} = \frac{\phi_j^T M_{tot} u_0}{\phi_j^T M_{tot} \phi_j}, \quad \dot{q}_{0j} = \frac{\phi_j^T M_{tot} \dot{u}_0}{\phi_j^T M_{tot} \phi_j}$$

$$q_0 = \begin{bmatrix} 0.132 \\ 0.261 \\ 0.365 \\ 0.209 \\ 0.034 \end{bmatrix}, \quad \dot{q}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

check:

$\Phi q_0 = u_0$ ✓

$\Phi \dot{q}_0 = \dot{u}_0$ ✓

Select:

$\Delta t = 0.015$

$NT = \dfrac{8 \cdot 2\pi/\omega_1}{\Delta t} = 272.8$ (need NT > 250 for extra part)

$i = 0..273$

✓

$$u_i := \sum_{j=0}^{4} \left[\Phi^{\langle j \rangle} \cdot \exp(-\zeta \cdot \omega_j \cdot i \cdot \Delta t) \cdot \left[q0_j \cdot \cos(\omega D_j \cdot i \cdot \Delta t) + \frac{\left(qd0_j + \zeta \cdot \omega_j \cdot q0_j\right)}{\omega D_j} \cdot \sin(\omega D_j \cdot i \cdot \Delta t) \right] \right]$$

$$u1_i := \Phi^{\langle 0 \rangle} \cdot \exp(-\zeta \cdot \omega_0 \cdot i \cdot \Delta t) \cdot \left(q0_0 \cdot \cos(\omega D_0 \cdot i \cdot \Delta t) + \frac{qd0_0 + \zeta \cdot \omega_0 \cdot q0_0}{\omega D_0} \cdot \sin(\omega D_0 \cdot i \cdot \Delta t) \right)$$

$$u2_i := \Phi^{\langle 1 \rangle} \cdot \exp(-\zeta \cdot \omega_1 \cdot i \cdot \Delta t) \cdot \left(q0_1 \cdot \cos(\omega D_1 \cdot i \cdot \Delta t) + \frac{qd0_1 + \zeta \cdot \omega_1 \cdot q0_1}{\omega D_1} \cdot \sin(\omega D_1 \cdot i \cdot \Delta t) \right)$$

$$u3_i := \Phi^{\langle 2 \rangle} \cdot \exp(-\zeta \cdot \omega_2 \cdot i \cdot \Delta t) \cdot \left(q0_2 \cdot \cos(\omega D_2 \cdot i \cdot \Delta t) + \frac{qd0_2 + \zeta \cdot \omega_2 \cdot q0_2}{\omega D_2} \cdot \sin(\omega D_2 \cdot i \cdot \Delta t) \right)$$

$$u4_i := \Phi^{\langle 3 \rangle} \cdot \exp(-\zeta \cdot \omega_3 \cdot i \cdot \Delta t) \cdot \left(q0_3 \cdot \cos(\omega D_3 \cdot i \cdot \Delta t) + \frac{qd0_3 + \zeta \cdot \omega_3 \cdot q0_3}{\omega D_3} \cdot \sin(\omega D_3 \cdot i \cdot \Delta t) \right)$$

$$u5_i := \Phi^{\langle 4 \rangle} \cdot \exp(-\zeta \cdot \omega_4 \cdot i \cdot \Delta t) \cdot \left(q0_4 \cdot \cos(\omega D_4 \cdot i \cdot \Delta t) + \frac{qd0_4 + \zeta \cdot \omega_4 \cdot q0_4}{\omega D_4} \cdot \sin(\omega D_4 \cdot i \cdot \Delta t) \right)$$

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HOMEWORK #10

Part Two (cont'd)

t=1.0 s

	M1	M2	M3	M4	M5	TOTAL
Floor 1	−0.0563	−0.0783	0.0408	0.0135	0.00040	
Floor 2	−0.062	−0.0487	−0.0153	−0.0192	−0.00098	
Floor 3	−0.0661	0.005331	−0.0485	−0.0305	0.00112	
Floor 4	−0.0686	0.0567	−0.0089	0.0209	−0.00075	
Roof	−0.0699	0.091	0.0705	−0.023	0.00045	

$$u_{100} = \begin{pmatrix} -0.0798 \\ -0.1463 \\ -0.1112 \\ -6.3433 \times 10^{-4} \\ 0.0691 \end{pmatrix}$$

⟶
decreasing influence on total deflection values.

not always

✓

t=2.5 s

	M1	M2	M3	M4	M5	TOTAL
Floor 1	−0.0503	0.00849	−0.0025	0.00068	-6.28×10^{-5}	
ROOF	−0.0625	−0.00987	−0.0043	−0.00115	-7.03×10^{-5}	

$$u_{250} = \begin{pmatrix} -0.0437 \\ -0.0499 \\ -0.057 \\ -0.0658 \\ -0.0779 \end{pmatrix}$$

✓

To calculate shears,

- put all u values into one matrix, u
- $f = K_{tot}\, u$
- solve for individual levels

t=1.0 s

$$f = \begin{pmatrix} -3.715 & -106.267 & 165.435 & 93.111 & 3.872 \\ -4.089 & -66.137 & -62.175 & -132.275 & -9.396 \\ -4.362 & 7.235 & -196.471 & -20.939 & 10.705 \\ -4.53 & 76.97 & -35.899 & 143.699 & -7.213 \\ -2.307 & 61.771 & 142.888 & -79.024 & 2.168 \end{pmatrix}$$

$$V_b := f^T \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad V_5 := f^T \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$V_b = \begin{pmatrix} -19.003 \\ -26.428 \\ 13.779 \\ 4.572 \\ 0.136 \end{pmatrix} k \qquad V_5 = \begin{pmatrix} -2.307 \\ 61.771 \\ 142.888 \\ -79.024 \\ 2.168 \end{pmatrix} k$$

✓

TOTAL: −26.944 k   125.496 k

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HOMEWORK #10

Part Two (cont'd)

Shears, t = 2.5s

$$f_1 := K_{tot} \cdot U_1$$

$$f_1 = \begin{pmatrix} -3.32 & 11.524 & -10.189 & 4.646 & -0.602 \\ -3.653 & 7.172 & 3.829 & -6.601 & 1.46 \\ -3.898 & -0.785 & 12.1 & -1.045 & -1.664 \\ -4.047 & -8.347 & 2.211 & 7.171 & 1.121 \\ -2.061 & -6.699 & -8.8 & -3.943 & -0.337 \end{pmatrix}$$

$$V_{b1} = \begin{pmatrix} -16.98 \\ 2.866 \\ -0.849 \\ 0.228 \\ -0.021 \end{pmatrix} \qquad V_{51} = \begin{pmatrix} -2.061 \\ -6.699 \\ -8.8 \\ -3.943 \\ -0.337 \end{pmatrix}$$

TOT: −14.756 K   −21.84 K

✓

After a longer period of time (t = 2.5s), the modal deflections were smaller, but less varied. The story shears are also smaller, but not necessarily less varied (by mode).

✓

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HOMEWORK #10

Part Three

$p_o = 5000\,K$

$f_o = 3\,Hz$

$\omega_o = 2\pi f_o = 18.85\,rad/s$

$$r = \omega_o/\omega_n = \begin{bmatrix} 4.641 \\ 1.023 \\ 0.592 \\ 0.455 \\ 0.385 \end{bmatrix} rad/s \xleftarrow{\text{very close to resonance}}$$

$$p(t) = \begin{bmatrix} 0 \\ 0 \\ 5000 \\ 0 \\ 0 \end{bmatrix}$$ sine forcing function

$$p_n(t) = \phi_n^T p(t) = \begin{bmatrix} 4.727\times10^3 \\ 292.8 \\ -3.437\times10^3 \\ 662.4 \\ 1.234\times10^4 \end{bmatrix}$$

$$q_{no} = \frac{p_{no}}{K_n(0)} = \begin{bmatrix} 18.898 \\ 0.112 \\ -0.619 \\ 0.04 \\ 0.087 \end{bmatrix}$$

✓

$q_n(t) = C\sin\omega_o t + D\cos\omega_o t$

$t = 1.0\,s$

$C = CC,\ D = DD$

$$CC_n := q_{no_n}\cdot\frac{1-(r_n)^2}{\left[1-(r_n)^2\right]^2+(2\cdot\zeta\cdot r_n)^2} \qquad DD_n := q_{no_n}\cdot\frac{-2\cdot\zeta\cdot r_n}{\left[1-(r_n)^2\right]^2+(2\cdot\zeta\cdot r_n)^2}$$

$$CC = \begin{pmatrix} -0.92 \\ -0.418 \\ -0.945 \\ 0.051 \\ 0.102 \end{pmatrix} \qquad DD = \begin{pmatrix} -0.021 \\ -0.906 \\ 0.086 \\ -2.903\times10^{-3} \\ -4.589\times10^{-3} \end{pmatrix} \qquad q = \begin{pmatrix} -0.0208 \\ -0.906 \\ 0.0862 \\ -2.9028\times10^{-3} \\ -4.5887\times10^{-3} \end{pmatrix}$$

$u_{dyn} = \phi_n q_n$

$$\begin{array}{cccccc} & \text{Mode 1} & 2 & 3 & 4 & 5 \\ udyn = & \left(\begin{array}{c} -0.0167 \\ -0.0184 \\ -0.0196 \\ -0.0204 \\ -0.0208 \end{array}\right. & \begin{array}{c} 0.7793 \\ 0.485 \\ -0.0531 \\ -0.5645 \\ -0.906 \end{array} & \begin{array}{c} 0.0499 \\ -0.0188 \\ -0.0593 \\ -0.0108 \\ 0.0862 \end{array} & \begin{array}{c} 0.0017 \\ -0.0024 \\ -3.8458\times10^{-4} \\ 0.0026 \\ -0.0029 \end{array} & \left.\begin{array}{c} -0.0041 \\ 0.0099 \\ -0.0113 \\ 0.0076 \\ -0.0046 \end{array}\right) in \end{array}$$

Total ✓

$$udyn_{tot} = \begin{pmatrix} 0.8101 \\ 0.4554 \\ -0.1437 \\ -0.5854 \\ -0.8481 \end{pmatrix} in.$$

✓

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HOMEWORK #10

Part Three (cont'd)

Displacements: t=1.0s

	Mode 1	M2	M3	M4	M5	TOTAL
Floor 1	-0.0167	0.7793	0.0499	0.0017	-0.0041	0.8101
Roof	-0.0208	-0.906	0.0862	-0.0029	-0.0046	-0.8481

Force calcs

$$F_s^{\langle n \rangle} = \omega_n^2 \cdot M_{tot} \cdot \phi_n \cdot q_n$$

$$F_s = \begin{pmatrix} -1.104 & 1.058 \times 10^3 & 202.242 & 11.754 & -39.294 \\ -1.215 & 658.252 & -76.008 & -16.698 & 95.348 \\ -1.296 & -72.006 & -240.183 & -2.643 & -108.636 \\ -1.346 & -766.071 & -43.886 & 18.14 & 73.201 \\ -0.686 & -614.797 & 174.679 & -9.976 & -22 \end{pmatrix}$$

Story shears = ΣF_s from that level up

$$F_{sb} = \begin{pmatrix} -5.647 \\ 263.03 \\ 16.844 \\ 0.577 \\ -1.383 \end{pmatrix} \qquad F_{sr} = \begin{pmatrix} -0.686 \\ -614.797 \\ 174.679 \\ -9.976 \\ -22 \end{pmatrix} \quad \begin{matrix} \text{Mode 1} \\ \text{M2} \\ \text{M3} \\ \text{M4} \\ \text{M5} \end{matrix}$$

273.42 K (base) | -472.78 K (roof) | TOTAL ✓

Largest influence is from the second mode, likely because it has a natural frequency very close to the forcing frequency of the applied force (r = 1.023). In that mode, the structure is near resonance. ✓

HOMEWORK #10

Part Four

$t_d = 0.6s$

$P_{ro} = 500k$ (r for rectangular)

$$\frac{t_d}{T_n} = \begin{bmatrix} 0.388 \\ 1.759 \\ 3.039 \\ 3.958 \\ 4.676 \end{bmatrix} \qquad P_{ro_n} = \phi_n^T P_r = \begin{bmatrix} 402.59 \\ -430.08 \\ 289.45 \\ -294.56 \\ 446.52 \end{bmatrix} k$$

P_r: $[5\times1]$, $P_{r_o} = 500k$

$$q_{ro} = \frac{P_{ro}}{K_n(0)} = \begin{bmatrix} 1.609 \\ -0.165 \\ 0.052 \\ -0.018 \\ 0.00313 \end{bmatrix}$$

$$u_{rn}(t) = q_{ro}\left[\cos(\omega_n(t-t_d)) - \cos(\omega_n t)\right], \quad t = 1.0s$$

$$u_{rect} = \phi_n u_{rn}$$

$$\begin{array}{c|ccccc|c} & \text{Mode 1} & \text{M2} & \text{M3} & \text{M4} & \text{M5} & \\ \hline & 0.7155 & -0.0628 & 0.0021 & 0.0019 & 0.0013 & \text{BASE} \\ \text{urect} = & 0.7874 & -0.0391 & -7.9737\times10^{-4} & -0.0026 & -0.0032 & \\ & 0.8401 & 0.0043 & -0.0025 & -4.1824\times10^{-4} & 0.0037 & \text{in} \\ & 0.8723 & 0.0455 & -4.6039\times10^{-4} & 0.0029 & -0.0025 & \\ & 0.8886 & 0.073 & 0.0037 & -0.0032 & 0.0015 & \text{ROOF} \end{array}$$

TOTAL

$$\text{urect}_{tot} = \begin{pmatrix} 0.658 \\ 0.7417 \\ 0.8452 \\ 0.9177 \\ 0.9636 \end{pmatrix}$$

$F_{sr} = \omega_n^2 M_{tot} \phi_n u_{rn}$, $F_{sr\,floor}$ = sum of floors above (in kips)

$$F_{srb} = \begin{pmatrix} 241.472 \\ -21.183 \\ 0.716 \\ 0.628 \\ 0.451 \end{pmatrix} \qquad F_{srr} = \begin{pmatrix} 29.316 \\ 49.512 \\ 7.426 \\ -10.849 \\ 7.18 \end{pmatrix}$$

TOTAL: 222.084 k (base) 82.585 k (roof)

First mode forces are most important at the first floor, but not at the fifth. This could be from the application of force to the 1st floor.

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HOMEWORK #10

Part Five

From data in .xls file:

$$PA = \text{pseudo}\cdot A = \begin{bmatrix} 0.486957 \\ 0.605248 \\ 0.494389 \\ 0.392795 \\ 0.381086 \end{bmatrix} \cdot g$$

PA values interpolated between data points in Excel file.

$$L_n = \sum_{j=0}^{4} M_{tot_{j,j}} \cdot \phi_{j,n}$$

$$\Gamma = \frac{L_n}{M_n} = \begin{bmatrix} 1.086 \\ -0.112 \\ 0.035 \\ -0.012 \\ 0.00212 \end{bmatrix}$$

$$D = \frac{PA_n}{\omega_n^2} \cdot 386\,\text{in/s}^2 = \begin{bmatrix} 11.395 \\ 0.689 \\ 0.188 \\ 0.088 \\ 0.061 \end{bmatrix} \text{in}$$

← much bigger than other modes

$$q_{earth} = \Gamma_n \cdot D_n, \qquad u_{earth}^{\langle n \rangle} = \phi_n \, q_{earth_n}$$

Since the peak displacement of each mode doesn't occur simultaneou

find $u_{o\ \text{floor, mode}} = \Gamma_{\text{mode}} \cdot \Phi_{\text{floor, mode}} \cdot D_{mo}$

Then, use SRSS rule.

$$q_{earth} = \begin{pmatrix} 12.379 \\ -0.077 \\ 6.625\times10^{-3} \\ -1.067\times10^{-3} \\ 1.298\times10^{-4} \end{pmatrix}$$

$$\begin{array}{cccccc} & M1 & M2 & M3 & M4 & M5 \\ u_{earth} = & 9.967 & 0.0661 & 0.0038 & 6.2875\times10^{-4} & 1.1594\times10^{-4} \\ & 10.9693 & 0.0411 & -0.0014 & -8.9322\times10^{-4} & -2.8132\times10^{-4} \\ & 11.7035 & -0.0045 & -0.0046 & -1.414\times10^{-4} & 3.2052\times10^{-4} \\ & 12.1517 & -0.0478 & -8.3225\times10^{-4} & 9.7036\times10^{-4} & -2.1597\times10^{-4} \\ & 12.3786 & -0.0768 & 0.0066 & -0.0011 & 1.2982\times10^{-4} \end{array}$$

TOTAL (in)

10.038
11.008
11.695
12.104
12.307

use SRSS rule

Almost all the displacement comes from the first mode — high Γ value, low ωn value.

12/12

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HOMEWORK #10

Part Five (cont'd)

$F_{se} = \omega_n^2 \cdot M_{tot} \cdot \phi_n \cdot q_{earth}$

$$F_{se} = \begin{pmatrix} 657.648 & 89.648 & 15.542 & 4.322 & 1.112 \\ 723.783 & 55.794 & -5.841 & -6.139 & -2.698 \\ 772.229 & -6.103 & -18.457 & -0.972 & 3.074 \\ 801.804 & -64.933 & -3.372 & 6.669 & -2.071 \\ 408.387 & -52.111 & 13.423 & -3.668 & 0.622 \end{pmatrix}$$

	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	TOTAL	
Floor 1	3363.85	22.295	1.294	0.212	0.039	3387.69	F_{seb}
Roof	408.387	-52.111	13.423	-3.668	0.622	366.654	F_{ser}

Sum of Squares

$F_{seb\,tot} = \sqrt{\sum_{j=0}^{4} F_{seb_j}^2}$ = 3363.926 K BASE SHEAR

$F_{ser\,tot} = \sqrt{\sum_{j=0}^{4} F_{ser_j}^2}$ = 411.934 K ROOF SHEAR

SRSS

✓ $u_{earthb\,tot} = \sqrt{\sum_{j=0}^{4} {u_{earth}^{\langle j \rangle}}_0^2}$ = 9.967 in BASE DISP.

✓ $u_{earthr\,tot} = \sqrt{\sum_{j=0}^{4} {u_{earth}^{\langle j \rangle}}_4^2}$ = 12.379 in ROOF DISP.

Earthquake excitation doesn't seem to cause high excitations in the higher level modes – displacement and force values are majorly controlled by mode 1.

To reduce deflections and deformations, a study could focus on mode 1 and the various impacts when changing mass, stiffness, and damping.

From the ground to the first floor – almost 10" of displacement. Floor one to five – under 2.5". The much lower stiffness of the first floor really hurts this system; bulking up or bracing those columns could do a lot to reduce deflections.

Homework No. 10

Assigned: April 25, 2006
Due: May 4, 2006

This homework will serve as a synthesis of nearly everything that you have learned in this course. Please turn in all program listings and calculations that support your answers to each question. For this homework, assume that this is an analysis that you carried out at a design office and that you are then required to present your results (actual quantitative values) to your peer or supervisor. In order to show him/her that you really know what the analyses tell you, you are also required to give some physical insights into your results. Comment, accordingly, on any issue that you think is noteworthy regarding the modeling, the excitation, the response and the relative importance of the different modes.

While grading this homework assignment, importance will be placed on presentation and on the comments that you make regarding your findings. The actual numerical answers are important but less so and all the answers are provided on the following pages.

The Structural Model

You are to analyze a five-story building for various dynamic excitations. The model for this building is shown in Figure 1. It is meant to represent a realistic structure – you may ignore axial deformations in all members (beams and columns), and may also neglect flexural deformations in the beams. Use any computer program (or write one) to carry out your calculations.

$I_{\text{typ}} = 4320 \text{ in}^4$ (for all columns)
$h_{\text{typ}} = 10$ ft (for floors 2 -5)
$h_1 = 2\, h_{\text{typ}}$ (for first floor)
$m_{\text{typ}} = 4 \text{ kip s}^2/\text{in}$ (for floors 1-4)
$m_5 = 0.5\, m_{\text{typ}}$ (for roof)
$E = 30000$ ksi

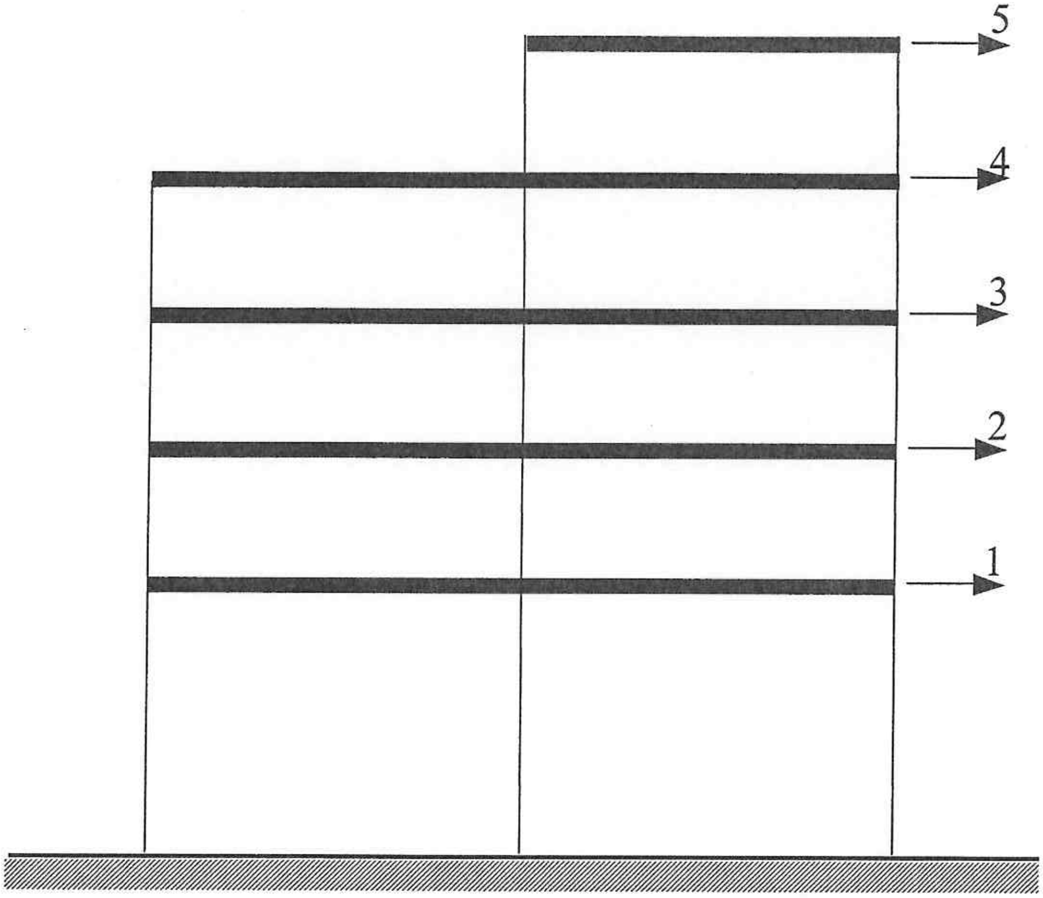


Figure 1

Part I

Natural vibration frequencies and mode shapes

Verify that the natural frequencies are as shown in the table below:

Mode	Natural Frequency (rad/sec)
1	4.061
2	18.419
3	31.829
4	41.452
5	48.962

Also, verify that the corresponding mode shapes are as shown in the table below:

Floor	Mode				
	1	2	3	4	5
1	0.805	-0.860	0.579	-0.589	0.893
2	0.886	-0.535	-0.218	0.837	-2.167
3	0.945	0.059	-0.687	0.132	2.469
4	0.982	0.623	-0.126	-0.909	-1.664
5	1.000	1.000	1.000	1.000	1.000

Draw rough sketches for each of these mode shapes.

For the mode shapes normalized so that the roof displacement is unity, show that the modal masses and modal stiffnesses are as given in the table below:

Mode, *i*	Modal Mass, M_i (kip-s^2/in)	Modal Stiffness, K_i (kip/in)
1	15.165	250.150
2	7.672	2603.005
3	5.484	5555.164
4	9.567	16438.818
5	59.426	142459.812

Part II

Free vibration

Assume that the physical damping for the given structure is such that it may be assumed to yield a diagonal modal damping matrix (i.e., this is "classical damping"), and that the modal damping ratio for each mode is 5% of critical. If as initial conditions to start off free vibrations, only the roof is moved to the right by 1.0 inch, and the initial velocity at each floor is zero, show that, at t = 1 sec, the physical displacements at the roof and first story levels are as follows:

Response at t = 1 sec	Mode					TOTAL
	1	2	3	4	5	
Roof displacement (in.)	-0.0699	0.0910	0.0705	-0.0230	0.0005	**0.0691**
First floor displacement (in.)	-0.0563	-0.0783	0.0408	0.0135	0.0004	**-0.0798**

Next, show that the fifth story shear and the base shear at t = 1 sec are as follows:

Response at t = 1 sec	Mode					TOTAL
	1	2	3	4	5	
Fifth story shear (kips)	-2.307	61.771	142.888	-79.024	2.168	**125.496**
Base shear (kips)	-19.003	-26.428	13.779	4.572	0.136	**-26.994**

(*Optional question*: Compare the above four response quantities at t = 2.5 sec. Comment on any differences relative to the response at t = 1 sec.)

Part III

Harmonic excitation

A sinusoidal force $p_0 \sin(2\pi f_0 t)$ is applied at the third floor level (where $p_0 = 5000$ kips, $f_0 = 3$ Hz). This force is along the direction of degree-of-freedom no. 3 of the model. Again, as in Part II, assume classical damping and that the modal damping ratio for each mode is 5% of critical. Ignore the transient response (i.e., consider only the steady-state response) and show that, at $t = 1$ sec, the physical displacements at the roof and first story levels are as follows:

Response at $t = 1$ sec	Mode					TOTAL
	1	2	3	4	5	
Roof displacement (in.)	-0.0208	-0.9060	0.0862	-0.0029	-0.0046	**-0.8481**
First floor displacement (in.)	-0.0167	0.7793	0.0499	0.0017	-0.0041	**0.8101**

Next, show that the fifth story shear and the base shear at $t = 1$ sec are as follows:

Response at $t = 1$ sec	Mode					TOTAL
	1	2	3	4	5	
Fifth story shear (kips)	-0.686	-614.797	174.679	-9.976	-22.000	**-472.780**
Base shear (kips)	-5.647	263.030	16.844	0.577	-1.383	**273.421**

Part IV

Short-duration pulse load

An explosion within the building causes a lateral load of very short duration at the level of the first floor. The force will be assumed to act along the direction of degree-of-freedom no. 1 of the model. This short-duration excitation may be approximated by a rectangular pulse load of magnitude 500 kips that lasts for 0.6 seconds. Ignore damping effects and show that, at t = 1 sec, the physical displacements at the roof and first story levels are as follows:

Response at t = 1 sec	Mode					TOTAL
	1	2	3	4	5	
Roof displacement (in.)	0.8886	0.0730	0.0037	-0.0032	0.0015	**0.9636**
First floor displacement (in.)	0.7155	-0.0628	0.0021	0.0019	0.0013	**0.6580**

Next, show that the fifth story shear and the base shear at t = 1 sec are as follows:

Response at t = 1 sec	Mode					TOTAL
	1	2	3	4	5	
Fifth story shear (kips)	29.316	49.512	7.426	-10.849	7.180	**82.585**
Base shear (kips)	241.472	-21.183	0.716	0.628	0.451	**222.084**

(*Optional question*: Compare the above four response quantities at t = 0.5 sec, i.e., during the blast. Comment on any differences relative to the response after the blast, i.e., at t = 1 sec.)

Part V

Earthquake Response Spectrum Analysis

You are required to analyze the five-story building for seismic loads that might have been experienced during a recent earthquake. As in Parts II and III above, assume that energy dissipation for this structure is such that it leads to a diagonal modal damping matrix (i.e., classical damping), and that the modal damping ratio for each mode is 5% of critical.

You may use the natural frequencies and mode shapes as well as the modal masses and stiffnesses from Part I to perform your analysis.

Maximum seismic response using a response spectrum

In Figure 2, a 5%-damped response spectrum is shown that was obtained from a ground acceleration record during an earthquake that occurred in Turkey in August 1999.

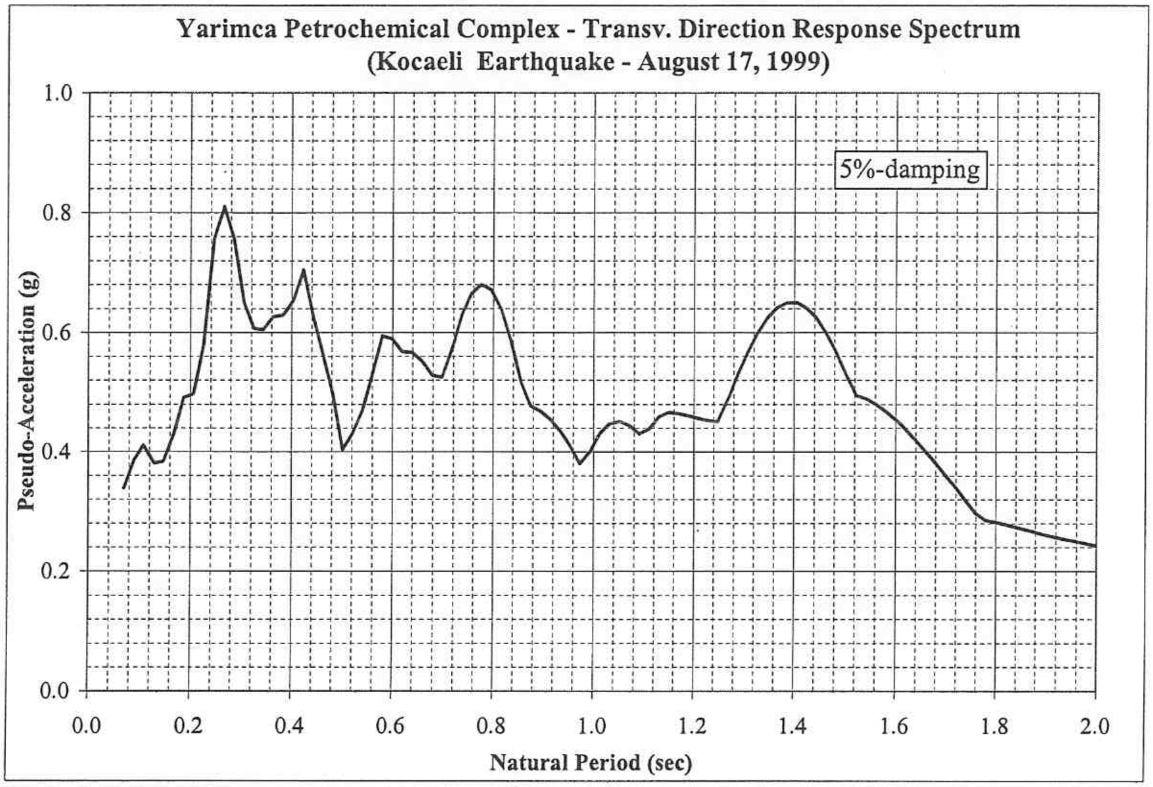
If you think you could use the actual data in electronic form for the response spectrum, you can find it in the YPTSaData tab of the Excel file that you can download from: *http://www.ce.utexas.edu/prof/Manuel/Spring2006_CE384P/Homeworks/HW10_Fig2Data.xls*

Often information about the ground acceleration record (time history) is summarized in the form of a response spectrum such as the one shown in Figure 2. Then, it is not possible to determine the *exact* values of any structural response measure. Rather, modal combination rules need to be used to estimate the maximum response – this may be done using response spectrum ordinates at the modal frequencies/periods.

Assuming that the modes of our 5-degree-of-freedom structure have well separated natural frequencies, use the SRSS (square-root-of-sum-of-squares) modal combination rule and show that the following are estimates of the various maximum response levels that might have occurred during the earthquake:

(a) maximum roof displacement = 12.39 in.
(b) maximum first story displacement = 9.98 in.
(c) maximum fifth story shear = 412.36 kips.
(d) maximum base shear = 3367.36 kips.

Comment briefly on the relative contribution of each mode to these maximum response estimates.



Yarimca Petrochemical Complex - Transv. Direction Response Spectrum
(Kocaeli Earthquake - August 17, 1999)

Figure 2

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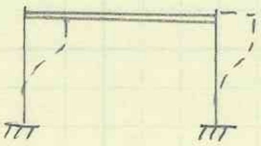
TEST #1

1.



$I_b >> I_c$ ← This already assumed 2 columns.

$k = 24 \frac{EI_c}{h^3} = 24(2k/in) = 48 k/in$

2 columns, $k_T = 96 k/in$ ✗

$w = 96 k$, $m = 0.249 \, k \cdot s^2/in$

$\omega_n = \sqrt{\frac{k_T}{m}} = 19.65 \, rad/s$ wrong k & Tn

$T_n = \frac{2\pi}{\omega_n} = 0.320 \, s$ ✗

i. $t_d = 0.05s$ $t_d/T_n = 0.156$ ✗
ii. $t_d = 0.10s$ $t_d/T_n = 0.313$ ✗
iii. $t_d = 0.5s$ $t_d/T_n = 1.56$ ✗

$$\left. \begin{aligned} u_g(t) &= 0.4gt^2 \\ \dot{u}_g(t) &= 0.8gt \\ \ddot{u}_g(t) &= 0.8g \end{aligned} \right\} t \leq t_d$$ ✓

$$(u_{st})_o = \frac{\ddot{u}_{go}}{\omega_n^2} = \frac{0.8(386 \, in/s^2)}{(19.65 \, rad/s)^2} = 15.72 \, in$$ off because of

In (a): The peak value occurs after t_d for i. and ii, and before t_d for iii.
($t_d/T_n \leq 0.5$)

I'm unsure how to get R_d for a quadratically varying function. With R_d, I would solve for u_o: | actually, load is a step function.

$$R_d (u_{st})_o = u_o$$ ✓

for each of the cases. R_d will depend soley on t_d/T_n.

a) +3

b) +5

c) +2

1	10
2	25
	35

b) Again, using R_d,

$$R_d \ddot{u}_{go} = \ddot{u}_o^t$$ ✓

where $\ddot{u}_{go} = 0.8g$ for all values of t.

c) Approximation methods are acceptable for $t_d/T_n \leq 1/4$. In this problem, (i) with $t_d/T_n = 0.156$ could be approximated ✓ ok.

using $\mathcal{I} = \int_0^{t_d} p(t)dt$ (the area under the curve of the forcing function).

$u_o = \frac{\mathcal{I}}{k} \cdot \frac{2\pi}{T_n}$. R_d could be calculated using u_o and $(u_{st})_o$, and used to find $\ddot{u}_o^t$.

Why? → can't you get $\mathcal{I}$?

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TEST #1

2. (cont'd)

$m = 0.05 \text{ k}\cdot\text{s}^2/\text{in}$

$\xi = 0.30$

$TR = 0.80$

From graph, $\omega/\omega_n = 1.56$ ✓

using the same equations:

$$\frac{(10\pi)^2}{1.56^2}(0.05\text{ k}\cdot\text{s}^2/\text{in}) = k = 20.28 \text{ k/in}$$

$$c = (0.30)(2)\left[(20.28\text{ k/in})(0.05\text{ k}\cdot\text{s}^2/\text{in})\right]^{1/2}$$

$$c = 0.604 \text{ k}\cdot\text{s/in}$$

$$\boxed{\begin{array}{l} \text{b.i. } k \le 20.28\text{ k/in} \\ c = 0.60\text{ k}\cdot\text{s/in} \end{array}}$$ ✓ ✓

$u \le 0.50\, u_{st_0}$

$R_d \le 0.50$ From graph, $\xi = 0.30$,

$\omega/\omega_n \ge 1.65$

$\omega \ge 1.65\,\omega_n$

$$\left(\frac{10\pi}{1.65}\right)^2 = \frac{k}{m}, \quad k \le 18.13 \text{ k/in}$$

$$c = 0.604\text{ k}\cdot\text{s/in} \cdot \left[\frac{18.13\text{ k/in}}{20.28\text{ k/in}}\right]^{1/2}$$

$$c = 0.571 \text{ k}\cdot\text{s/in}$$

$$\boxed{\begin{array}{l} \text{b.ii. } k \le 18.13\text{ k/in} \\ c = 0.57\text{ k}\cdot\text{s/in} \end{array}}$$ ✓ ✓

* It is necessary only to check $\omega = 10\pi$ ($f = 5$ hz) because the limits are for minimum ω/ω_n values. Increasing ω will result in a higher ratio, and acceptable conditions. ✓

Pg. out of order.

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3/3

TEST #1

2.

$m = 0.05 \; k \cdot s^2/in$

$f = 5hz, 10hz, 20hz$

$p_o = 40 \; k$

$TR \le 80\%$ ✓

a) $\xi = 0.60$

$$TR = \left[\frac{1+(2\xi\,\omega/\omega_n)^2}{(1-(\omega/\omega_n)^2)^2+(2\xi\,\omega/\omega_n)^2}\right]^{1/2}$$

$f = \frac{\omega}{2\pi}$, $\omega = 10\pi, 20\pi, 40\pi$

$\omega_n = \sqrt{\frac{k}{m}}$

$$0.64\left[(1-(\omega/\omega_n)^2)^2 + (2\xi\,\omega/\omega_n)^2\right] = 1 + (2\xi\,\omega/\omega_n)^2$$

From graph, TR = 80% at $\omega/\omega_n = 1.72$

$\omega = 10\pi, 20\pi, 40\pi$

$\omega_n = \sqrt{\frac{k}{0.05}}$

$\frac{\omega}{\omega_n} \ge 1.72$

$\omega \ge 1.72\sqrt{\frac{k}{0.05}}$ smallest ω value controls ✓

$$\left(\frac{10\pi}{1.72}\right)^2 (0.05) \ge k$$

$$\boxed{k \le 16.68 \; k/in}$$ ✓

$$\xi = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}}$$

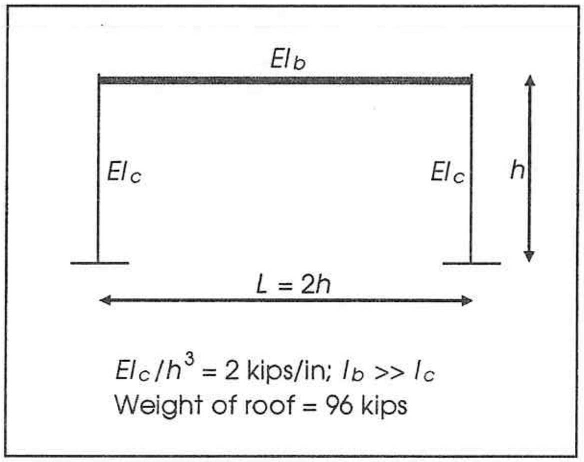
$$(0.60)(2)\sqrt{(16.68k/in)(0.05\,k\cdot s^2/in)} = c$$

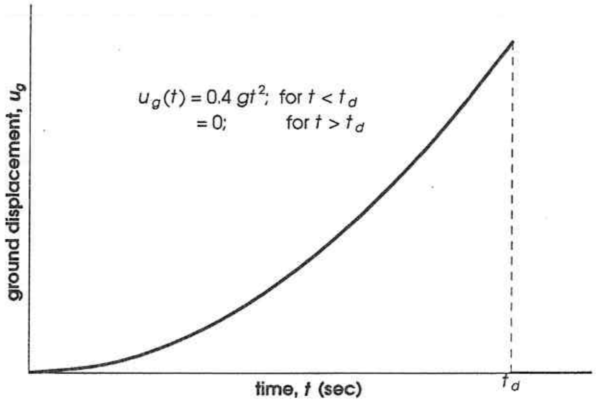
$c = 1.096$

$$\boxed{c = 1.10 \; k \cdot s/in}$$ ✓

CE 384P – Dynamic Response of Structures
Exam No. 1 – March 2, 2006 (75 minutes)
Open Text Book PLUS a single 8.5"×11" page of notes is permitted

1. Consider the frame shown below.





(Note: Despite the appearance of the graph, assume that the fall-off in $u_g(t)$ at $t=t_r$ takes a very small non-zero time and that the fall-off is a smooth function so that no large forces are mobilized at $t=t_r$.)

It is subjected to the limited-duration ground displacement, $u_g(t)$, shown where in the expression given in the figure, g refers to acceleration due to gravity (32.2 ft/sec^2).

Neglect damping and consider three cases for the duration, t_d, of ground shaking:
(i) $t_d = 0.05$ sec; (ii) $t_d = 0.10$ sec; (iii) $t_d = 0.5$ sec.

(a) What is the peak (relative) deformation, u_0, in each case? In each case, does the peak value occur during or after the ground shaking has ceased?

(b) What is the peak (total) acceleration, $\ddot{u}_0^t$, in each case?

(c) In which of the three cases can you use any suitable approximation to estimate peak displacements and accelerations? Repeat Parts (a) and (b) for any such cases and compare with your previous answers. Comment very briefly.

[25 points]

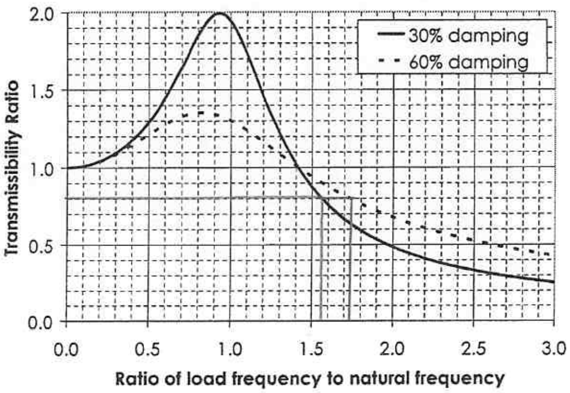
2. Consider a structure idealized as a single-degree-of-freedom that has a mass of 0.05 kip-sec^2/in. At different times, harmonic loadings are expected at three different frequencies: (i) 5 Hz; (ii) 10 Hz; and (iii) 20 Hz. The load amplitude is 40 kips.

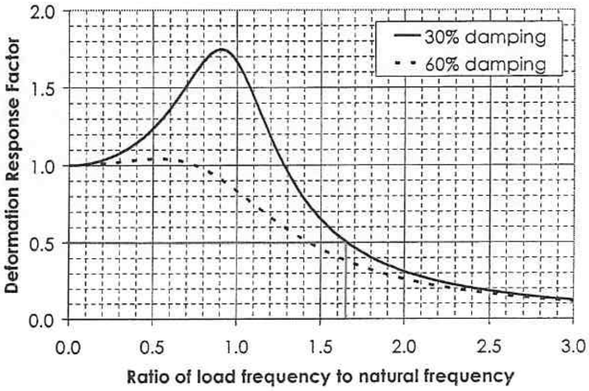
   Of interest is the design of this structure that can assure that only 80% of the applied load is transmitted to the supports.

   (a) Consider a system with 60% of critical damping.
   Determine the largest stiffness of the structure that can meet the requirement for transmitted loads no matter which load frequency is used.
   Also, determine the viscous damping coefficient, *c*.

   (b) Next, consider a system with 30% of critical damping.
   (i) Again, for the same restrictions on the transmitted load, determine the largest acceptable stiffness as in Part (a). Then, determine *c*.
   (ii) Suppose there is an additional requirement now that the dynamic displacement must be at most 50% of the static displacement, how would you change your stiffness, *k*, and damping coefficient, *c*, so as to meet both the transmitted force and the displacement criteria?

   For convenience, you may use the plots available below.





[25 points]

CE384P: DYNAMICS

CATHERINE HOVELL
21 March 06

EXAM #1 REVISITED

1.

$\frac{EI_c}{h^3} = 2\ K/in$

$I_b >> I_c$

$W = 96\ K$

For $I_b >> I_c$,

$K = 24\ \frac{EI_c}{h^3} = 48\ K/in$ ✓

i. $t_d = 0.05s$
ii. $t_d = 0.10s$
iii. $t_d = 0.5s$

•2 counting two columns twice

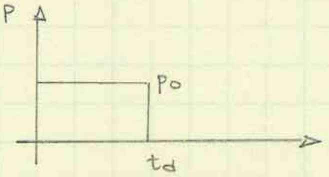
$u_g(t) = 0.4gt^2 \quad t < t_d$

$\dot{u}_g = 0.8gt$

$\ddot{u}_g = 0.8g$ constant acceleration

→ constant force

$$\omega_n = \sqrt{\frac{k}{m}} = \left[\frac{(48K/in)(2)}{96K/386 in/s^2}\right]^{1/2}$$



$\omega_n = 19.66\ rad/s$

$T_n = \frac{2\pi}{\omega_n} = 0.320\ s$ ×

$P_o = \ddot{u}_g m$

$P_o = (0.8g)\frac{96K}{g} = 76.8\ K$ ✓

i. $t_d/T_n = 0.156$
ii. $t_d/T_n = 0.313$
iii. $t_d/T_n = 1.56$

×

$$u_{st_o} = \frac{P_o}{k} = \frac{76.8\ K}{2(48\ K/in)} = 0.8\ in$$

If $t_d/T_n > 1/2$, $u_{o\,max} = 2u_{st_o}$ (case iii)

If $t_d/T_n < 1/2$, $u_{o\,max} = 2u_{st_o}\sin(\pi\, t_d/T_n)$

$u_o = 2\ (0.8\ in)\sin\left[\pi\ t_d/0.320s\right]$ for $t_d = 0.05$
0.10
0.50s

1	23
2	25
	48

Nice improvement!

a. $u_{o_i} = 0.75$ in – after shaking ✓
$u_{o_{ii}} = 1.33$ in – after shaking ✓
$u_{o_{iii}} = 1.6$ in – during shaking ✓

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EXAM #1, TAKE 2

1. (cont'd)

$$\ddot{u}_o^t = R_d \ddot{u}_{go}, \quad \ddot{u}_{go} = u_{st_o} \cdot \omega_n^2, \quad R_d = \frac{u_o}{u_{st_o}}$$

✓ right idea.

$$\ddot{u}_o^t = u_o \omega_n^2, \quad \omega_n = 19.66 \text{ rad/s}$$

using u_o values from (a),

$$\boxed{\begin{aligned} \text{b. } \ddot{u}_{o\,i}^t &= 292 \text{ in/s}^2 \\ \ddot{u}_{o\,ii}^t &= 514 \text{ in/s}^2 \\ \ddot{u}_{o\,iii}^t &= 618 \text{ in/s}^2 \end{aligned}}$$

×

(c) The approximations work best when $t_d/T_n < 0.5$; the smaller the ratio, the better. So, for $t_d = 0.05$ s and $t_d = 0.10$ s:

✓

$$u_o = \frac{I\, 2\pi}{k \cdot T_n}, \quad I = \text{area beneath the load (p) curve}$$

right idea

$$I = p_o t_d = 76.8\text{k}\,(t_d)$$

$$u_o = \frac{76.8\text{k}\,(2\pi)}{2(48\text{k/in})} \frac{t_d}{T_n} = 5.03\, t_d/T_n$$

$$\boxed{\begin{aligned} \text{c.(a) } u_{o\,i} &= 0.79 \text{ in} \\ u_{o\,ii} &= 1.57 \text{ in} \end{aligned}}$$

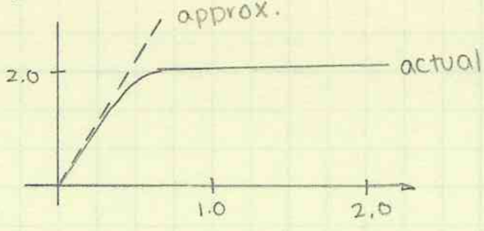
×

$$\ddot{u}_o^t = u_o \omega_n^2$$

$$\boxed{\begin{aligned} \text{c.(b) } \ddot{u}_{o\,i}^t &= 304 \text{ in/s}^2 \\ \ddot{u}_{o\,ii}^t &= 608 \text{ in/s}^2 \end{aligned}}$$

×

This approximation overestimates the actual response (which is conservative) because it is a linear approximation, whereas the actual response curves over as t_d/T_n increases through 1/2.



The decrease in accuracy can be seen between case i and case ii; in i, the approximation is only 5% high, whereas it is 18% high for case ii.

✓ ✓

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EXAM #1 REVISITED

2. $m = 0.05\ k\cdot s^2/in$
$P_o = 40K$

i. $f = 5Hz$
ii. $f = 10Hz$
iii. $f = 20Hz$

For $\zeta = 0.60$, $TR \leq 0.80$, $\omega/\omega_n \geq 1.75$

consider $f = 5Hz$ ($\omega = 31.4\ rad/s$)
(larger f values will not control) ✓

$$\omega_n = \sqrt{\frac{K}{m}}, \quad \omega_n(1.75) \leq \omega = 31.4\ rad/s$$

$$K \leq \left(\frac{\omega}{1.75}\right)^2 m$$

$$K \leq \left[\frac{31.4\ rad/s}{1.75}\right]^2 (0.05\ k\cdot s^2/in)$$

$$K \leq 16.1\ K/in$$ ✓

$$c = \zeta 2\sqrt{km}$$

$$c = 0.60(2)\left[(16.1\ k/in)(0.05\ k\cdot s^2/in)\right]^{1/2}$$

$$c \leq 1.08\ k\cdot s/in$$

$$\boxed{\begin{array}{l} a.\ K \leq 16.1\ k/in \\ \quad c \leq 1.08\ k\cdot s/in \end{array}}$$ ✓

For $\zeta = 0.30$, $TR \leq 0.80$, $\omega/\omega_n \geq 1.55$ ✓

$$K \leq \left[\frac{31.4\ rad/s}{1.55}\right]^2 (0.05\ k\cdot s^2/in) = 20.52\ k/in$$

$$c = (0.30)(2)\left[(20.52\ k/in)(0.05\ k\cdot s^2/in)\right]^{1/2}$$

$$c = 0.61\ k\cdot s/in$$

$$\boxed{\begin{array}{l} b.i\ \ K \leq 20.5\ k/in \\ \quad c \leq 0.61\ k\cdot s/in \end{array}}$$ ✓

If $R_d \leq 0.50$, $\omega/\omega_n \geq 1.65$ ✓

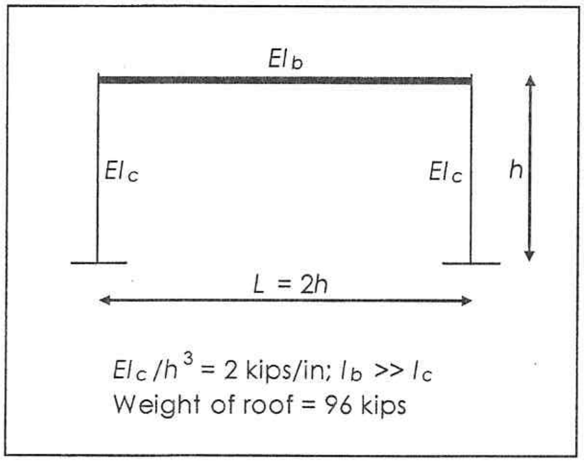
$$K \leq \left[\frac{31.4\ rad/s}{1.65}\right]^2 (0.05\ k\cdot s^2/in) = 18.11\ k/in$$

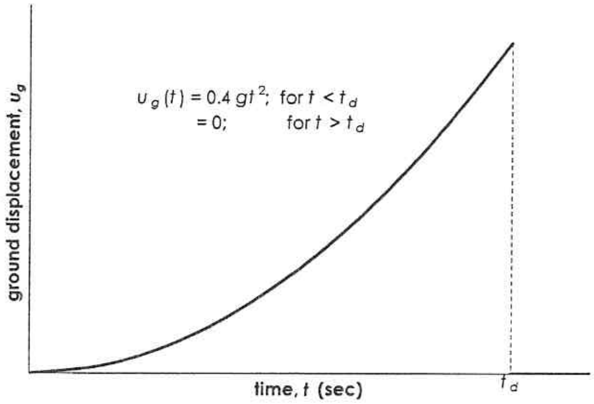
$$c = 0.61\left[\frac{18.11\ k/in}{20.52\ k/in}\right]^{1/2} = 0.57\ k\cdot s/in$$

$$\boxed{\begin{array}{l} b.ii\ \ K \leq 18.1\ k/in \\ \quad c \leq 0.57\ k\cdot s/in \end{array}}$$ ✓

CE 384P – Dynamic Response of Structures
Exam No. 1 REVISITED – The Take-Home Version
Date/Time Out: 11:30, Monday; Date/Time In (within 24 hrs): :

1. Consider the frame shown below.





(Note: Despite the appearance of the graph, assume that the fall-off in $u_g(t)$ at $t=t_r$ takes a very small non-zero time and that the fall-off is a smooth function so that no large forces are mobilized at $t=t_r$.)

It is subjected to the limited-duration ground displacement, $u_g(t)$, shown where in the expression given in the figure, g refers to acceleration due to gravity (32.2 ft/sec^2).

Neglect damping and consider three cases for the duration, t_d, of ground shaking:
(i) $t_d = 0.05$ sec; (ii) $t_d = 0.10$ sec; (iii) $t_d = 0.5$ sec.

(a) What is the peak (relative) deformation, u_0, in each case? In each case, does the peak value occur during or after the ground shaking has ceased?

(b) What is the peak (total) acceleration, $\ddot{u}_0^t$, in each case?

(c) In which of the three cases can you use any suitable approximation to estimate peak displacements and accelerations? Repeat Parts (a) and (b) for any such cases and compare with your previous answers. Comment very briefly.

[25 points]

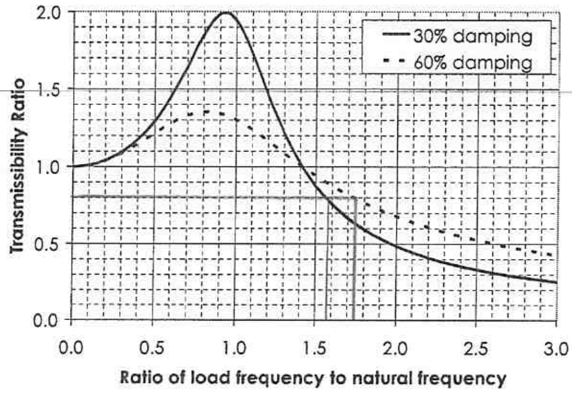
2. Consider a structure idealized as a single-degree-of-freedom that has a mass of 0.05 kip-sec^2/in. At different times, harmonic loadings are expected at three different frequencies: (i) 5 Hz; (ii) 10 Hz; and (iii) 20 Hz. The load amplitude is 40 kips.

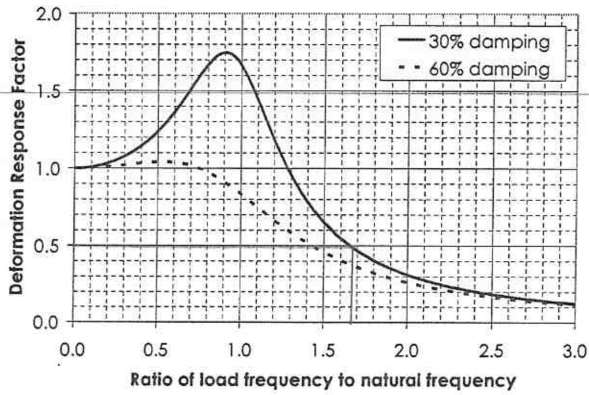
Of interest is the design of this structure that can assure that only 80% of the applied load is transmitted to the supports.

(a) Consider a system with 60% of critical damping.
Determine the largest stiffness of the structure that can meet the requirement for transmitted loads no matter which load frequency is used.
Also, determine the viscous damping coefficient, *c*.

(b) Next, consider a system with 30% of critical damping.
(i) Again, for the same restrictions on the transmitted load, determine the largest acceptable stiffness as in Part (a). Then, determine *c*.
(ii) Suppose there is an additional requirement now that the dynamic displacement must be at most 50% of the static displacement, how would you change your stiffness, *k*, and damping coefficient, *c*, so as to meet both the transmitted force and the displacement criteria?

For convenience, you may use the plots available below.





[25 points]

[redacted]

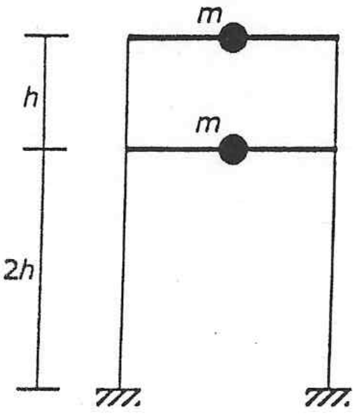
CE 384P – Dynamic Response of Structures
Exam No. 2 – April 21, 2005
2-hour exam; 2 pages of notes permitted.

$\boxed{50}$ Excellent work!!

---

1. The 2-story shear building shown in Figure 1 is to be analyzed as a generalized SDOF system. Neglect axial deformations in all members and flexure in the beams.
   (a) Given the shape vector, $\underline{\psi}^T = \{2/3, 1\}$, express the natural frequency, ω_n, of the system in terms of E, I, m, and h?
   (b) The building is subject to ground shaking that has a rolling quality such that the ground displacement feels like a sinusoidal motion, i.e., $u_g(t) = u_{g0} \sin\omega t$. Assume that the damping ratio is ζ, and determine the peak displacement at both floors in terms of u_{g0}, ω, ω_n, and ζ. Next, determine the peak base shear in terms of m, u_{g0}, ω, ω_n, and ζ. You may ignore initial conditions (transients).
   (c) Assume that $\zeta = 0.05$, that $EI/h^3 = 14.2561$ kip/in, that $\omega/\omega_n = 0.9$, and that $u_{g0} = 2$ inches. If the mass, m, equals 1 kip-sec^2/in, determine the peak top-floor displacement and the peak base shear during the ground shaking.
   (d) A different shape vector, $\underline{\psi}^T = \{1/2, 1\}$, is selected. Does it lead to a more accurate natural frequency than the original one? Why?

[25 points]



E and *I* are the same for all columns. The 2nd story stiffness is equal to $24EI/h^3$.

Figure 1

1	25
2	25
	50

2. Consider two very light wood-frame single-story structures shown in Figure 2. Neglect axial deformations ~~in the~~ beams and columns of both structures ~~and flexure in the~~ beams. Both structures are located very close to each other and can be idealized as SDOF systems. Each has the same mass, m, equal to 2 kip-sec^2/in. The natural periods for Structures A and B are 2.0 and 1.0 seconds, respectively.
Our interest is in assessing the performance of the structures in a past earthquake that excited both of them. A 5%-damped pseudo-acceleration response spectrum for this site and for this earthquake is shown in Figure 3.
   (a) Estimate the peak ground acceleration of the ground shaking at this site.
   (b) What is the shear force in each column of the two structures?
   (c) Assume that the buildings are separated by a distance, Δ. If Δ is too small, the buildings might strike each other at the top. What spacing, Δ, would guarantee that the two building did not strike each other during the earthquake?
   (d) The right column of Structure B has a height αh. What is the value of α?
   (e) If the ground acceleration time history were available to you, you could easily obtain the response for either structure. If you are asked to do so for Structure B using the central difference method, what time step would you use? Why?
   If Newmark's method with $\gamma = 1/2$ and $\beta = 5/24$ were to be used instead, what time step would you use now? Why?

[25 points]

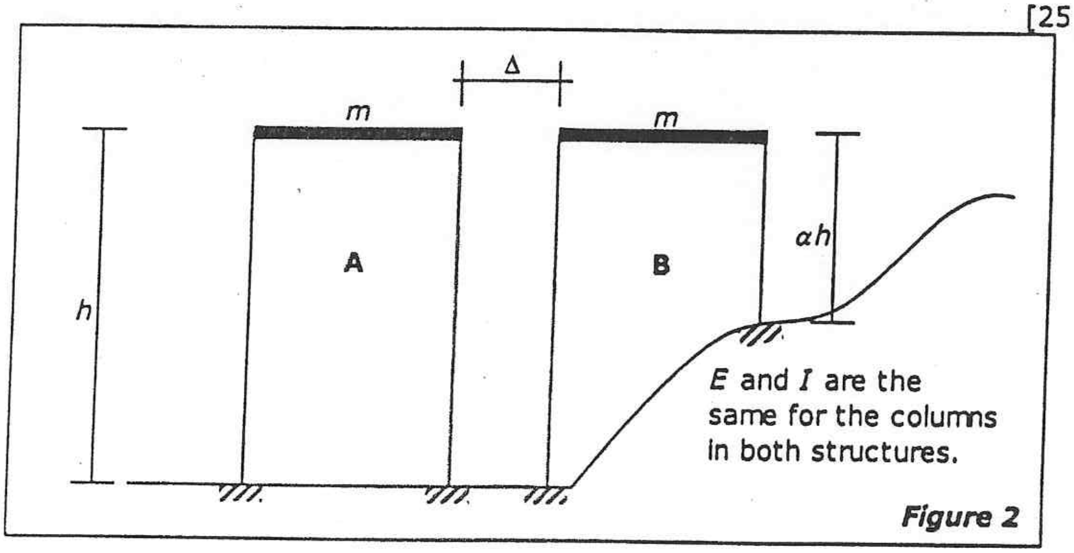


Figure 2

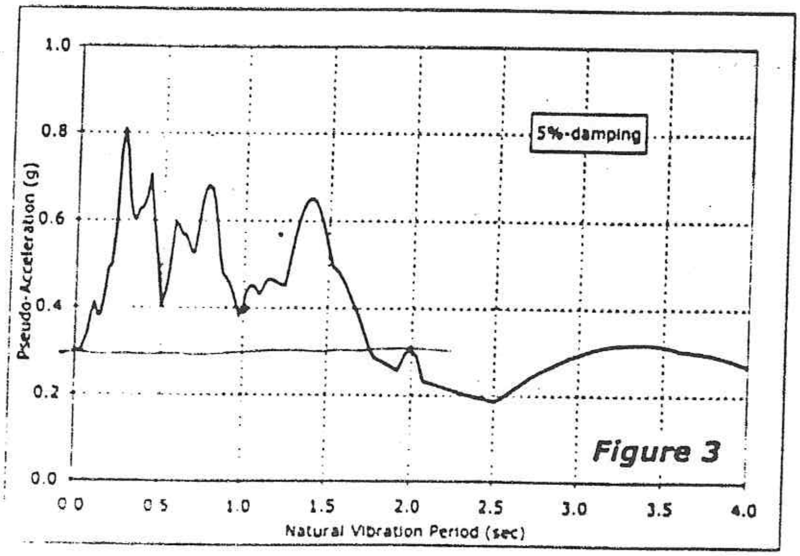


Figure 3

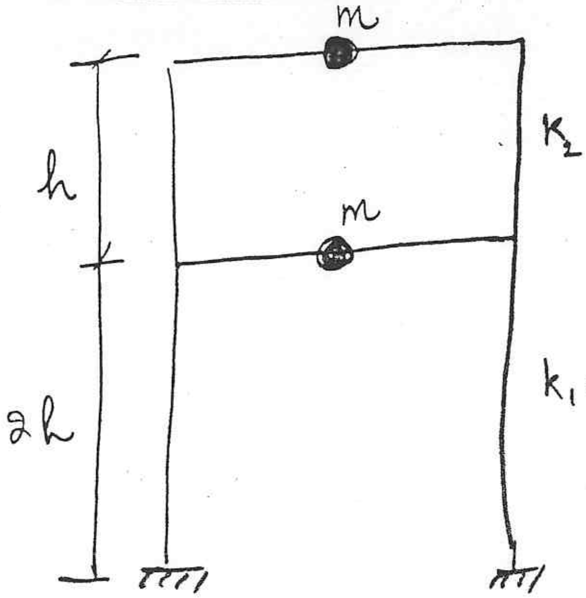
384P
am No. 2
/2005

CE384P EXAM 2

[redacted]

⑪ pages

s. 1.



$k_2 = \frac{24EI}{h^3} = k$ (say)

$k_1 = \frac{24EI}{(2h)^3} = \frac{24EI}{8h^3} = \frac{k}{8}$ ✓

$\left(= \frac{3EI}{h^3} \right)$ ✓

) $\psi = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

$\tilde{m} = \sum_{j=1}^{2} m_j \psi_j^2 = m \left(\frac{2^2+3^2}{3^2} \right) = \boxed{1.44m} \longrightarrow (1)$ ✓

$\tilde{k} = \sum_{j=1}^{2} k_j (\psi_j - \psi_{j-1})^2 = \left(\frac{k}{8} \right) (2/3)^2 + k (1/3)^2$

$\boxed{\tilde{k} = 0.167k} \longrightarrow (2)$ ✓

$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \sqrt{\frac{0.167k}{1.44m}} = \sqrt{\frac{0.167}{1.44} \left(\frac{24EI}{h^3} \right)} \quad \left(\because k = \frac{24EI}{h^3} \right)$

$\boxed{\omega_n = 1.668 \sqrt{\frac{EI}{mh^3}}}$ ✓ NATURAL FREQUENCY $\Longrightarrow$ (Ans a) $\longrightarrow$ (3)

also $\tilde{L} = \sum_{j=1}^{2} m_j \psi_j = m \left(\frac{2+3}{3} \right) = \frac{5m}{3} = \boxed{1.67m} \longrightarrow (4)$ ✓

[redacted]

$$u_g(t) = u_{go} \sin \omega t \longrightarrow (6)$$

$$\dot{u}_g(t) = \omega\, u_{go} \cos \omega t \longrightarrow (7)$$

$$\ddot{u}_g(t) = -(\omega^2 u_{go}) \sin \omega t \longrightarrow (8)$$

The generalized equation of motion for such a structure is given by

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = -\tilde{L}\,\ddot{u}_g(t)$$

$$\boxed{\therefore \tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{L}(u_{go}\omega^2)\sin\omega t.} \quad \text{from (8)} \longrightarrow (9)$$

The above equation has the general form

$$\boxed{\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{p}(t)} \longrightarrow (10)$$

where $\tilde{p}(t) = \tilde{L}(u_{g_o}\omega^2)\sin\omega t$

$$\therefore \boxed{\tilde{p}_o = \tilde{L}\, u_{g_o}\omega^2.} \quad \text{from (9) \& (10)} \longrightarrow (11)$$

Now, for a harmonic response to a sinusoidal exciting force given by $p(t) = p_o \sin\omega t$, we know that

$$R_d = \frac{1}{[(1-r^2)^2 + (2\xi r)^2]^{1/2}} \quad \text{where } r = \omega/\omega_n$$

(ignoring transients)

$$\therefore z_o = (z_{st})_o R_d$$

$$z_o = \frac{\tilde{p}_o}{\tilde{k}} . R_d$$

$$(\because \tilde{k} = \tilde{m}\omega_n^2)$$

$$z_o = \frac{\tilde{p}_o}{\tilde{m}\omega_n^2} . R_d.$$

$\tilde{P}_0 = (\tilde{L} u_{g_0} \omega^2)$.

$$\therefore \quad z_0 = \boxed{\frac{\tilde{L}}{\tilde{m}}} u_{g0} \boxed{\frac{\omega^2}{\omega_n^2}} \frac{1}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\xi(\omega/\omega_n)\right]^2\right\}^{1/2}}$$

– from(11)

$$z_0 = \frac{u_{g0} \tilde{\Gamma} (\omega/\omega_n)^2}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\xi(\omega/\omega_n)\right]^2\right\}^{1/2}}$$ ✓

$$\boxed{z_0 = \frac{1.157\, u_{g0} r^2}{\left[(1-r^2)^2 + (2\varepsilon r)^2\right]^{1/2}}}$$

where $r = \dfrac{\omega}{\omega_n}$ → (12)

PEAK TOP FLOOR DISPLACEMENT

Max base shear

$$= \sum_{j=1}^{2} \omega_n^2 m_j \psi_j z_0$$

$$= \omega_n^2 z_0 \sum m_j \psi_j$$

$$(\because \sum m_j \psi_j = \tilde{L})$$

$$= \omega_n^2 z_0 \tilde{L}$$ ✓

$$V_{b0} = \frac{(\cancel{\omega_n^2})(1.67m)(1.157)u_{g0}(\omega/\cancel{\omega_n})^2}{\left[(1-r^2)^2 + (2\varepsilon r)^2\right]^{1/2}}.$$

$$\boxed{\therefore V_{b0} = \frac{1.93\, m u_{g0} \omega^2}{\left[(1-r^2)^2 + (2\varepsilon r)^2\right]^{1/2}}}$$

where

$r = \omega/\omega_n$ → (Ans b)

→ (13)

PEAK BASE SHEAR

$\zeta = 0.05 \quad EI/h^3 = 14.2561 \text{ k/in} \; ; \; \omega/\omega_n = 0.9 \; ; \; u_{g_0} = 2''$ ④

[redacted]

$m = 1 \text{ kip-sec}^2/\text{in}.$

$\therefore \boxed{\omega_n = 6.31 \text{ rad/sec}}$ from eq (3)

$\therefore \omega = (0.9)\,\omega_n = 0.9 \times 6.31$

$\boxed{\therefore \omega = 5.68 \text{ rad/sec}}$ ✓

$\therefore$ Peak top floor deflection

$$Z_0 = \frac{(1.157)(2)(0.9)^2}{\left[(1-0.9^2)^2 + [2(0.05)(0.9)]^2\right]^{1/2}}$$ from eq (12)

$\therefore \boxed{Z_0 = 8.91''}$ ✓ ⟶ TOP FLOOR DEFLECTION (maximum) (Ans.c)

$$V_{b0} = \frac{(1.93)(1)(2)(5.68)^2}{\left[(1-0.9^2)^2 + [(2)(0.05)(0.9)]^2\right]^{1/2}}$$

$\therefore \boxed{V_{b0} = 592.34 \text{ kips}}$ ✓ ⟶ BASE SHEAR (Ans c)

☺ [redacted]

$$\psi = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

2

$$\tilde{m} = m \left[\frac{1^2 + 2^2}{2^2} \right] = 1.25m$$

$$\tilde{k} = \left(\frac{k}{8}\right)\left(\frac{1}{2}\right)^2 + k\left(\frac{1}{2}\right)^2 = 0.281k$$

$$\omega_{n_2} = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \sqrt{\frac{0.281k}{1.25m}} = \sqrt{\left(\frac{0.281}{1.25}\right)\left(\frac{24EI}{mh^3}\right)}$$

$$\therefore \boxed{\omega_{n_2} = 2.323\sqrt{\frac{EI}{mh^3}}} \left(> 1.668\sqrt{\frac{EI}{mh^3}} = \omega_{n_1} \right) ✓$$

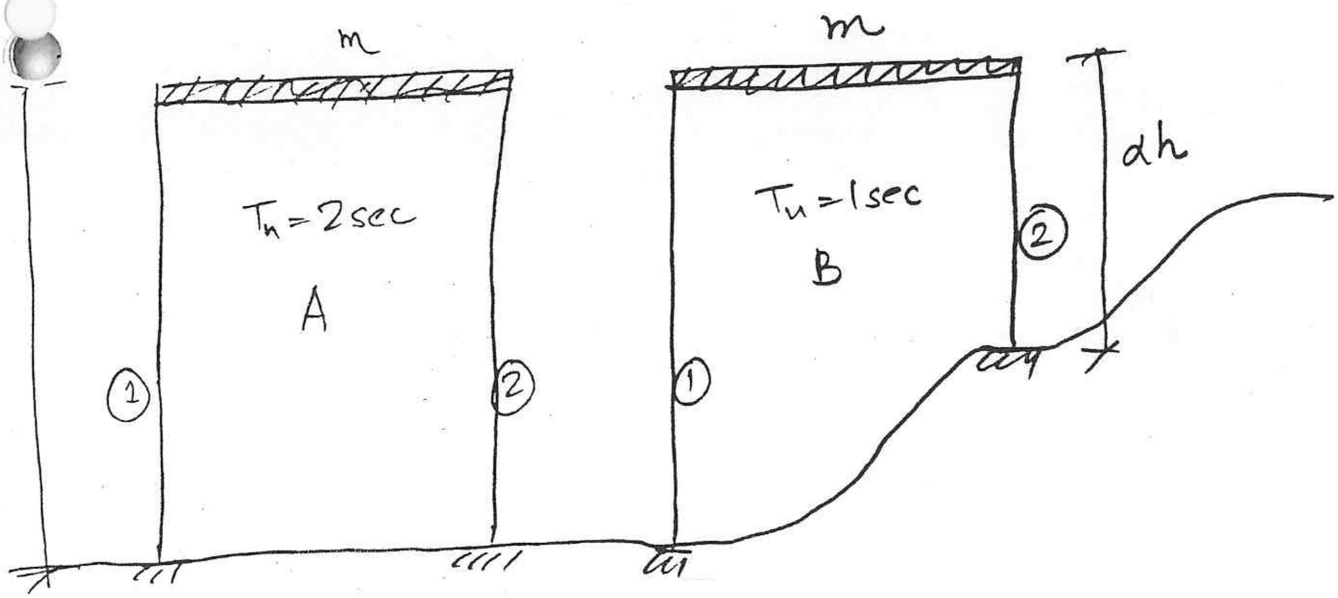
This gives a higher value of ω_n; therefore it is less accurate, Because an approximately calculated value of natural frequency by an assumed shape functions, is always greater than or equal to exact natural frequency. ∴ 2^n shape function does not lead to a more accurate natural frequency. ✓

Also, on casual approximation, we can see that the 1st story is only $\frac{1}{8}$th as stiff as second story. ✓ ∴ Its deflection should be more than half of total deflection. This is not true of the 2nd shape function, therefore it is less accurate.. Very good observation

The shape function should resemble deflected shape of the structure. ✓

= 2 kip·sec²/in $\quad T_{n_A} = 2\,\text{sec}, \; T_{n_B} = 1\,\text{sec}.$

$\zeta_A = \zeta_B = 0.05$



a) Estimate peak ground acceleration:

For very small values of T_n, we know that

$A \approx \ddot{u}_{go}$

From the response spectrum, as $T_n \to 0$, $A \to 0.3g$.

$$\boxed{\ddot{u}_{go} = 0.3g} \quad ✓$$

b) Shear Force in columns:

Structure A:

$T_n = 2\,\text{secs} \Rightarrow$ from fig 3, $\boxed{A = 0.3g}$ ✓

$\therefore A = 0.3 \times 386 = 115.8\ \text{in/s}^2$

now $T_n = 2\pi\sqrt{\dfrac{m}{k}} \Rightarrow 2 = 2\pi\sqrt{\dfrac{2}{k}} \Rightarrow k = \dfrac{(4\pi^2)(2)}{(2)^2}$

$\Rightarrow$ K = [illegible] kip/in

$$\omega_n = \frac{2\pi}{T_n} = \frac{2\pi}{2} = \boxed{3.14 \text{ rad/sec.}}$$

$$\boxed{\therefore D = \frac{A}{\omega_n^2} = \frac{115.8}{3.14^2} = 11.75 \text{ in } (\leftrightarrow) = u_{oA}}$$

$$V_{bo} = mA = 115.8 \times 2 \text{ kips} \left(\text{also } V_{bo} = ku_o\right)$$

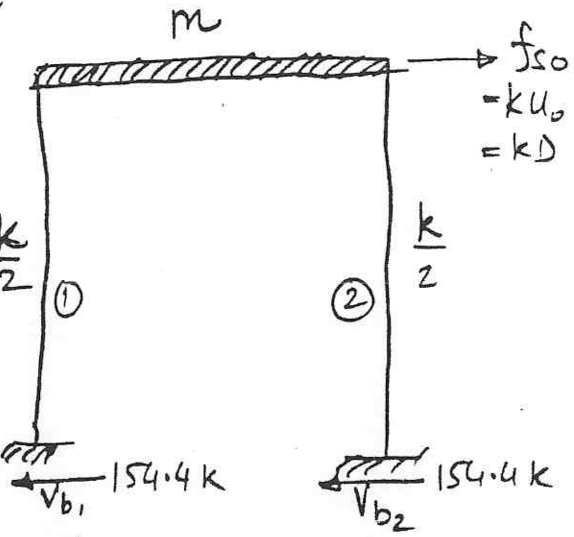
$$\boxed{\therefore V_{bo} = 231.6 \text{ kips}} \longrightarrow (\text{Ans b(i)})$$

✓

Now, for structure A, both columns have equal stiffness. Each takes half of the total base shear

✓

$$\boxed{\therefore V_{b_1} = V_{b_2} = \frac{V_{bo}}{2} = 115.8 \text{ kips}} \longrightarrow (\text{Ans b(ii)})$$



FREE BODY DIAGRAM

te:-
column, $\frac{12EI}{h^3} = \frac{k}{2} = 9.87$ k/in (needed later)

ucture B:

$T_n = 1$ sec $\Rightarrow$ from fig. 3 $A = 0.4g = 154.4$ in/s².

$$\omega_n = \frac{2\pi}{T_n} = 6.28 \text{ rad/sec} \qquad D = \frac{A}{\omega_n^2} = \frac{154.4}{6.28^2} = 3.91''$$

lso $\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_n^2 = (2)(6.28)^2$

$$\boxed{k = 78.9 \text{ kips/in}}$$

$$\therefore \boxed{u_{oB} = D = 3.91''} \quad ✓$$

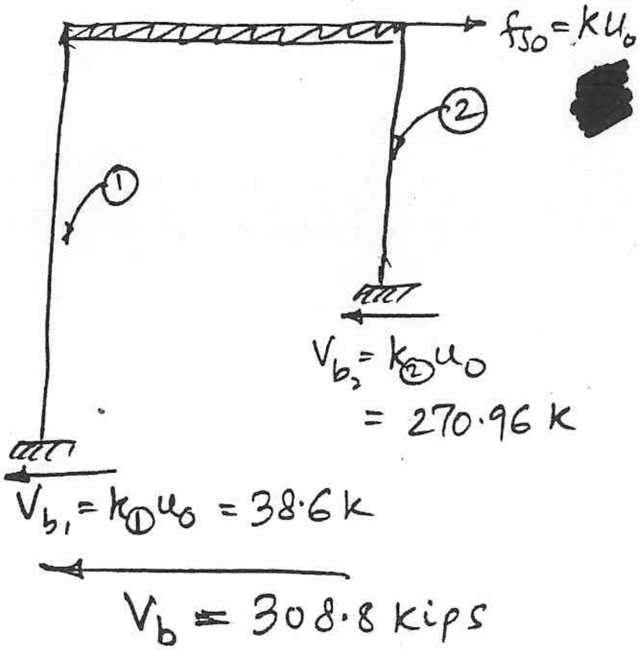
w for base shear

$$V_{bo} = mA = 2 \times 154.4$$

$$\boxed{V_{bo} = 308.8 \text{ kips}} \quad ✓$$

ation of force in each column:

nis is slightly more complicated
his case.
The total base shear is shared by
he column in proportion to their
stiffnesses.



[redacted]

ow $k_{①} = \dfrac{12EI}{h^3} = \boxed{9.87 \text{ k/in}}$

mn ①

✓

r entire structure

$k = 78.9$ k/u

$\therefore k_{②} = 78.9 - 9.87$

$\boxed{k_{②} = 69.03 \text{ k/in}}$ ✓

$V_{b_1} = k_{①} u_0$

$= 9.87 \times 3.91$

$\boxed{\therefore V_{b_1} = 38.6 \text{ kips}}$ ✓

$V_{b_2} = k_{②} u_0$ ✓

$= (69.03)(3.91)$

$\boxed{V_{b_2} = 270.96 \text{ k}}$

heck:

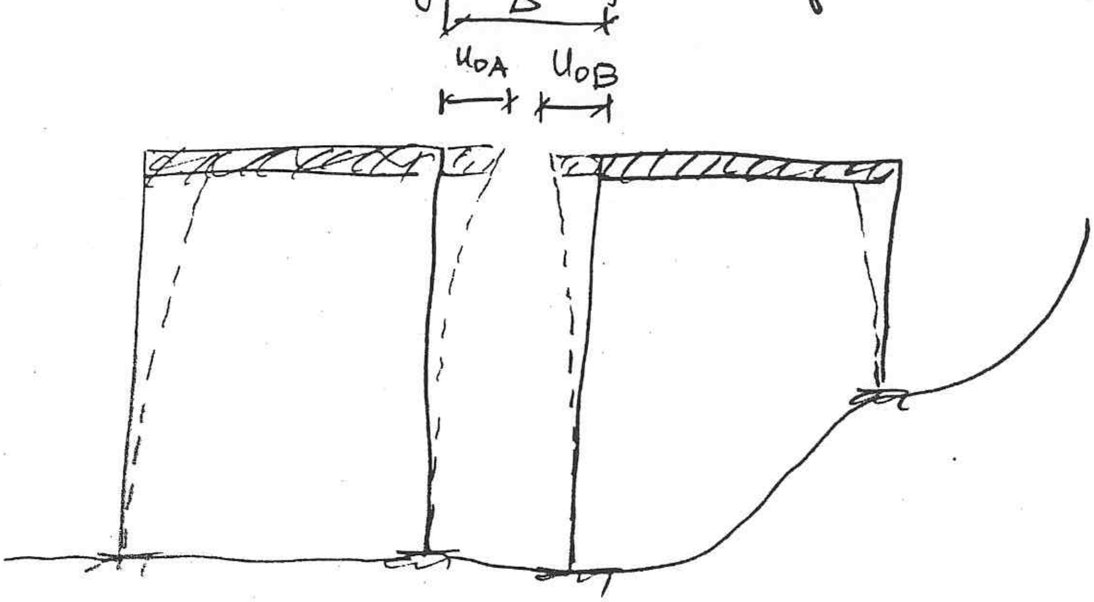
$V_{b_1} + V_{b_2} = 38.6 + 270.96$ ✓

$= 309.6 \;(\approx V_{b0} = 308.8 \text{ k})$ $\therefore$ OK good!

ote:

inspection, we can see that
olumn ① is a lot <u>less</u> stiffer, and will therefore pick up
force (shear) ✓

Consider the following arrangement of the structures:



the buildings not to hit each other during an rthquake,

$$\Delta \geq u_{OA} + u_{OB} \checkmark$$

$$\therefore \Delta_{min} = u_{OA} + u_{OB}$$

$$= 11.75 + 3.91$$

$$\boxed{\therefore \Delta_{min} = 15.66''}$$

would recommend to provide spacing $\Delta = 20''$ at least.

Now, the stiffness of the second column of structure B

$$k_{②} = \frac{24EI}{(\alpha h)^3}$$

$$= \frac{24EI}{\alpha^3 h^3}$$ ✓

But we know $k_{②} = 69.03$ k/in, & $\frac{24EI}{h^3} = 9.87$ k/in

$$\therefore \quad 69.03 = \frac{9.87}{\alpha^3}$$

$$\therefore \quad \alpha^3 = \frac{9.87}{69.03}$$ ✓

$$\boxed{\therefore \alpha = 0.523}$$ ✓

) for structure B, $T_n = 1$ sec.

Central Difference Method is stable if $\frac{\Delta t}{T_n} \leq 0.318$

& ∴ to analyse structure B, we would need a

$$\frac{\Delta t}{T_n} \leq 0.318$$

$$\Rightarrow \frac{\Delta t}{1} \leq 0.318$$

$$\boxed{\Delta t \leq 0.318}$$ ✓

(Central Diff method is prone to P.E. errors.)

I would recommend a value of $\Delta t = 0.1$s for reasonable convergence

[illegible] to further reduce errors due to period

(contd.)

ewmark's Method is stable if

$$\frac{\Delta t}{T_n} \le \frac{1}{\pi\sqrt{2}}\frac{1}{\sqrt{\gamma-2\beta}}$$

$$\therefore \frac{\Delta t}{1} \le \frac{0.225}{\sqrt{\gamma-2\beta}}$$ ✓

given $\gamma = 1/2,\ \beta = 5/24$

$$\boxed{\therefore \Delta t \le 0.78 \text{ secs}}$$ ✓

For reasonable convergence, and to reduce P.E. errors, I would recommend

$$\boxed{\Delta t = 0.1 \text{ sec}}$$ ✓

CATHERINE HOVELL
18 APRIL 2006

CE 384P – Dynamic Response of Structures
Exam No. 2 – April 18, 2006 (2 hours)
Open Text Book PLUS a single 8.5″×11″ page of notes is permitted

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1. The three-story building shown in Fig. 1(a) is to be analyzed as a generalized single-degree-of-freedom system. Neglect axial deformations in all members and flexure in the beams. Use the shape vector, $\underline{\Psi}^T = \{1/3, 2/3, 1\}$. Assume damping is negligible. Given: E =30,000 ksi, I = 2.4 in^4, h = 10 ft, and m = 1.0856×10^{-2} kip-s^2/in. A time-varying load is applied at the second floor level (see Fig. 1(b)).
   (a) Determine the peak displacement of the second floor.
   (b) Determine the peak base shear and the peak shear in each column of the structure on all floors.
   (c) Determine the peak base overturning moment.

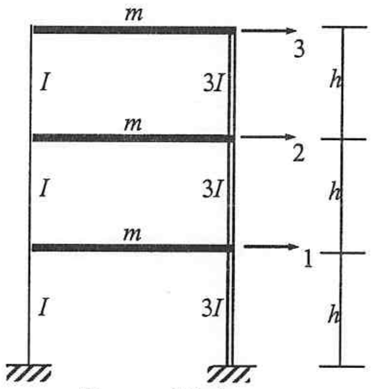

Figure 1(a)

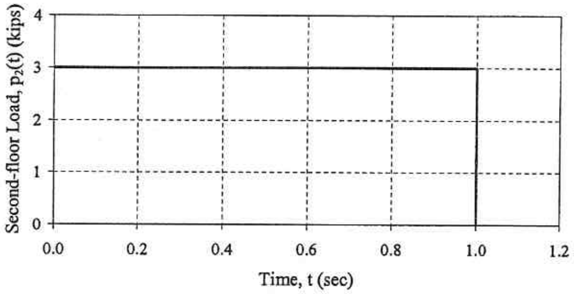

Figure 1(b)

   (d) If an impulse of magnitude 3 kip-sec is applied instead of the load shown in Fig. 1(b), what is the peak displacement of the second floor? What is the peak shear in the second-story columns?

   (e) A different shape vector, $\underline{\Psi}^T = \{1/3, 1/2, 1\}$, is tried next. Is the resulting frequency closer to the unknown exact fundamental frequency of the building? Why?

[25 points]

1 | ~~19~~ 20
2 | 23
| ~~42~~ 43
✓

2. Consider two structures located very close to each other so that they both experience a peak ground acceleration of 0.8g and a peak ground displacement of 5 inches in ground shaking during an earthquake.
   Structure A has a stiffness of 10 kip/in and a natural period of 20 seconds.
   Structure B has a natural period of 0.02 seconds.

   (a) Estimate the peak deformation and base shear for Structure A.
   What must the mass of Structure B have been if it experienced the same peak base shear as Structure A?
   Assume 5% damping for both structures, if you need to.

   The same earthquake caused a 5%-damped pseudo-acceleration response of 1.294g for a natural period of 1 second. Structure C, also located close to the other two, has a natural period of 1 second, a stiffness of 3.9478 kip/in, and 5% damping.

   (b) Estimate the peak deformation and base shear of Structure C.

   (c) An increase in damping for each of the three structures is proposed by an engineer. Comment briefly on the effectiveness of such a change.

   (d) Consider Structure C again. Assume zero damping this time. The structure is subjected to free vibration with an initial displacement of 1 inch and a zero initial velocity. Use the minimum amount of calculations possible and estimate the displacement at *t*=0.1 seconds using Newmark's average acceleration method. What is the percent error in your answer? Are there any issues such as stability, convergence, etc. that you need to be concerned about here?

[25 points]

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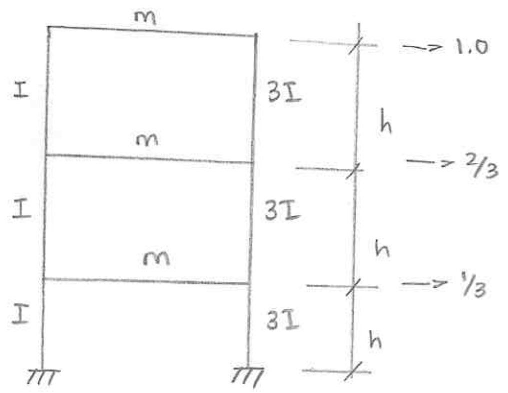
18 APRIL 2006

TEST #2

1.



$$\psi = \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$$

$E = 30{,}000\ \text{ksi}$
$I = 2.4\ \text{in}^4$
$h = 10\ \text{ft}$
$m = 1.0856 \times 10^{-2}\ \text{k}\cdot\text{s}^2/\text{in}$

$$K_{floor} = \frac{12EI}{h^3} + \frac{12E(3I)}{h^3} = \frac{48EI}{h^3} = K \quad ✓$$

$$K = \frac{48(30000\text{ksi})(2.4\text{in}^4)}{(10\text{ ft} \times 12\,\text{in}/\text{ft})^3} = 2.0\ \text{k/in} \quad ✓$$

$u_o = \psi(x)\, z_o$

$$\tilde{m} = \sum m_i \psi_i^2 = m\left[(1/3)^2 + (2/3)^2 + (1)^2\right] = \frac{14}{9}m$$

$$\tilde{k} = \sum k_j (\psi_j - \psi_{j-1})^2 = K\left[(1/3)^2 + (1/3)^2 + (1/3)^2\right] = \frac{k}{3}$$

$$w_n^2 = \frac{\tilde{k}}{\tilde{m}} = \frac{2.0\text{k/in}/3}{\frac{14}{9}(1.0856\ldots)} = 39.48, \quad w_n = 6.28\,\frac{\text{rad}}{\text{s}}\ (\sim 2\pi)$$

$$T_n = \frac{2\pi}{w_n} = 1.0\ \text{s}$$

t_d from step force $= 1.0$ s

$$t_d/T_n = \frac{1.0\text{s}}{1.0\text{s}} = 1.0 \quad ✓$$

using shock spectra from figure 4.10.1 for a step function,

$$t_d/T_n = 1.0, \quad R_d = \frac{z_o}{z_{st_o}} = 2.0 \quad ✓$$

$\tilde{p}_o \neq 3$ $\quad \tilde{p} = \sum P_i \psi_i$ $\quad$ (−3)

$$z_{st_o} = \frac{\tilde{p}_o}{\tilde{k}} = \frac{3.0\text{k}}{2.0\text{k/in}/3} = 4.5\ \text{in}$$

$$z_o = R_d z_{st_o} = (2.0)(4.5\text{in}) = 9.0\text{in}$$

$$u_{o_2} = \psi(2)\, z_o = \frac{2}{3}(9.0\text{in}) = 6.0\text{in}$$

$$\boxed{\text{a. } u_o(\text{2nd floor}) = 6.0\text{ in}}$$ ×

TEST #2

1. (cont'd)

$V_b = \sum k \psi z_o$ × ← incorrect formula

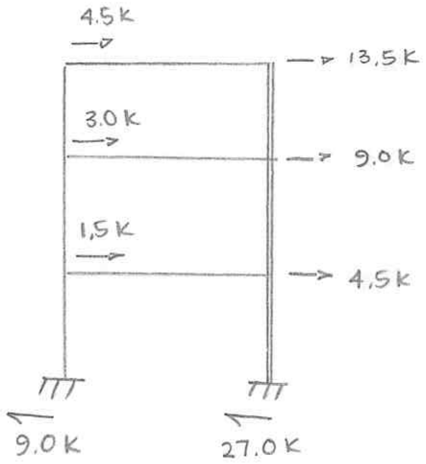
$V_b = k z_o \left[1.0 + 2/3 + 1/3 \right] = (2 k/in)(9 in)(2.0) = 36.0 k$

shear is split between columns,
proportional to stiffness

$k_{left} = \frac{1}{4} k$ ✓

$k_{right} = \frac{3}{4} k$ ✓ stiffer column carries more of the shear

b.



Peak shear values

incorrect formula here.

~~-3~~ (-2)

$M_{bo} = \sum (h_j - h_i) f_{jo}$, $f_{jo} = k_j \psi_j z_o = \begin{bmatrix} 6.0 \\ 12.0 \\ 18.0 \end{bmatrix} k$ → static appln of these forces won't give the dispkls you found.

×

$M_{bo} = (6.0 k) h + 12.0 k (2h) + 18.0 k (3h) = 84.0 h.$ right idea

$\boxed{c.\ M_{bo} = 840 k \cdot ft}$ ×

$k_j (\psi_j - \psi_{j-1}) z_o$

$k z_o (1/3) = \frac{4}{3} k z_o$

$k (6 in)(1/3) = 2k = 4$

$k(6)(1/3) = 4$

4 8 12

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TEST #2

1. (cont'd)

If $\tilde{p}$ were actually an impulse,

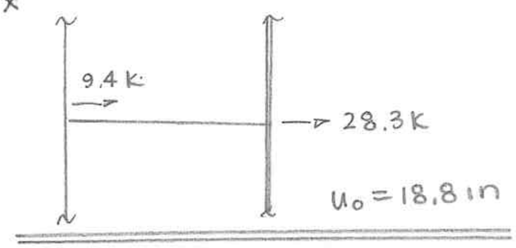
$I = 3\,k\cdot s$ ← needs to change to 2 kip-sec.

$$z_o = \frac{I}{\tilde{m}\,\omega_n} = \frac{\boxed{3.0\,k\cdot s}}{\left(\frac{14}{9}\,m\,\frac{k\cdot s^2}{in}\right)\left(6.28\,rad/s\right)} = 28.27\,in$$

$$u_{oz} = \psi(2)\,(28.3\,in) = \tfrac{2}{3}\,(28.3\,in) = 18.8\,in$$ ✗ right idea

$$\underline{V_{jo} = k\,\psi\,z_o} = 2\,k/in\,(2/3)(28.3\,in) = 37.7\,k$$ ✗

d.



impulse response is over triple the step force response

$$\psi = \begin{bmatrix} 1/3 \\ 1/2 \\ 1 \end{bmatrix}$$

$$\tilde{m} = m\left[(1/3)^2 + (1/2)^2 + (1)^2\right] = \frac{49}{36}\,m$$ ✓

$$\tilde{k} = k\left[(1/3)^2 + (1/6)^2 + (1/2)^2\right] = \frac{7}{18}\,k$$

$$\omega_n^2 = \frac{7/18\,(2.0\,k/in)}{\frac{49}{36}\left(1.0856\times10^{-2}\,\frac{k\cdot s^2}{in}\right)} = 52.6\,rad^2/s^2$$ ✓

$$\omega_n = 7.25\,rad/s > \omega_n(\psi_1)$$ ✓

e. The resulting frequency is larger, and thus less accurate. The first ψ matrix was better, most likely because the distribution was uniform (+1/3 with each floor). Since k and m are constant, there's no reason for the shape function not to be linear. ✓ can't say this

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TEST #2

2. $\ddot{u}_{go} = 0.8g$

$u_{go} = 5 \text{ in}$

$k_A = 10 \text{ k/in}$, $T_{n_A} = 20 \text{ s}$

$T_{n_B} = 0.02 \text{ s}$

$$m_A = \frac{k T_n^2}{(2\pi)^2} = \frac{10 \text{ k/in } (20 \text{ s})^2}{(2\pi)^2} = 101.3 \text{ k} \cdot \text{s}^2/\text{in}$$

peak deformation for long periods, $T_n > 15 \text{ s}$,

$$D \sim u_{go} \quad , \quad D = 5 \text{ in}$$

$$A = \omega_n^2 D = \left(\frac{2\pi}{T_n}\right)^2 (5 \text{ in}) = 0.493 \text{ in/s} = 0.0013g$$

$$V_{bo} = mA = kD = (10 \text{ k/in})(5 \text{ in}) = 50 \text{ k}$$

a. Structure A

$u_o = 5 \text{ in}$ ✓

$V_{bo} = 50 \text{ k}$ ✓

$V_{bo_2} = 50 \text{ k} = m \omega_n^2 D$

For systems with small T_n ($< 0.035 \text{ s}$),

$A = \ddot{u}_{go}$

$50 \text{ k} = mA = m\ddot{u}_{go}$

$$m = \frac{50 \text{ k}}{0.8g} = 0.1619 \text{ k} \cdot \text{s}^2/\text{in}$$

a. Structure B

$m = 0.16 \text{ k} \cdot \text{s}^2/\text{in}$

✓

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TEST #2

2. (cont'd)

$\zeta = 5\%$, $A = 1.294g$, $T_n = 1.0s$, $k = 3.9478\, k/in$

$$D = A\left(\frac{T_n}{2\pi}\right)^2 = 1.294g\left(\frac{1.0s}{2\pi}\right)^2 \quad , \; g = 386\, in/s^2$$

$D = 12.65$ in $= u_o$, peak deformation

$V_{bo} = kD = (3.9478\, k/in)(12.65\, in) = 49.95\, k$

$$\boxed{\begin{array}{l} \text{b. } D = 12.65\text{ in} \; ✓ \\ V_{bo} = 49.95\text{ K} \; ✓ \\ \text{for Structure C} \end{array}}$$

c. Increasing the damping would be most effective in Structure C ($T_n = 1.0$ s) and least effective in structure A ($T_n = 20$ s). Considering Fig. 6.9.7, increasing damping reduces A, V, and D. At high and low periods, the effect is much smaller. ✓

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TEST #2

2.(cont'd) $\zeta = 0$, $T_n = 1.0\ s$, $k = 3.9478\ k/in$

$u_o = 1\ in$, $\dot{u}_o = 0\ in/s$

For stability,

$\frac{\Delta t}{T_n} \leq \infty$, for $T_n = 1.0\ s$, Δt can be anything
(the smaller, the more accurate)

$\boxed{-2}$

$\gamma = 1/2$, $\beta = 1/4$, $\Delta t = 0.02\ s$ — chosen small for accuracy, large for ease of calculations

math error

initial calcs:

$$\ddot{u}_o = \frac{p_o - c\dot{u}_o - ku_o}{m} \qquad m = \frac{k}{\omega_n^2} = \frac{kT_n^2}{(2\pi)^2} = 0.01\ k \cdot s^2/in\ ✓ \times \quad 0.1\ k\text{-}s^2/in$$

$$\ddot{u}_o = \frac{-3.9478\ k/in\ (1.0\ in)}{0.01\ k \cdot s^2/in} = -39.48\ in/s^2\ ✓$$

$\Delta t = 0.02\ s \rightarrow$ not necessary to be < 0.1 sec

$$\hat{k} = k + \frac{\gamma}{\beta \cancel{\Delta t}}\cancel{c} + \frac{m}{\beta(\Delta t)^2} = 3.9478\ k/in + \frac{0.01\ k \cdot s^2/in}{1/4\ (0.02\ s)^2} = 103.9478\ k/in$$

$$a = \frac{m}{\beta \Delta t} + \frac{\gamma}{\beta}\cancel{c} = \frac{0.01\ k \cdot s^2/in}{1/4(0.02\ s)} = 2\ k/in \quad /$$

$$b = \frac{m}{2\beta} = \frac{0.01\ k \cdot s^2/in}{2(1/4)} = 0.02\ k \cdot s^2/in \quad /$$

only need to compute one time step using $\Delta t = 0.1$

t_i	p_i	$\ddot{u}_i$	Δp_i	$\Delta\hat{p}_i$	Δu_i	$\Delta\dot{u}_i$	$\Delta\ddot{u}_i$	$\dot{u}_i$	u_i
0.0	0	-39.48	0	-0.7896	-0.007596	-0.7596	3.0	0	1.0
0.02		-36.48		-2.2488	-0.02163	-0.6438	8.58	-0.7596	0.992404
0.04		-27.9		-3.3648	-0.03237	-0.4302	12.78	-1.4034	0.97077
0.06		-15.12		-3.9696	-0.03819	-0.1516	15.06	-1.8336	0.9384
0.08		-0.06		-3.9716	-0.0382	0.1496	15.16	-1.9852	0.90021
0.10		15.22						-1.8356	0.86201

free vibration

$$\Delta\hat{p}_i = \Delta p_i + a\dot{u}_i + b\ddot{u}_i$$

$$\Delta u_i = \frac{\Delta\hat{p}_i}{\hat{k}}$$

$$\Delta\dot{u}_i = \frac{\gamma\,\Delta u_i}{\beta\Delta t} - \frac{\gamma}{\beta}\dot{u}_i + \Delta t\left(1 - \frac{\cancel{\gamma}}{2\beta}\right)\ddot{u}_i$$

always 0

$$\Delta\ddot{u}_i = \frac{\Delta u_i}{\beta(\Delta t)^2} - \frac{\dot{u}_i}{\beta\Delta t} - \frac{\ddot{u}_i}{2\beta}$$

$u_{i+1} = u_i + \Delta u_i$

$\dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i$

$\ddot{u}_{i+1} = \ddot{u}_i + \Delta\ddot{u}_i$

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TEST #2

2. (cont'd)

d. $u_i = 0.862$ in in five steps ($\Delta t = 0.02$ s), Stability is not a problem with Newmark's average acceleration method, as the equation simplifies to $\Delta t / T_n \leq \infty$. Inaccuracies result with very large timesteps, however; using $\Delta t = 0.01$ s would yield a better result. As $\Delta t / T_n$ increases, period elongation will cause the answer to diverge from accurate (see Fig. 5.5.2 (c)). ✓

Exact sol'n, free vibration:

$u(t) = A\cos\omega_n t + B\sin\omega_n t$

$u'(t) = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t$

using $u(0) = 1$in, $\dot{u}(0) = 0$:

$B = 0$, $A = 1$

$u(t) = \cos\omega_n t$, $\omega_n = \frac{2\pi}{1s} = 2\pi$

$u(0.1s) = \cos(2\pi(0.1s)) = 0.809$ in ✓

Error = 6.5% ← actually error is a lot smaller. you had an error in m

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INTRODUCTION

overview

http://www.ce.utexas.edu/prof/manuel/spring2006_CE384P/home.htm

electronic reserve page: strudyn

http://reserves.lib.utexas.edu/eres/coursepage.aspx?cid=3261

(or, use link from syllabus online)

why study dynamics?

- excitations / loads are time-varying

previously, $ku = P$

scalar equations (or vector), solve for deflections

Now, ~~$ku(t) = P(t)$~~

↑ loads, deflections change with time

however. that implies that u is in phase with P

| NEVER HAPPENS.

- quasi-static solution ignores inertia (m) and dissipative (damping) forces (c)
- approach one: take quasi-static $\left[u(t) = \frac{P(t)}{k}\right]$ solution and multiply by a dynamic amplification factor
  - often highly conservative!
  - allows for high laziness
- more accurate approach

$m\ddot{u} + c\dot{u} + ku = P(t)$; result tends to resemble

$$u(t) = u_0 \sin(\omega t - \phi)$$

$u_0 \hookrightarrow \neq \frac{P_0}{k}$ $\phi \hookrightarrow \neq 0$

scattered observations

Resonance condition

$u_0 \rightarrow \infty$, $\omega = \sqrt{\frac{k}{m}}$ avoid that applied frequency

(only in the case of $c = 0.0$)

$\hookrightarrow$ unrealistic total assumption

Quasi-static solution is ok when dynamic amplification is small

$m\ddot{u}, c\dot{u} \ll ku$.

or, with a very rigid system, or very lightweight system

or, when loads are applied very slowly

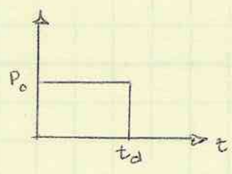
$\omega \rightarrow 0$ $u_0 = \frac{P_0}{k - \omega^2 m}$, becomes $\frac{P_0}{k}$

Natural frequency

$\omega_n = 2\pi\sqrt{m/k}$ most important: $\frac{T}{T_n}$ (natural period)

want/need relative values

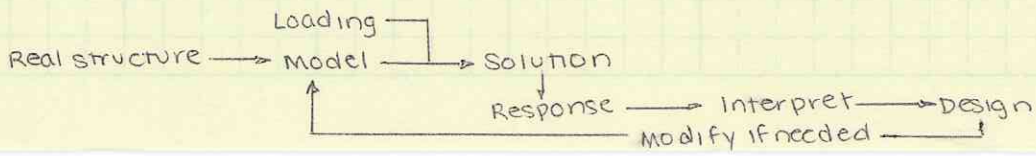
Stepped load



most important: t_d / T_n

matters more than m, c, k of the system

How?



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STRUCTURAL DYNAMICS

considering structures

real structure → model; need m, c, k for SDOF

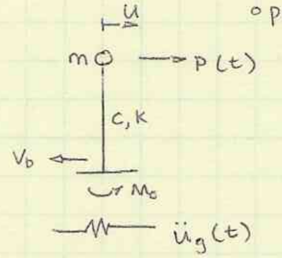
loading → simplified representation

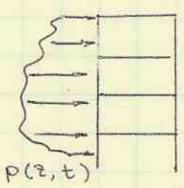
solution methods: classical D.E. solutions
- Duhamel integral
- frequency domain methods

→ response: $u, \dot{u}, \ddot{u}$; V_b, M_o; σ, δ; μ

interpretation of response

- peak response is usually of most interest





ASCE
7 - wind loads

simple structures
- full, empty water tanks
- single story frame with mass at roof

simple → complex

Excitation: Free vibration, Harmonic, Periodic, Step loads (finite duration), Arbitrary (earthquake)

Structure: SDOF m, c, k — generalized SDOF: shape $\psi(x)$; m^*, k^*, c^* related to $\psi(x)$ — MDOF $\underline{M}, \underline{C}, \underline{K}$

Analytical tools: classical sol'n of ODEs — Duhamel integral (time step integration) — modal analysis, response spectrum analysis

superposition

Fast Fourier transform

Interpretation: Time histories
Dynamic magnification factors; phase
Response spectra
Frequency content
Modal contributions

Peak values of
$u, \dot{u}, \ddot{u}$
V_b, M_o
σ, δ
μ

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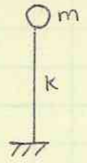
BOOK LEARNIN'

Chapter One

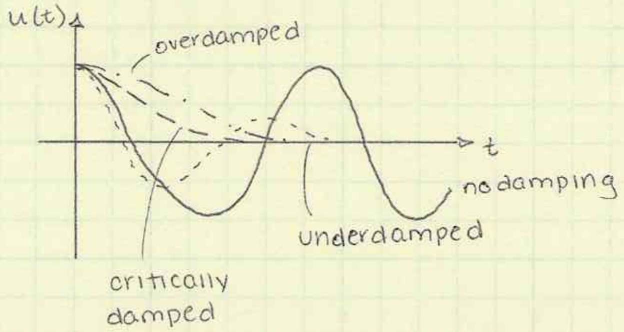
Simple structures – single DOF systems

idealized by a lumped mass (m), supported by a massless structure with stiffness k.



with no damping (c), ~~and~~ once excited, it would oscillate forever – UNREALISTIC.

need to dissipate energy



Ideal	Real
• Mass m at roof level	• all elements contribute to mass (roof, walls, columns)
• k comes from massless frame	• energy dissipation
• viscous damping c	

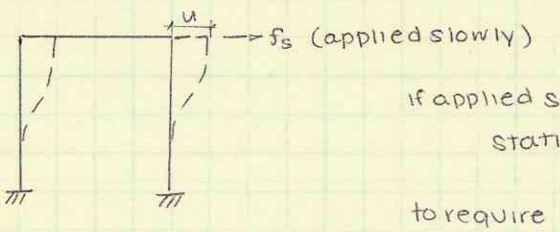
energy dissipation

to heat – concrete and steel

crack opening in concrete

friction

single degree of freedom system



if applied slowly, $f_s \propto u$ (by K)

static displacement – 3 DOFs

to require a dynamic analysis, the response of the structure must waver from that trend

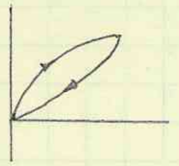
for SDOF systems, all we care about is the horizontal displacement of the mass

No. of independent coordinates required to define the position of all masses relative to their original position = No. of degrees of freedom for a dynamic analysis

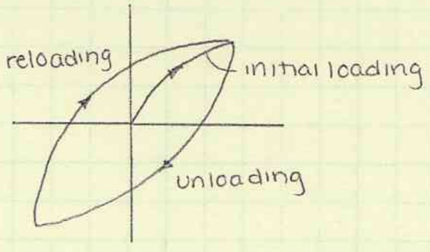
INTRO. TO DYNAMICS

Non-linear behavior

$F \propto u$ is linear; not always the case



non-linear, but still elastic

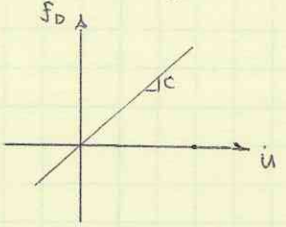


non-linear, inelastic

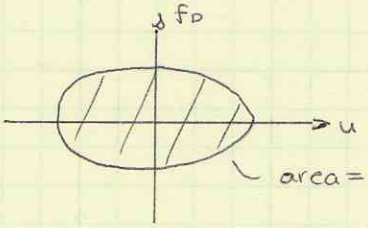
$f_s(u,\dot{u})$ is not a single-valued function of u

strength degradation — doesn't get up as high upon reloading

Viscous Damping



$f_D = c\dot{u}$ (c has units of $F/\dot{u}$)

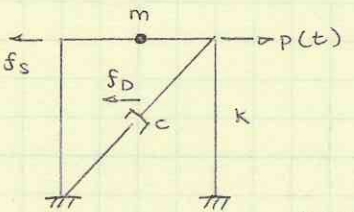


area = energy dissipated through damping

shape could change based on linearity of $\dot{u}, f_D$ relationship

model using an ellipse with the same area enclosed

Idealized structure



$p(t) - f_S - f_D = m\ddot{u}$ (F = ma)

OR, $\underline{m\ddot{u} + c\dot{u} + ku = p(t)}$

Net imbalance of forces causes acceleration

dynamic equilibrium
- uses fictitious inertia force : $m\ddot{u}$
- D'Alembert's principle
- result is the same equation

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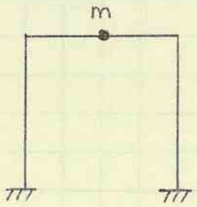
SINGLE DOF

Example Problem



$$\underset{\sim}{K} = \frac{EI}{h^3}\begin{bmatrix} 24 & 6h & 6h \\ 6h & 6h^2 & h^2 \\ 6h & h^2 & 6h^2 \end{bmatrix}$$

written out in detail on handout 2 — downloadable off library site

$$\underset{\sim}{F} = \begin{bmatrix} f_s \\ 0 \\ 0 \end{bmatrix} \qquad \underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\frac{24EI}{h^3} u_1 \cancel{\#} + \frac{6EI}{h^2}(u_2 + u_3) = f_s$$

$$u_2 = u_3 = \frac{-6}{7h} u_1$$

$$f_s = \underbrace{\frac{96}{7}\frac{EI}{h^3}}_{K} u_1$$

note that K is not just $\frac{24EI}{h^3}$

damping, acceleration (inertia) change things

value also (mainly) depends on relative stiffness of beam

column K will vary between 6 and 24 (EI/h^3).

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Example 1.1

Calculate the lateral stiffness for the frame shown in Fig. E1.1a, assuming the elements to be axially rigid.

Solution This structure can be analyzed by any of the standard methods, including moment distribution. Here we use the definition of stiffness influence coefficients to solve the problem.

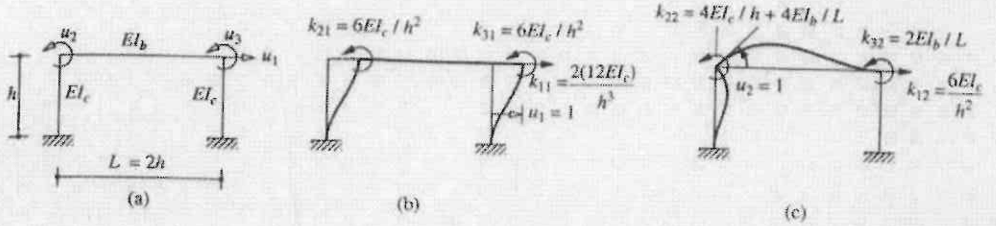


Figure E1.1

The system has the three DOFs shown in Fig. E1.1a. To obtain the first column of the 3×3 stiffness matrix, we impose unit displacement in DOF u_1, with $u_2 = u_3 = 0$. The forces k_{i1} required to maintain this deflected shape are shown in Fig. E1.1b. These are determined using the stiffness coefficients for a uniform flexural element presented in Appendix 1. The elements k_{i2} in the second column of the stiffness matrix are determined by imposing $u_2 = 1$ with $u_1 = u_3 = 0$; see Fig. E1.1c. Similarly, the elements k_{i3} in the third column of the stiffness matrix can be determined by imposing displacements $u_3 = 1$ with $u_1 = u_2 = 0$. Thus the 3×3 stiffness matrix of the structure is known and the equilibrium equations can be written. For a frame with $I_b = I_c$ subjected to lateral force f_S, they are

$$\frac{EI_c}{h^3}\begin{bmatrix} 24 & 6h & 6h \\ 6h & 6h^2 & h^2 \\ 6h & h^2 & 6h^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_S \\ 0 \\ 0 \end{Bmatrix} \qquad \text{(a)}$$

From the second and third equations, the joint rotations can be expressed in terms of lateral displacement as follows:

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = -\begin{bmatrix} 6h^2 & h^2 \\ h^2 & 6h^2 \end{bmatrix}^{-1}\begin{bmatrix} 6h \\ 6h \end{bmatrix} u_1 = -\frac{6}{7h}\begin{bmatrix} 1 \\ 1 \end{bmatrix} u_1 \qquad \text{(b)}$$

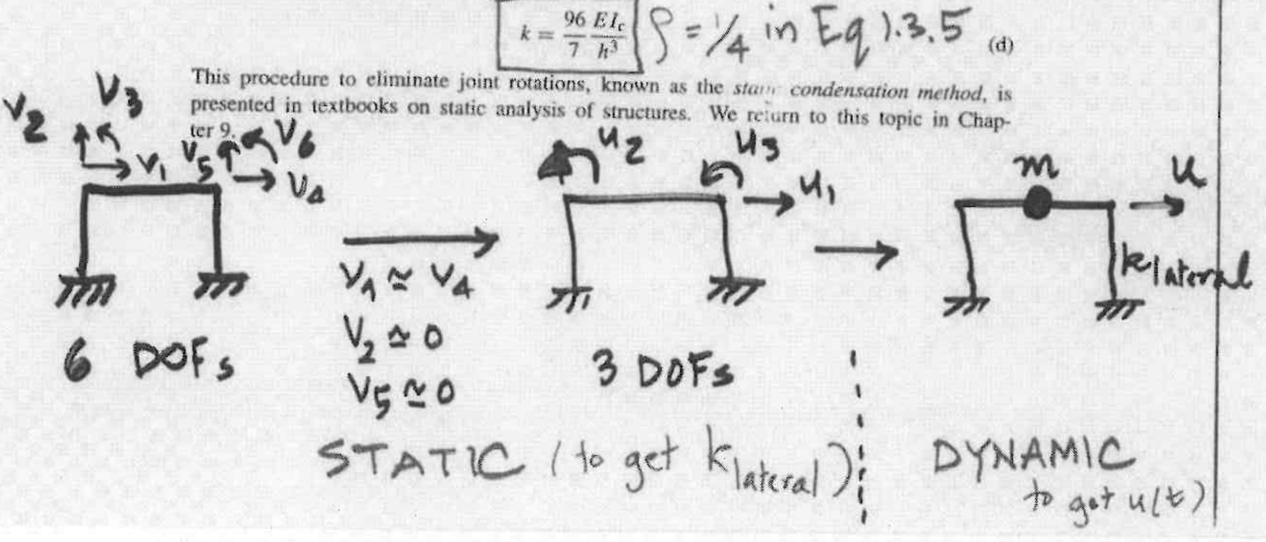
Substituting Eq. (b) into the first of three equations in Eq. (a) gives

$$f_S = \left(\frac{24EI_c}{h^3} - \frac{EI_c}{h^3}\frac{6}{7h}\langle 6h \quad 6h\rangle\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) u_1 = \frac{96}{7}\left(\frac{EI_c}{h^3}\right) u_1 \qquad \text{(c)}$$

Thus the lateral stiffness of the frame is

$$\boxed{k = \frac{96}{7}\frac{EI_c}{h^3}} \quad \rho = 1/4 \text{ in Eq 1.3.5} \qquad \text{(d)}$$

This procedure to eliminate joint rotations, known as the *static condensation method*, is presented in textbooks on static analysis of structures. We return to this topic in Chapter 9.



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More generally,

for $L = 2h$

$$\frac{I_b}{4I_c} = \rho$$

Lateral stiffness of frame, $k = \dfrac{24EI_c}{h^3}\left(\dfrac{12\rho + 1}{12\rho + 4}\right)$

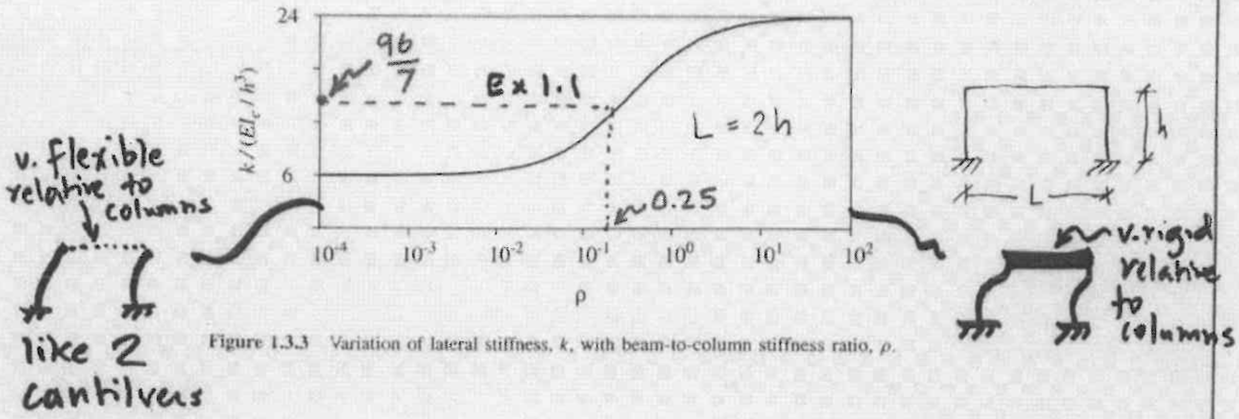


Figure 1.3.3 Variation of lateral stiffness, k, with beam-to-column stiffness ratio, ρ.

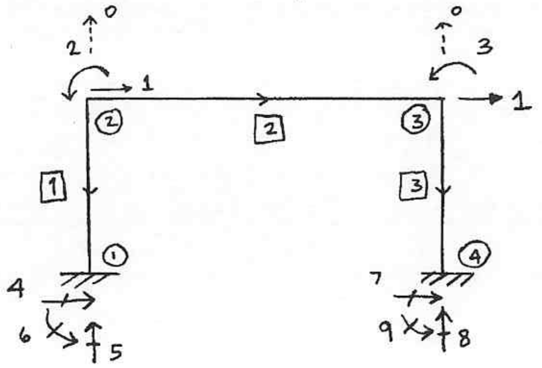
Finally,

if $h = \alpha L$ (i.e., any aspect ratio of frame)

$$\rho = \frac{I_b}{4I_c}$$

Lateral stiffness, $\underline{\underline{k = \dfrac{24EI_c}{h^3}\left(\dfrac{24\alpha\rho + 1}{24\alpha\rho + 4}\right)}}$

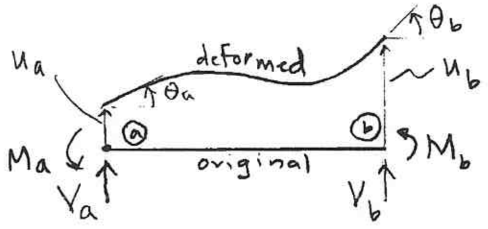
More notes on Example 1.1 & Stiffness Matrices



Define structure with degrees of freedom and restraints indexed by nos. (node nos.)

○ → joint no., □ → member no.

For any beam member (axial deformations neglected),



← Beam member of length, L

$$\begin{Bmatrix} V_a \\ M_a \\ V_b \\ M_b \end{Bmatrix} = \underbrace{\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}} \begin{Bmatrix} u_a \\ \theta_a \\ u_b \\ \theta_b \end{Bmatrix}$$

Member stiffness matrix for a beam member

We need to assemble all the member stiffness matrices using our numbering system in order to obtain the structure stiffness matrix, $\underline{K}$

$\underline{K}$ is of size 3x3 in our example

Members 1 and 3

$$\underline{K}_1 = \underline{K}_3 = \frac{EI}{h^3}\begin{array}{c|cccc|cc}
\boxed{3} & 1 & 3 & 7 & 9 & & \\
\boxed{1} & 1 & 2 & 4 & 6 & & \\
\hline
 & 12 & 6h & -12 & 6h & 1 & 1 \\
 & 6h & 4h^2 & -6h & 2h^2 & 2 & 3 \\
 & -12 & -6h & 12 & -6h & 4 & 7 \\
 & 6h & 2h^2 & -6h & 4h^2 & 6 & 9 \\
\hline
 & & & & & \boxed{1} & \boxed{3}
\end{array}$$

Member 2

$$\underline{K}_2 = \frac{EI}{(2h)^3}\begin{array}{|cccc|c}
0 & 2 & 0 & 3 & \\
\hline
12 & 6(2h) & -12 & 6(2h) & 0 \\
6(2h) & 4(2h)^2 & -6(2h) & 2(2h)^2 & 2 \\
-12 & -6(2h) & 12 & -6(2h) & 0 \\
6(2h) & 2(2h)^2 & -6(2h) & 4(2h)^2 & 3
\end{array}$$

Collect terms with code nos. 1, 2, 3 (our degrees of freedom) For structure $\underline{K}$

$$\boxed{1} \qquad\qquad \boxed{2} \qquad\qquad \boxed{3}$$

$$K_{11} = \frac{12EI}{h^3} + 0 + \frac{12EI}{h^3} = \frac{24EI}{h^3}$$

$$K_{12} = \frac{6EI}{h^2} + 0 + 0 = \frac{6EI}{h^2}$$

$$K_{13} = 0 + 0 + \frac{6EI}{h^2} = \frac{6EI}{h^2}$$

$$K_{22} = \frac{4EI}{h} + \frac{2EI}{h} + 0 = \frac{6EI}{h}$$

$$K_{23} = 0 + \frac{EI}{h} + 0 = \frac{EI}{h}$$

$$K_{33} = 0 + \frac{2EI}{h} + \frac{4EI}{h} = \frac{6EI}{h}$$

other terms of $\underline{K}$ obtained by symmetry.

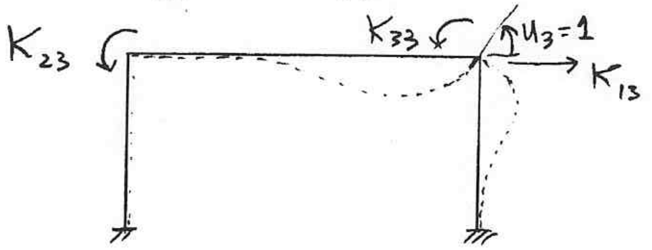
$$\underline{K} = \frac{EI}{h^3}\begin{bmatrix} 24 & 6h & 6h \\ 6h & 6h^2 & h^2 \\ 6h & h^2 & 6h^2 \end{bmatrix}$$

Note: alternative way of obtaining $\underline{K}$ is to generate one column at a time from first principles

K_{ij} = Force in direction of dof i due to unit displacement in direction of dof j (while all other dofs are set to zero)

Use appendix to determine coefficients.

Example to get K_{i3} (3rd column of $\underline{K}$)



$u_3 = 1$
$u_1 = u_2 = 0$

$K_{13} = \frac{6EI}{h^2}$ only member 3 shear

$K_{23} = \frac{2EI}{2h}$ only member 2 moment at left end

$$K_{33} = \underset{\substack{\uparrow \\ \text{moment at} \\ \text{right end} \\ \text{of member 2}}}{\frac{4EI}{2h}} + \underset{\substack{\nwarrow \text{moment at} \\ \text{top of} \\ \text{member 3}}}{\frac{4EI}{h}}$$

The latter approach (deriving structure stiffness matrix from first principles) is more general and more intuitive.

Static Condensation

$\underline{u}_t$ = translation dofs

$\underline{u}_r$ = rotation dofs

$$\underset{\substack{\uparrow \\ \underline{K}\ \text{(structure stiffness matrix}}}{\begin{bmatrix} \underline{K}_{tt} & \underline{K}_{tr} \\ \underline{K}_{rt} & \underline{K}_{rr} \end{bmatrix}} \begin{Bmatrix} \underline{u}_t \\ \underline{u}_r \end{Bmatrix} = \begin{Bmatrix} \underline{f}_s \\ \underline{0} \end{Bmatrix} \quad \text{—(*)}$$

For us, $\underline{u}_t$ is of size 1×1 (scalar)

$\underline{f}_s$ is of size 1×1 (scalar)

Expanding 2nd row of (*),

$$\underline{K}_{rt}\, \underline{u}_t + \underline{K}_{rr}\, \underline{u}_r = \underline{0}$$

$$\Rightarrow \quad \underline{u}_r = -\underline{K}_{rr}^{-1}\, \underline{K}_{rt}\, u_t$$

1st row $\Rightarrow$ $\left[\underline{K}_{tt} - \underline{K}_{tr}\, \underline{K}_{rr}^{-1}\, \underline{K}_{rt}\right] \underset{\leftarrow}{u_t} = \underset{\substack{\uparrow \\ \text{scalar}}}{F_s}$

$$\frac{f_s}{u_t} = \underset{\substack{\uparrow \\ \text{frame} \\ \text{lateral} \\ \text{stiffness}}}{k} = \underset{\substack{\downarrow \\ \text{stiffness} \\ \text{of two} \\ \text{columns}}}{\underline{K}_{tt}} - \underbrace{\underline{K}_{tr}\, K_{rr}^{-1}\, K_{rt}}_{\substack{\uparrow\ \text{reduction from } \frac{24EI}{L^3} \\ \text{for our frame} \\ \text{to account for flexural} \\ \text{rigidity of beam.}}}$$

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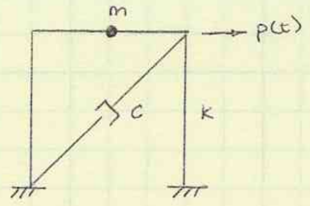
DYNAMICS

Homework

- use mathcad or other program
- make K matrix

Dynamic behavior

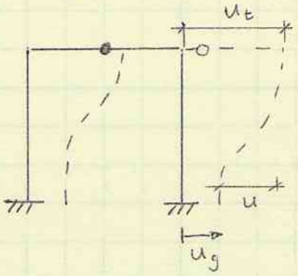


$m\ddot{u} + c\dot{u} + ku = p(t)$

$m\ddot{u} + c\dot{u} + f(\dot{u}, u) = p(t)$ inelastic

Earthquake vibrations

ground excitation, as a function of time



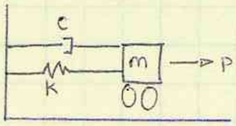
$u^t = u + u_g$

$m\ddot{u}^t = m\ddot{u}_g + m\ddot{u}$

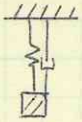
$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g(t)$

earthquake excitation at base

common example



$m\ddot{u} + c\dot{u} + ku = p(t)$



- settles under self-weight, δ_{st}
- displaces under force, u
- total movement, u^t

$m\ddot{u}^t + ku^t = p(t) + mg$

$mg = k\delta_{st}$

$u^t = \delta_{st} + u$

$m\ddot{u} + \cancel{mg} + ku = p(t) + \cancel{mg}$

Structure on handout

E-W direction

$4\left[\frac{12EI_w}{h^3}\right]$ = very small, due to I_w

add stiffness from cross-bracing

$2\left[\frac{AE}{L}\right]\cos^2\theta \rightsquigarrow p = \frac{AE}{L}\delta, \ \delta = u\cos\theta$

$f_s = p\cos\theta$

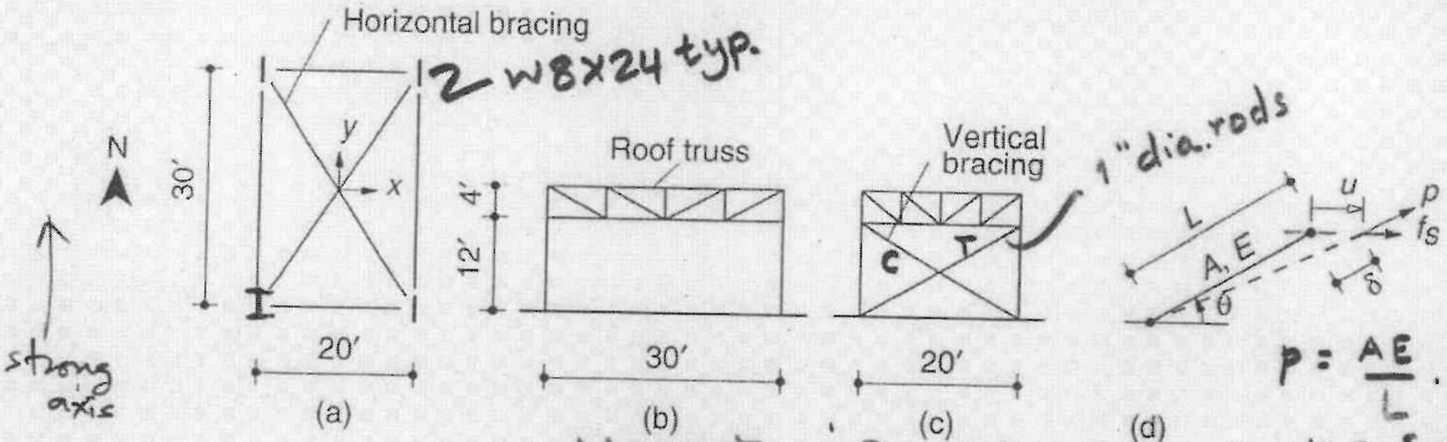
Roof wt: 30lb/ft²

Derive Eqn of Motion
for N-S & E-W directions

Mass: $m = 20 \times 30 \times \frac{30}{386}$

Example 1.2

$\Rightarrow m = .047\ \text{k-s}^2/\text{in}$



Moment frame N-S : Braced Frame E-W

$$p = \frac{AE}{L} \cdot \delta$$

$$u = \delta / \cos\theta$$

$$f_s = p \cos\theta$$

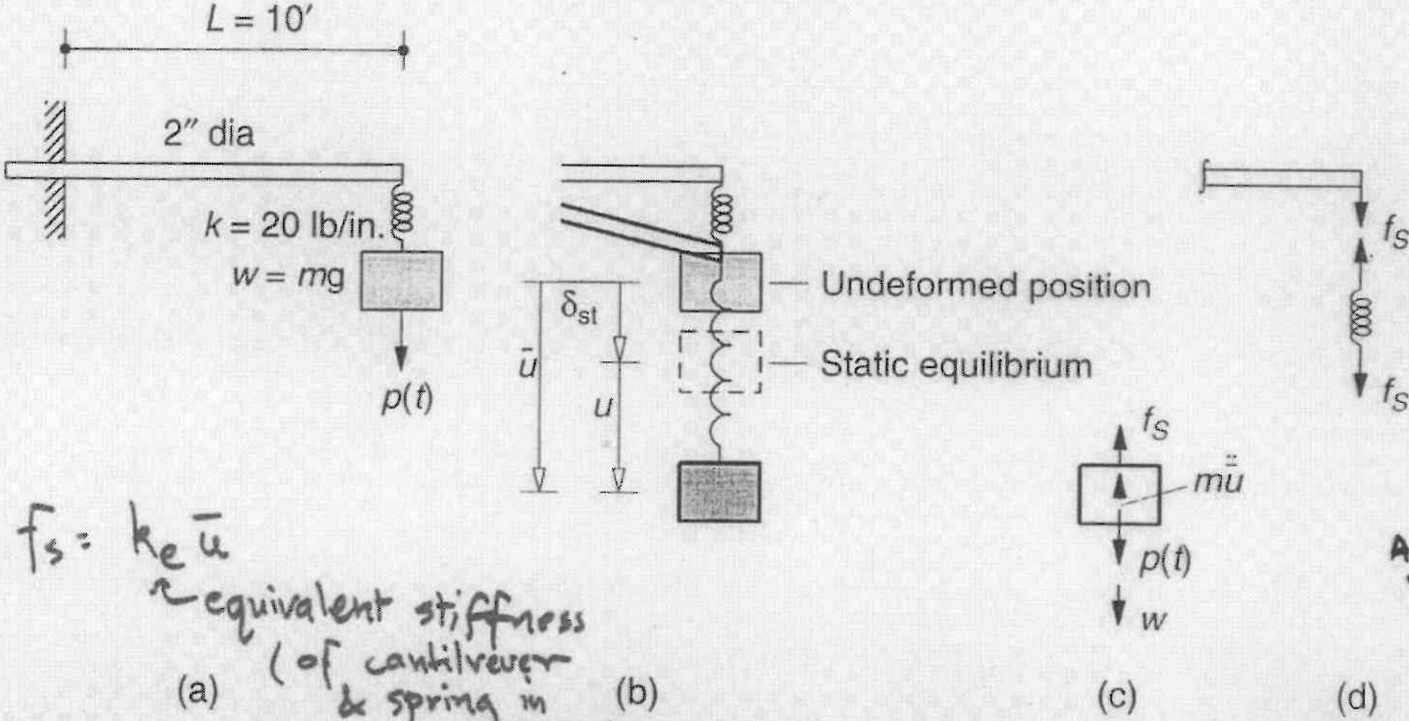
Figure E1.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Stiffness: $4\left(\frac{12EI_x}{h^3}\right)$: $\overset{\text{truss}}{2\left(\frac{AE}{L}\right)\cos^2\theta} + \overset{\text{frame}}{4\left(\frac{12EI_y}{h^3}\right)}$

4 columns : 2 cross-braces in tension ↑ small <10% of E-W frames stiffness.

$$k_{N-S} \ll k_{E-W}$$

Example 1.~~3~~4



Same force f_s in spring AND at end of cantilever

$f_s = k_e \bar{u}$ ← equivalent stiffness (of cantilever & spring in series)

$\bar{u} = \delta_{\text{cantilever}} + \delta_{\text{spring}}$

$$k_e = \frac{k_{\text{cantilever}} \cdot k_{\text{spring}}}{k_{\text{cantilever}} + k_{\text{spring}}}$$

⇒ Series system

Figure E1.~~3~~4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

12a

Example 1.~~4~~ 7

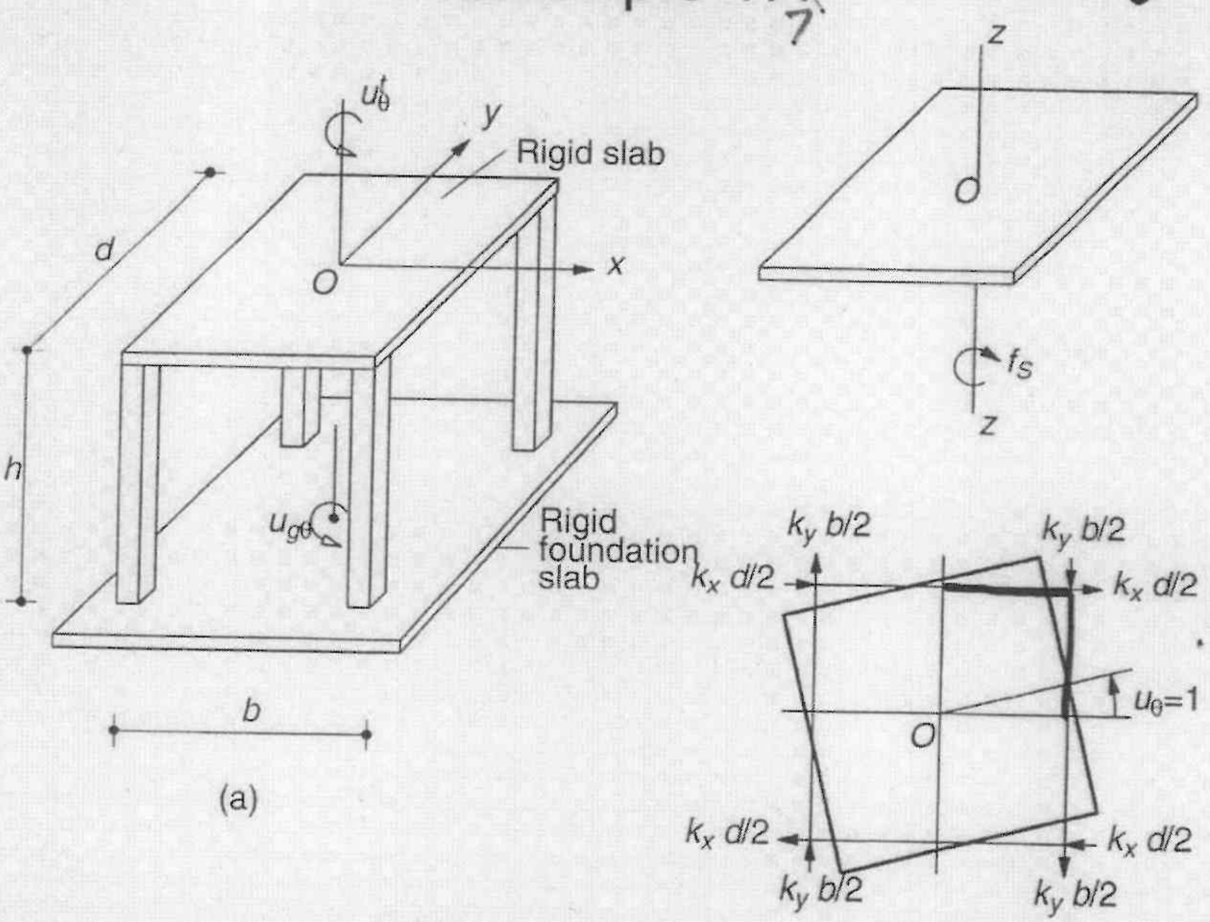
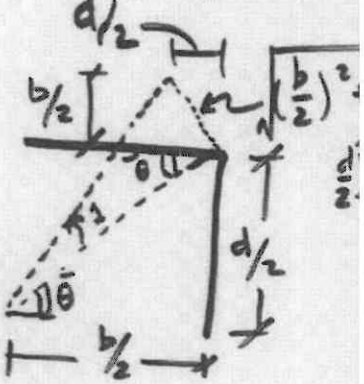


Figure E1.~~4~~ 7 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

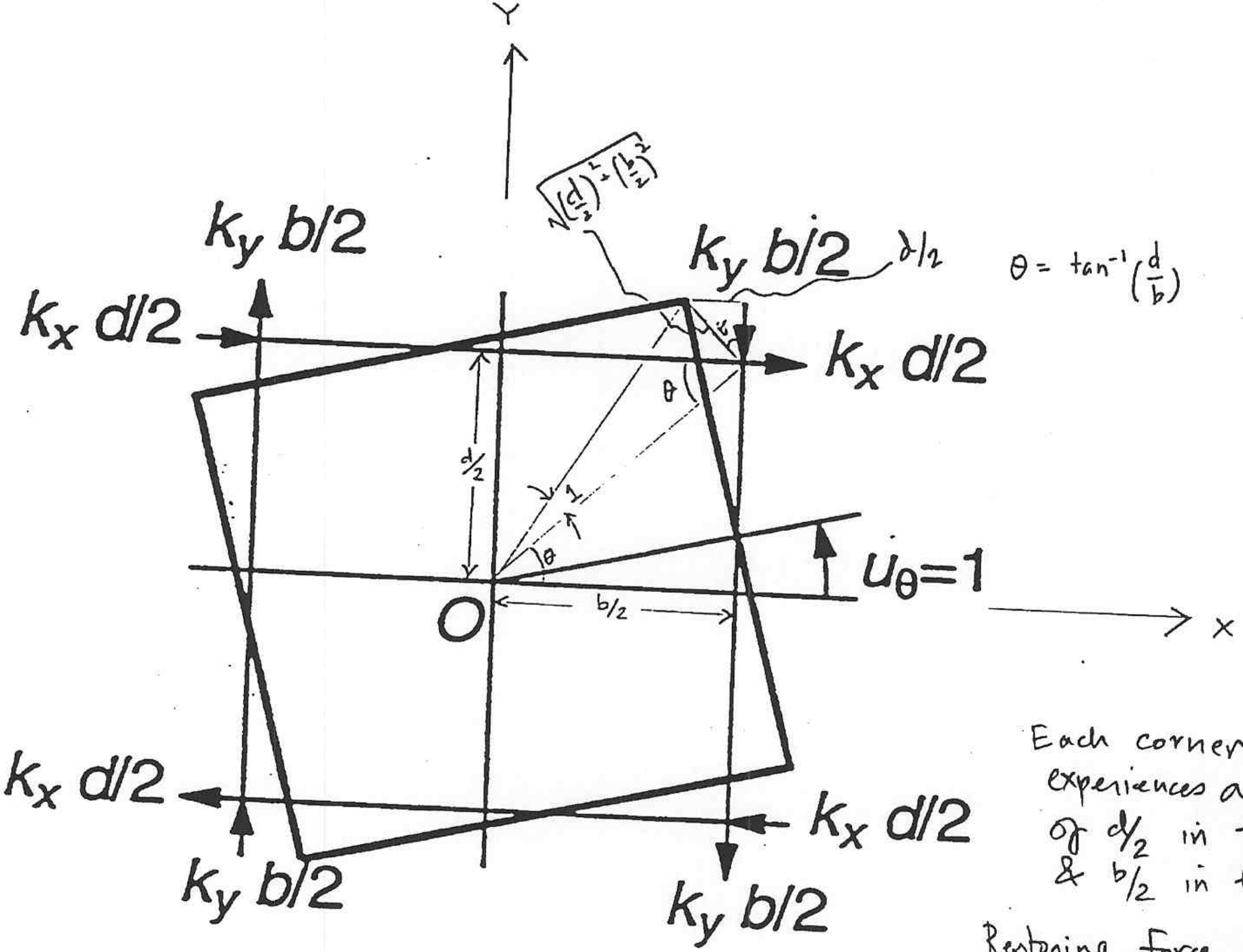
$$u_\theta^t = \underset{\nearrow \text{rotation of foundation}}{u_{g\theta}} + \underset{\nwarrow \text{rotation of slab relative to foundation}}{u_\theta}$$



Restoring Torque on each column

$$= k_y \frac{b}{2} \cdot \frac{b}{2} + k_x \cdot \frac{d}{2} \cdot \frac{d}{2}$$

Multiply by 4 (No. of columns) to get k_θ



$\theta = \tan^{-1}\left(\frac{d}{b}\right)$

Each corner experiences a displacement of $d/2$ in the X direction & $b/2$ in the Y direction.

Restoring force at a corner

= $k_x \frac{d}{2}$ in X direction

= $k_y\ b/2$ in Y direction

Restoring torque for a corner

$$= \left(k_x \frac{d}{2}\right) \times \frac{d}{2} + \left(k_y \frac{b}{2}\right) \times \frac{b}{2}$$

CHAPTER ONE

Steps in Analysis

Solve for $\underline{u}(t)$

(1) work element-by-element

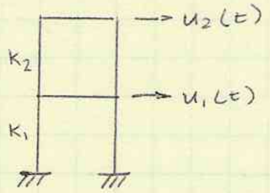
ex. if you know displacements and rotations at member ends, get $V_a, M_a, V_b, M_b \ldots$ using properties of element

(2) get equivalent static forces

$$\underline{f_s} = \underline{k}\,\underline{u}$$

↑ static forces that could produce dynamic $u(t)$.

Example



option 1:

$$V_2(t) = k_2(u_2 - u_1)$$
$$V_1(t) = k_1(u_1)$$

ks are stiffnesses of stories

option 2:

$$\underline{k} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \qquad \underline{f_s} = \left[\underline{k}\right]\underline{u}$$

$$\underline{f_s} = \begin{bmatrix} (k_1 + k_2)u_1 - k_2 u_2 \\ -k_2 u_1 + k_2 u_2 \end{bmatrix}$$

$V_{s2} = f_{s2}$, $V_{s1} = f_{s1} + f_{s2}$ ✓

(check to opt. 1)

inelastic systems

use incremental analysis —

solve for u_i (Δu_i)

$$f_{i+1} = f_i + k_L \Delta u_i$$

restoring force at $t = i+1$

↑ approximation to true value

$$k_L u_i \cong f_s(u_i, \dot{u}_i)$$

$$u_{i+1} = u_i + \Delta u_i$$

* cannot (typically) just add static and dynamic responses (eg, spring hanging)

SOLVING METHODS

Solution methods of the Eq. of motion

(I) classical method

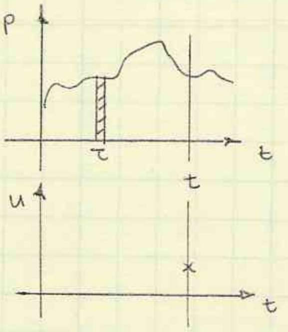
$$u(t) = \underbrace{u_c(t)}_{\substack{\text{complementary}\\ \text{(homogeneous)}\\ p(t)=0}} + \underbrace{u_p(t)}_{\substack{\text{particular}\\ \text{(often a guess)}}}$$

apply initial (boundary) conditions to obtain undetermined coefficients in $u_c(t)$

(II) Duhamel integral

convolution of force, stiffness

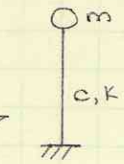
→ response at any time, t, is the ~~response~~ sum of all of the responses of impulses (that make up p(t)) and that precede time t



impulse at $t=\tau$

causes free vibration of structure as does each impulse $\tau<t$

$$u(t) = \int_0^t \underbrace{h(t-\tau)}_{\substack{\text{unit impulse}\\ \text{response function}\\ \text{(depends on system props.)}}} \cdot p(\tau)\,d\tau$$



more on this in chapt. 4

(III) Transform methods

$$\begin{tikzcd} \text{Real} & p(t) \arrow[r, "\times"] \arrow[d] & u(t) \\ \text{complex} & \underline{P}(\omega) \arrow[r] & \bar{U}(\omega) \arrow[u] \end{tikzcd}$$

easier, cheaper to do three steps than one; but, uses complex numbers

$p(t)$ to $\underline{P}(\omega)$ involves Fourier transforms

$\underline{P}$ to $\bar{U}$: $H(\omega)$, transfer function, "filter" ($\underline{P} \times H = \bar{U}$)

$\bar{U}$ to u : inverse transform

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SOLUTION METHODS

Quick review

1. classical method
2. Duhamel integral
3. transform methods

4. Numerical methods (chapt. 5)
   allows nonlinear / elastic characterizations
   most general case

5. State-space methods (not in book)

$m\ddot{u} + c\dot{u} + ku = p(t)$ — scalar ODE of 2nd order

state: $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$ — vector ODE of 1st order

$$\underline{z}(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix}, \quad \dot{\underline{z}}(t) = \underline{A}\,\underline{z}(t) + \underline{F}(t) = \begin{bmatrix} \dot{u}(t) \\ \ddot{u}(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{\underline{A}} \begin{bmatrix} u \\ \dot{u} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{p(t)}{m} \end{bmatrix}}_{\underline{F}(t)}$$

$$\underline{z}(t) = e^{\underline{A}(t-t_0)}\underline{z}(t_0) + e^{\underline{A}t}\int_{t_0}^{t} e^{\underline{A}s}\underline{F}(s)\,ds$$

with $t > t_0$

Simplifying Things

$m\ddot{u} + c\dot{u} + ku = p(t) \longrightarrow m\ddot{u} + ku = 0$ (no damping, forces)

initial conditions: $u(0), \dot{u}(0)$

2nd order ODE, homogeneous, constant coefficients

$$u(t) = e^{st}, \quad \ddot{u}(t) = s^2 e^{st}$$

$$e^{st}(ms^2 + k) = 0 \qquad e^{st} = 0 \text{ trivial solution}$$

$$ms^2 + k = 0 \qquad s = \sqrt{\frac{-k}{m}}$$

$$s_{1,2} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_n$$

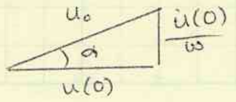
$$u(t) = a_1 e^{i\omega_n t} + a_2 e^{-i\omega_n t}$$

using de Moivre's theorem:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$u(t) = A\cos\omega_n t + B\sin\omega_n t$$

$$u(t) = u(0)\cos\omega t + \frac{\dot{u}(0)}{\omega}\sin\omega t = u_0\left[\underbrace{\frac{u(0)}{u_0}}_{\cos\alpha}\cos\omega t + \underbrace{\frac{\dot{u}(0)/\omega}{u_0}}_{\sin\alpha}\sin\omega t\right]$$



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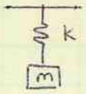
EQUATION DERIVATION

Standard Equations

Period, $T_n = 2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\omega}$



$\delta_{st} = \frac{mg}{k}$, $T_n = 2\pi\left[\delta_{st}/g\right]^{1/2}$

Damping

overdamped $\xi = 2$

underdamped $\xi = 0.1$

critically damped $\xi = 1.0$

Golden Gate periods of vibration

transverse - 18.2 s

vertical - 10.9 s

longitudinal - 3.8 s

Equations of Motion

consider damping (realistic)

$\xi = \frac{c}{2m\omega_n}$ damping ~~coefficient~~ [illegible] ratio

$m\ddot{u} + c\dot{u} + ku = 0$

$\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = 0$

c_{cr} = critical damping coefficient = $2\sqrt{km}$

$$\xi : \begin{array}{ll} <1 & \text{underdamped} \\ =1 & \text{critically} \\ >1 & \text{over damped} \end{array}$$

non-oscillatory

damping ratio = $c/c_{cr} = \xi$

$(s^2 + 2\xi\omega_n s + \omega_n^2)e^{st} = 0$

Free vibration of a system without damping

Amplitude, u_o

$$u_o = \sqrt{u^2(0) + \left(\frac{\dot{u}(0)}{\omega_n}\right)^2}$$

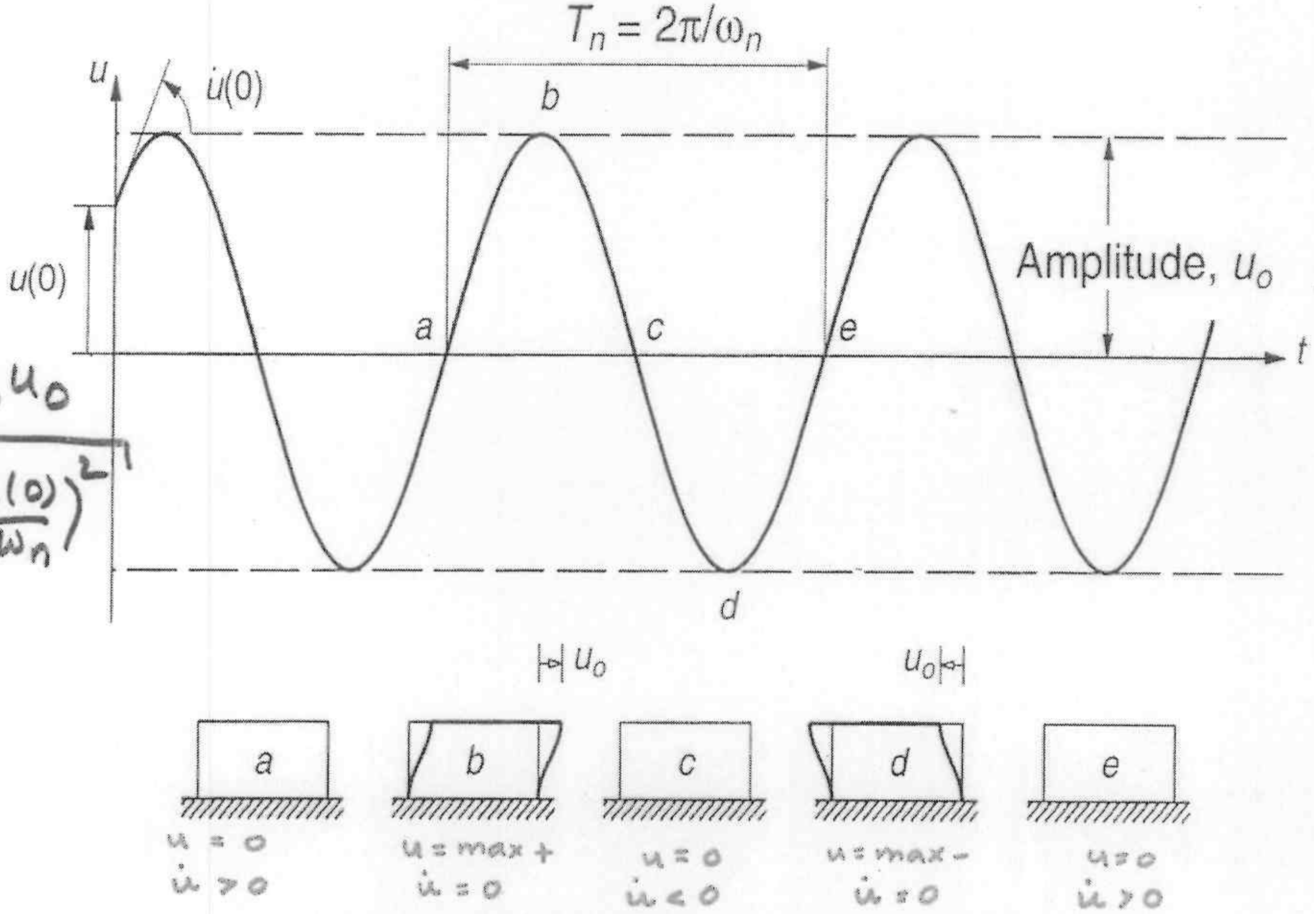
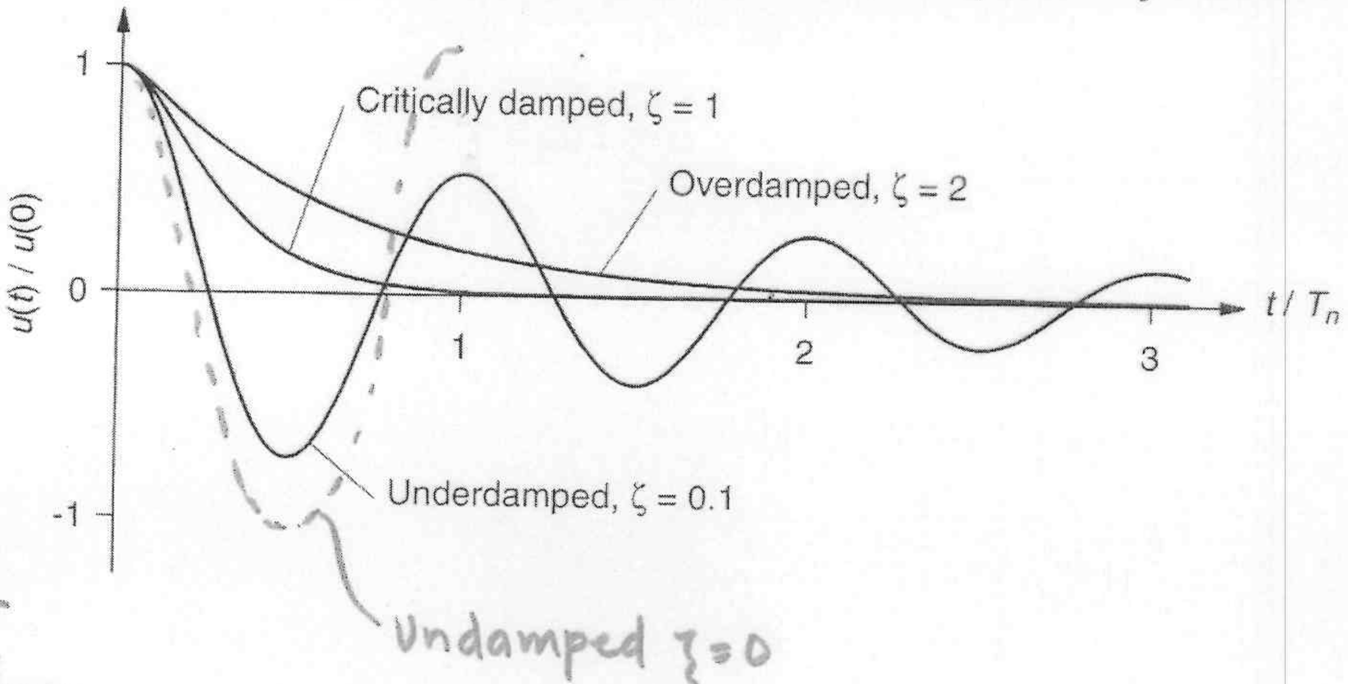


Figure 2.1.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Free vibration of underdamped, critically-damped, and overdamped systems



$\xi = \frac{c}{c_{cr}}$
$= \frac{c}{\sqrt{km}}$

Figure 2.2.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Handout No. 4

List of Interesting Structures and their Natural Periods (from Chopra, 2001)

Structure	*Natural Period (s)*	*How measured?*
Nuclear Power Plants		
Nuclear Power Plant Containment Structure San Onofre, California	0.15 (fixed base) 0.50 (incl. SSI/soil flexibility)	Computational analysis (Fig 1.10.1)
Dams		
Morrow Point Dam (465 ft. arch dam) Gunnison River, Colorado	0.27 (reservoir partially full) 0.30 (reservoir full)	Forced vibration tests (Fig. 1.10.2)
Pine Flat Dam (400 ft. concrete gravity dam) Fresno, California	0.29 (reservoir at 310 ft) 0.31 (reservoir at 345 ft)	Forced vibration tests (Fig. 2.1.2d)
Buildings		
Medical Center Building (3-story steel frame) Richmond, California	0.63 (long direction) 0.74 (short direction) 0.46 (torsional)	During Loma Prieta Earthquake (1989) (Fig. 2.1.2c)
Alcoa Building (26-story steel building) San Francisco, California	1.67 (N-S) 2.21 (E-W) 1.12 (torsional)	Force vibration tests (Fig. 2.1.2a)
Transamerica Building (60-story steel building) San Francisco, California	2.90 (N-S and E-W)	Force vibration tests (Fig. 2.1.2b)
Chimneys		
Reinforced concrete chimney (250 m. high) Aramon, France	3.57	Wind-induced vibration (Fig. 2.1.2f)
Bridges		
Golden Gate Bridge (4,200 ft. main span) Suspension bridge	18.2 (transverse) 10.9 (vertical) 3.8 (longitudinal) 4.4 (torsional)	Ambient conditions (wind, traffic) (Fig. 2.1.2e)

Increasing Period (Decreasing Stiffness Relative to Mass) ↓

FREE VIBRATION

Damping and vibration

$C_{cr} = 2\sqrt{km}$ critically damped

$c = 0$ undamped

$c < C_{cr}$ underdamped] oscillatory

$c > C_{cr}$ overdamped

non oscillatory

$$\zeta = \frac{c}{2m\omega_n} = \frac{c}{C_{cr}}$$

free vibration:

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = 0$$

solution $u = e^{st}$

$$s_{1,2} = \omega_n\left[-\zeta \pm i\sqrt{1-\zeta^2}\right]$$

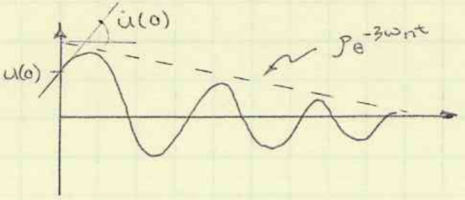
$$\omega_D = \omega_n\sqrt{1-\zeta^2}$$ with no damping, $\zeta = 0$, $\omega_D = \omega_n$

$$u = e^{-\zeta\omega_n t}\left[a\cos\omega_D t + b\sin\omega_D t\right]$$

given $u(0)$ and $\dot{u}(0)$,

$$u = e^{-\zeta\omega_n t}\left[u_0\cos\omega_D t + \frac{\dot{u}(0)+\zeta\omega_n u(0)}{\omega_d}\sin\omega_D t\right]$$ [illegible] ρ

$$u(t) = \rho e^{-\zeta\omega_n t}\cos(\omega_D t - \alpha)$$



$$T_D = \frac{2\pi}{\omega_D} > T_n$$

logarithmic decrement

$$\delta = \ln\frac{u_j}{u_{j+1}} = \ln\left[e^{2\pi\zeta/\sqrt{1-\zeta^2}}\right] = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$\zeta = \frac{0.11}{j}$, where j = the number of cycles to drop the value to half

in critically damped scenario,

$$u(t) = \rho e^{-\omega_n t}\left[u(0) + (\dot{u}(0) + \omega_n \cancel{\zeta} u(0))t\right]$$ doesn't oscillate

Chapter Three

Harmonic force

$$m\ddot{u} + ku = p_0\sin\omega t$$

$u(t) = u_c(t) + u_p(t)$

↳ $A\cos\omega_n t + B\sin\omega_n t$

↳ $c\sin\omega t$

* first two terms oscillate at natural frequency. Third is at forced freq.

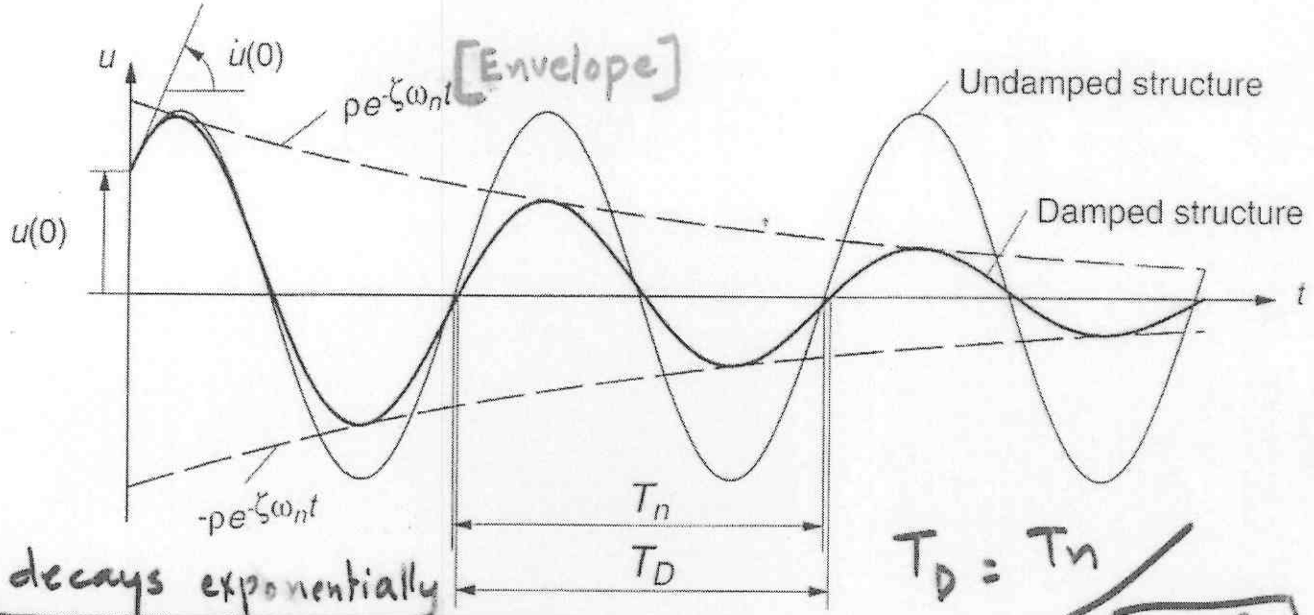
$u(t) = A\cos\omega_n t + B\sin\omega_n t + c\sin\omega t$, with $u(0)$, $\dot{u}(0)$

$(-m\omega^2 + k)c\sin\omega t = p_0\sin\omega t$, so $C = \frac{P_o}{k}\frac{1}{1-(\omega/\omega_n)^2}$ $r = \frac{\omega}{\omega_n}$

$A = u(0)$

~~B=~~ $\dot{u}(0) = \omega_n B + \frac{P_o}{k}\omega\frac{1}{1-r^2}$

Effects of damping on free vibration



Envelope decays exponentially

$$\rho = \sqrt{[u(0)]^2 + \left[\frac{\dot{u}(0) + \zeta\omega_n u(0)}{\omega_D}\right]^2}$$

$$T_D = T_n \Big/ \sqrt{1-\zeta^2}$$

$$T_D > T_n$$

Figure 2.2.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

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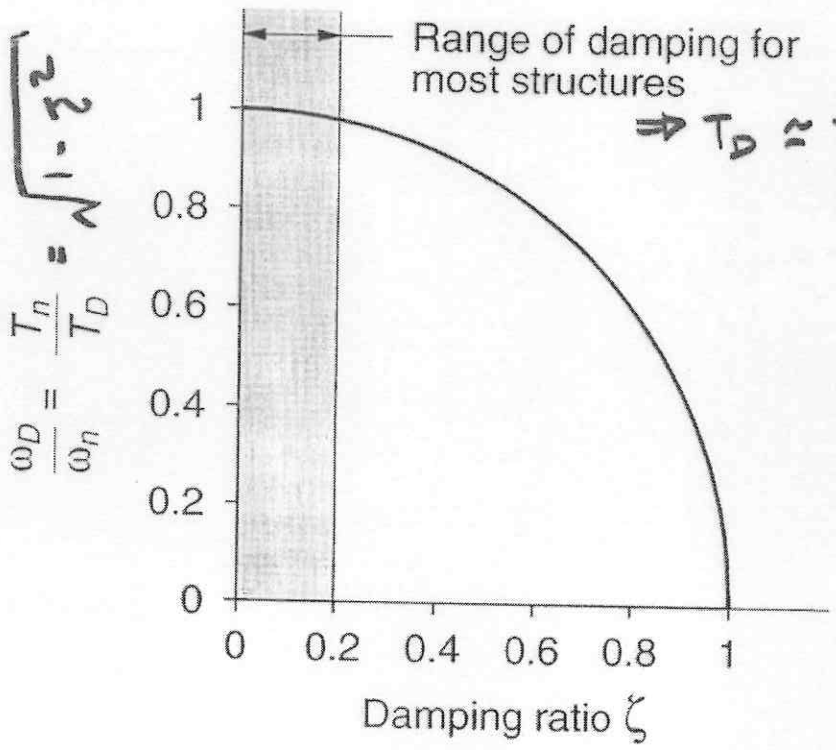
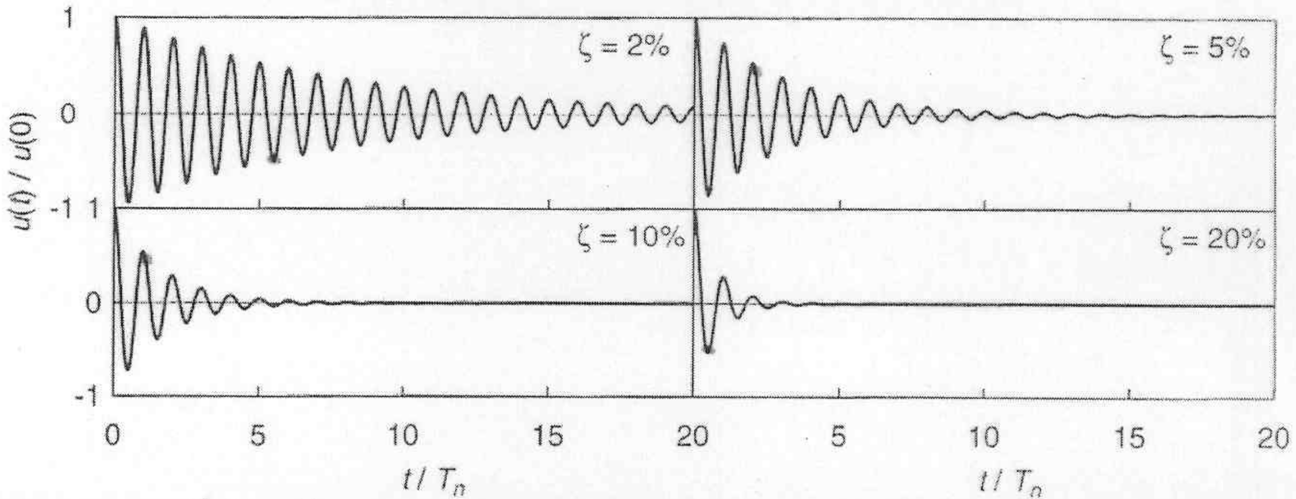
Effects of damping on the natural vibration frequency



$\Rightarrow T_D \approx T_n$ for most structures

Figure 2.2.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Free vibration: ζ = 2,5,10 and 20%



Larger damping (ξ) ⇒ faster decay of FREE VIBRATIONS

Figure 2.2.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Logarithmic decrement versus damping ratio

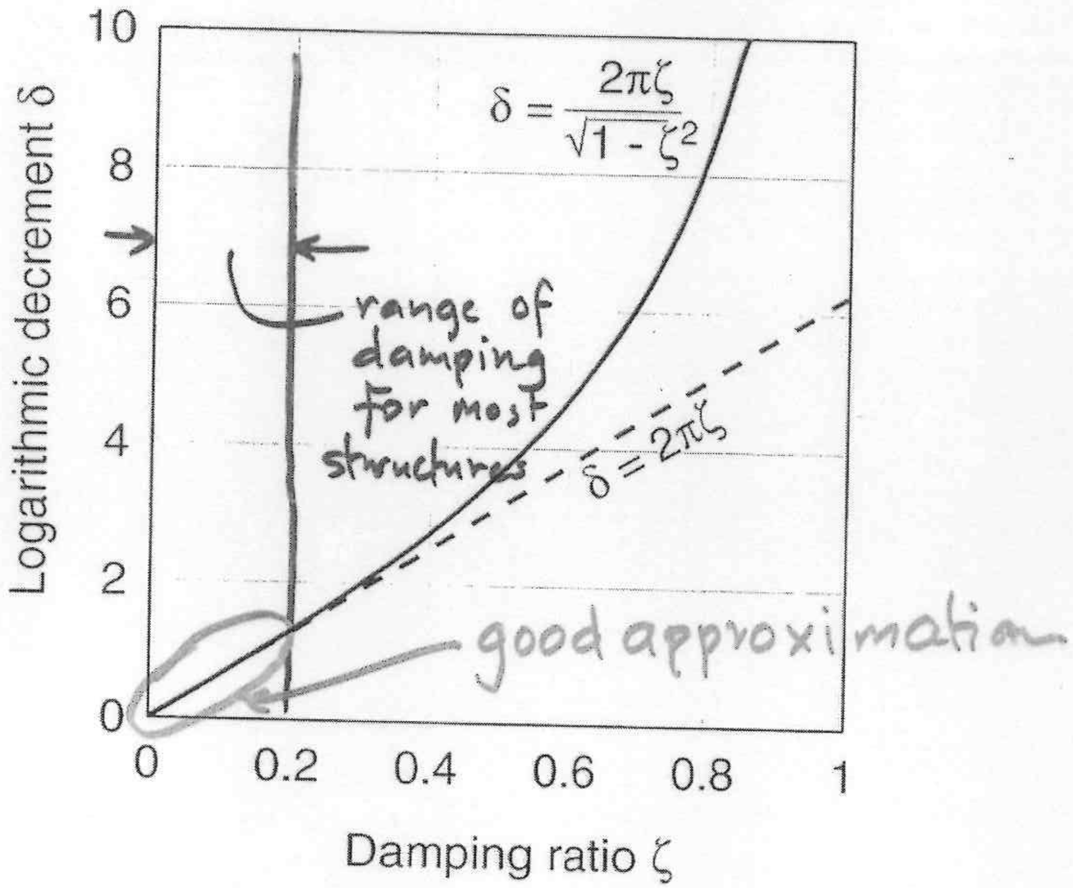
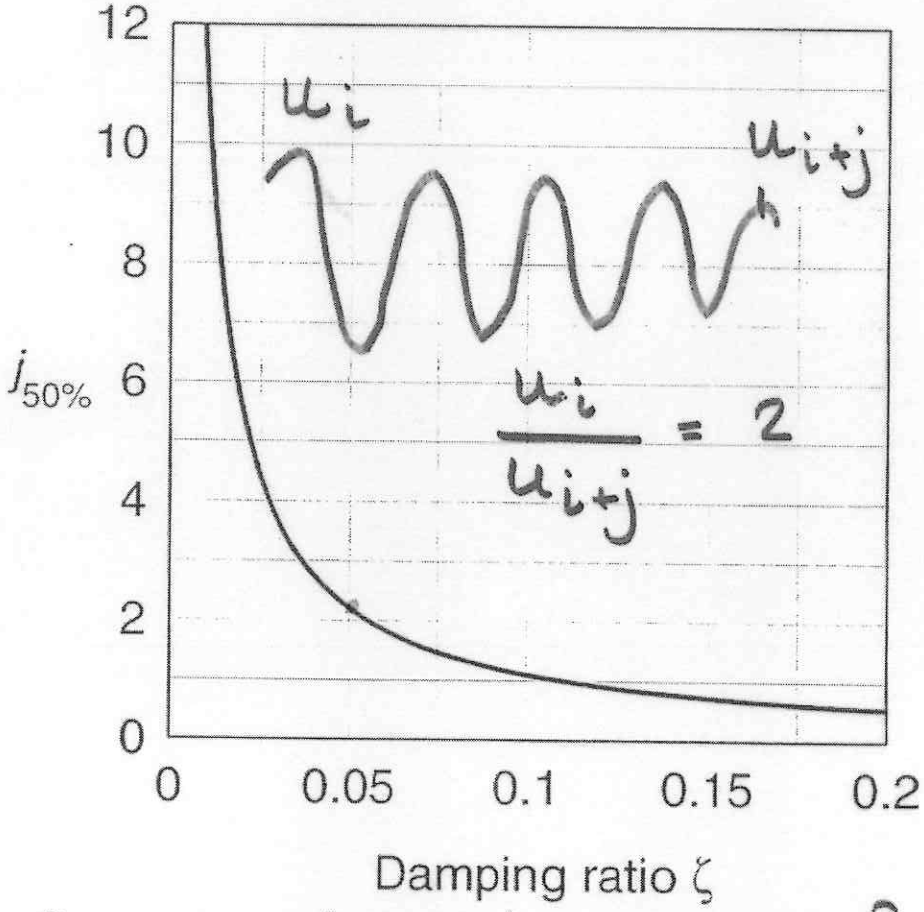


Figure 2.2.6 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Number of cycles for 50% amplitude reduction

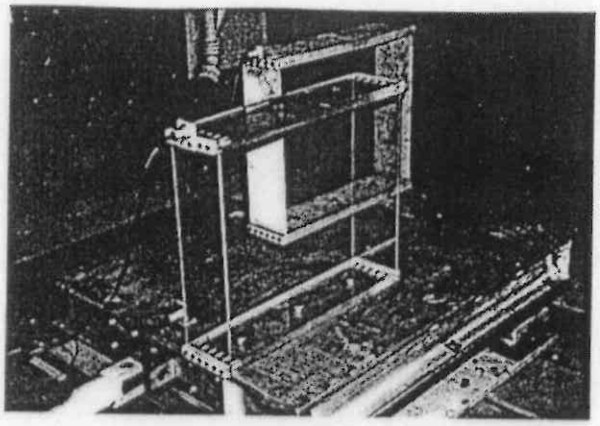


$$\frac{u_i}{u_{i+j}} = 2$$

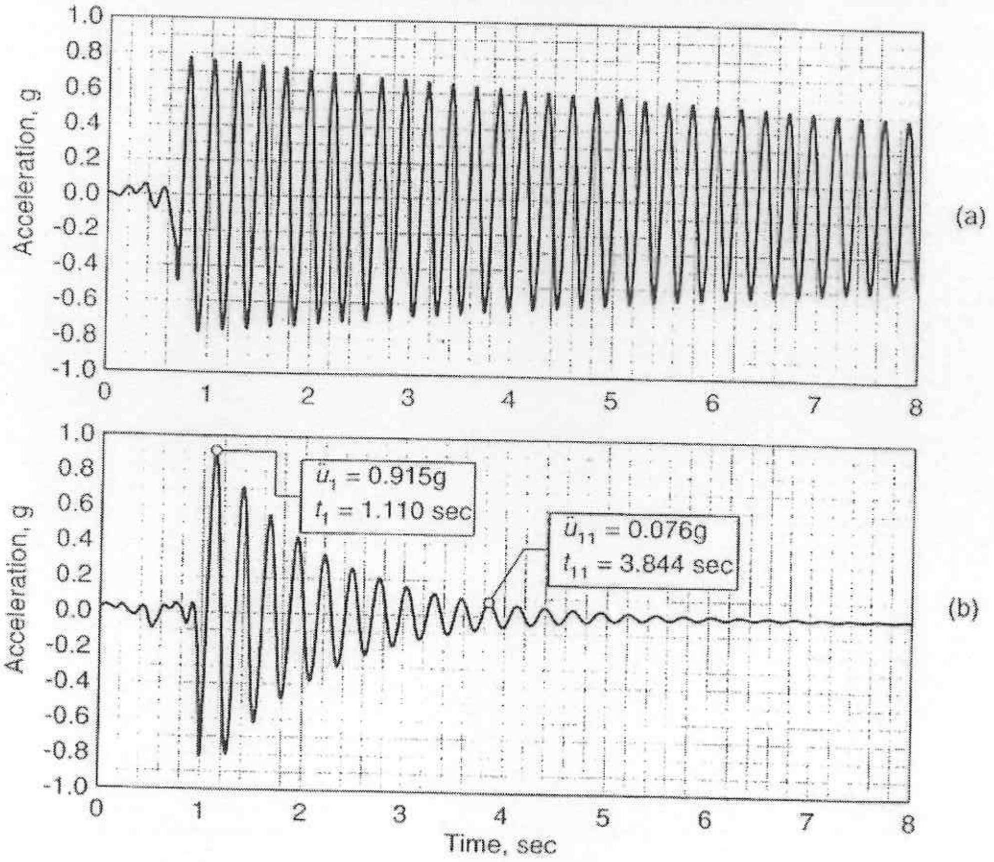
Smaller ζ ⇒ Larger no. of cycles to reduce by 50% (Larger $j_{50\%}$)

Figure 2.2.7 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$j_{50\%} \approx 2.2$ for $\zeta = 0.05$



Free vibration records:
(a) Aluminum model; (b) Plexiglass model



SHAKING TABLE TESTS

Figure 1.1.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

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EXCITED MOTION

Chapter Three

$m\ddot{u} + ku = p_0 \sin \omega_n t$

solution = transient + steady·state response

$$u(t) = \underbrace{u(0)\cos\omega_n t + \left[\frac{\dot{u}(0)}{\omega_n} - \frac{p_0}{k}\frac{\omega/\omega_n}{1-(\omega/\omega_n)^2}\right]\sin\omega_n t}_{\text{transient response}} + \underbrace{\frac{p_0}{k}\frac{1}{1-(\omega/\omega_n)^2}\sin\omega t}_{\text{steady·state}}$$

transient response
dies out over time, if damping is present
- oscillates at natural frequency of system
- depends on initial conditions
- decays over time

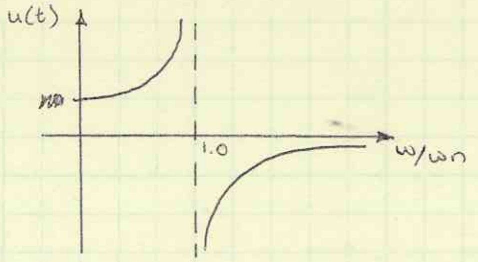
steady·state response due to forced excitation. continues always
- oscillates at forcing frequency
- doesn't depend on I.C.

in the case of $u(0)=0$, $\dot{u}(0)=0$,

$$u(t) = \frac{p_0}{k}\frac{1}{1-(\omega/\omega_n)^2}\left(\sin\omega t - {}^{\omega}/_{\omega_n}\sin\omega_n t\right)$$

---

relationship between ω/ω_n



as $\frac{1}{1-(\omega/\omega_n)^2}$ asymptotes at ${}^{\omega}/_{\omega_n} = 1.0$

$u(t) \propto$

when $\omega < \omega_n$, $\phi = 0°$
the two responses are in phase

when $\omega > \omega_n$, $\phi = 180°$
responses are 180° out of phase

steady·state response

$$u(t) = (u_{st})_0 \frac{1}{1-(\omega/\omega_n)^2}\sin\omega t \quad , (u_{st})_0 = \frac{p_0}{k}$$

or,

$$u(t) = (u_{st})_0 R_d \sin(\omega t - \phi)$$

$$R_d = \frac{1}{|1-(\omega/\omega_n)^2|}, \quad \phi = \begin{cases} 0 & \omega < \omega_n \\ 180 & \omega > \omega_n \end{cases}$$

If $\omega = \omega_n$, resonance occurs, deflections grow huge

→ ultimately controls, but first peak (with transient) could be the maximum.

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EXCITED MOTION

Resonant Response

$\omega/\omega_n = 1.0$ $$\frac{u(t)}{(u_{st})_0} = \frac{-1}{2}\left[2\pi\frac{t}{T_n}\cos 2\pi\frac{t}{T_n} - \sin 2\pi\frac{t}{T_n}\right]$$

deflection grows to infinity, but it takes an infinite time to get there

for $u(0)=0$ and $\dot{u}(0)=0$,
curve on pg 71 (fig. 3.1.4)

deformation amplitude grows by
$\pi\frac{p_o}{k}$ with each cycle

Damped, forced vibration

$m\ddot{u} + c\dot{u} + ku = p_o \sin\omega t$ OR $\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = \frac{p_o}{m}\sin\omega t$

$$u(t) = \underbrace{e^{-\zeta\omega_n t}(A\cos\omega_D t + B\sin\omega_D t)}_{\text{transient}} + \underbrace{C\sin\omega t + D\cos\omega t}_{\text{steady state}}$$

A, B derived through initial conditions $u(0), \dot{u}(0)$ — not necessarily easy or clean, and aren't too important, as transient solution falls off quickly

$$C = \frac{p_o}{k}\frac{1-(\omega/\omega_n)^2}{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}$$

$$D = \frac{p_o}{k}\frac{-2\zeta\omega/\omega_n}{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}$$

resonant case, $\omega = \omega_n$

$$u(t) = (u_{st})_0\frac{1}{2\zeta}\left[e^{-\zeta\omega_n t}\left(\cos\omega_D t + \frac{\zeta}{[1-\zeta^2]^{\frac{1}{2}}}\sin\omega_D t\right) - \cos\omega_n t\right]$$

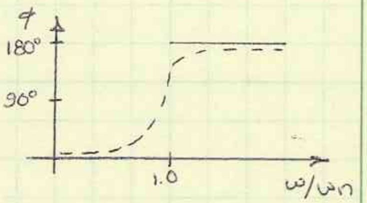
$$R_d = \frac{1}{2\zeta}$$

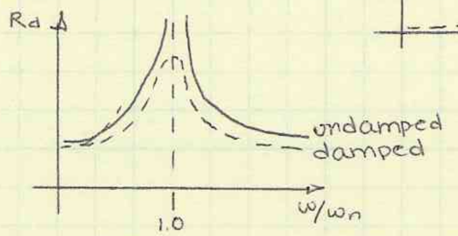
Steady-state response

$u(t) = (u_{st})_0 R_d \sin(\omega t - \phi)$, where

$$R_d = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta\omega/\omega_n]^2}}$$, and

$$\phi = \tan^{-1}\left[\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right]$$





- height of R_d peak increases as ζ decreases
- slope of ϕ curve increases as ζ decreases

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EXCITED MOTION

General characteristics

For $\omega/\omega_n \ll 1.0$

R_d is only slightly larger than 1.0, becomes independent of damping

$$u_o \cong (u_{st})_o = \frac{p_o}{k}$$

ϕ is close to zero, displacement is in phase with excitation force

For $\omega/\omega_n \gg 1.0$ – inertia dominates

R_d tends to zero, damping again doesn't control much

$$u_o \cong (u_{st})_o \frac{\omega_n^2}{\omega^2} = \frac{p_o}{m\omega^2}$$

ϕ is close to 180°

displacement is out of phase with force

For $\omega/\omega_n = 1.0$

R_d is very sensitive to damping – for small damping ratios, $R_d \gg 1.0$, making deflections very great

$$u_o = \frac{(u_{st})_o}{2\zeta} = \frac{p_o}{c\omega_n}$$

$\phi = 90°$ for all values of ζ

displacement reaches a peak when the force hits zero

Damping effects over time

$$\frac{|u_j|}{u_o} = 1 - e^{-2\pi\zeta j}$$

lightly damped, resonant systems

$$u(t) \simeq (u_{st})_o \frac{1}{2\zeta} \left(e^{-\zeta\omega_n t} - 1\right) \cos\omega_n t$$

(II) Damped Case

$$m\ddot{u} + c\dot{u} + ku = p_0 \sin\omega t \quad ; \quad u(0)\ ,\ \dot{u}(0) \text{ given} \qquad \text{—(I)}$$

Classical Solution: $u(t) = \underset{\substack{\text{Free}\\ \text{Vibration}\\ \text{Solution}}}{u_c(t)} + \underset{\substack{\text{Particular}\\ \text{Solution}}}{u_p(t)}$

We know $u_c(t) = e^{-\xi\omega_n t}\left[A\cos\omega_D t + B\sin\omega_D t\right]$ — (II)

where $\omega_D = \omega_n\sqrt{1-\xi^2}$

Particular solution $u_p(t) = C\sin\omega t + D\cos\omega t$ -(III)

If $u_p(t)$ is to satisfy our equation of motion,

$$\left.\begin{aligned} C &= \frac{p_0}{k}\cdot\frac{1-r^2}{\left[(1-r^2)^2+(2\xi r)^2\right]^{1/2}} \\ D &= \frac{p_0}{k}\cdot\frac{-2\xi r}{\left[(1-r^2)^2+(2\xi r)^2\right]^{1/2}} \end{aligned}\right\} \text{—(IV)}$$

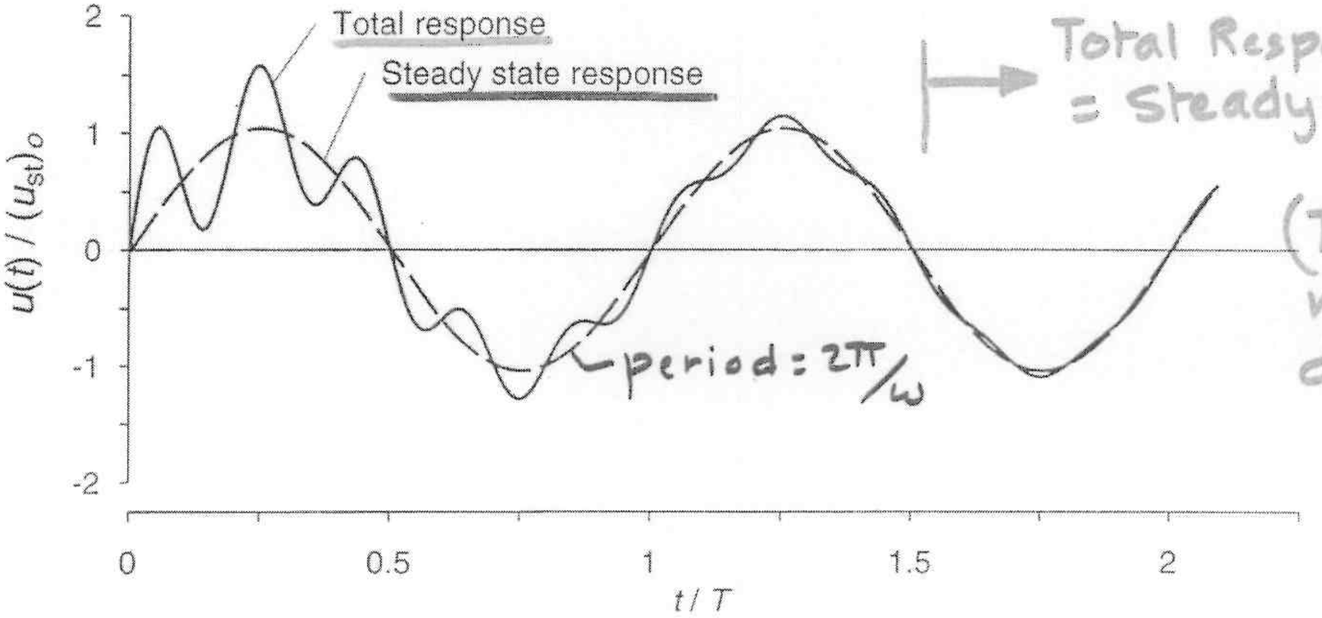
$(r = \omega/\omega_n)$

$$\Rightarrow \quad u(t) = \underbrace{e^{-\xi\omega_n t}\left[A\cos\omega_D t + B\sin\omega_D t\right]}_{\text{TRANSIENT}} + \underbrace{C\sin\omega t + D\cos\omega t}_{\text{STEADY STATE}}$$

TRANSIENT	STEADY STATE
- depends on ICs since A & B can be determined using $u(0)$, $\dot{u}(0)$ values	- does not depend on ICs. C & D are given by Eqn (IV)
- decays exponentially	- remains as long as force is applied

We'll study STEADY STATE response — this is generally of greater interest. (Note, though, that maximum response might occur before transient response has died out.)

Response of damped system to harmonic force; $\omega / \omega_n = 0.2$, $\zeta = 0.05$, $u(0) = 0$, and $\dot{u}(0) = \omega_n p_o / k$

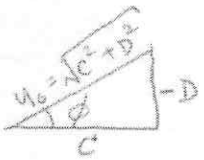


Total Response = Steady-state response (Transient response dies out.)

Figure 3.2.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Steady State Response

$$u(t) = C\sin\omega t + D\cos\omega t \quad ; \quad C \;\&\; D \text{ given by Eqn (IV)}$$



$$= \sqrt{C^2+D^2}\left[\frac{C}{\sqrt{C^2+D^2}}\sin\omega t - \frac{(-D)}{\sqrt{C^2+D^2}}\cos\omega t\right]$$

$$= u_0 \sin(\omega t - \phi)$$

$$\left(u_0 = \sqrt{C^2+D^2} \quad ; \quad \phi = \tan^{-1}(-D/C) \right)$$

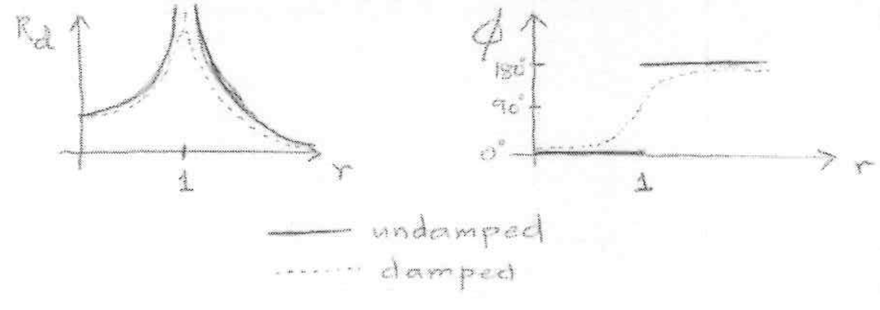
$$u_0 = \sqrt{C^2+D^2} = \frac{p_0}{k}\cdot\frac{1}{\left[(1-r^2)^2 + (2\xi r)^2\right]^{1/2}}$$

$$\Rightarrow \quad R_d = \frac{u_0}{(u_{st})_0} = \frac{1}{\left[(1-r^2)^2 + (2\xi r)^2\right]^{1/2}} \quad \text{—(V)}$$

$$\phi = \tan^{-1}\left(\frac{-D}{C}\right) = \tan^{-1}\left[\frac{2\xi r}{1-r^2}\right] \quad \text{—(VI)}$$

$$\underline{u(t) = (u_{st})_0 \cdot R_d \cdot \sin(\omega t - \phi)} \quad \text{—(VII)}$$

Compare R_d and ϕ for undamped and damped cases



R_d and ϕ depend on ξ and r

Steady state harmonic response ($\zeta = 0.2$)

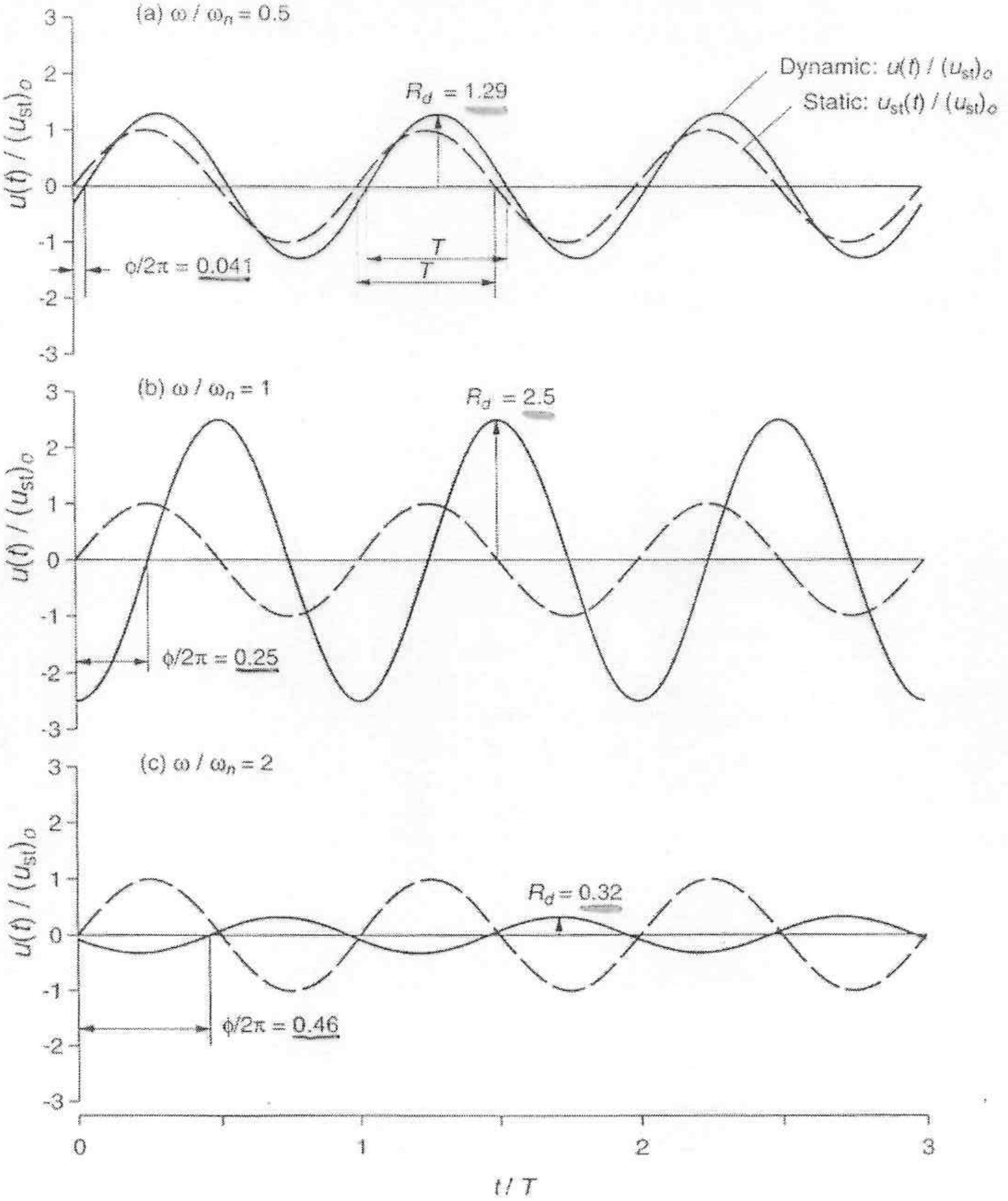


Figure 3.2.5 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Figure 3.2.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall

Deformation response factor and phase angle

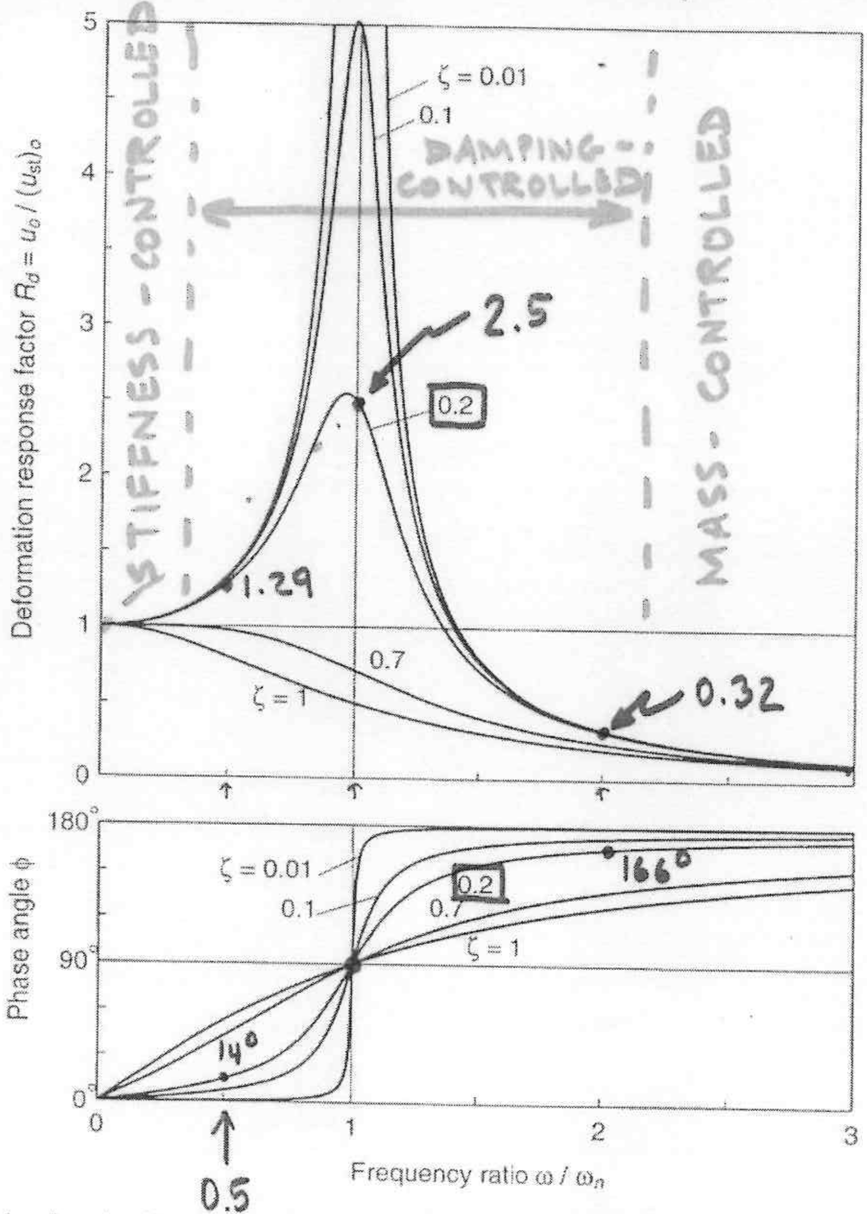


Figure 3.2.6 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Frequency-response curve
showing R_d v/s r

Limiting Cases and Useful Insights

$r << 1$
$r \simeq 0$
$\omega << \omega_n$

$R_d \rightarrow 1$, $u_o \rightarrow (u_{st})_o$ since force is "slowly varying"

$\phi \rightarrow 0$ $\Rightarrow$ displacement is in phase with force

Quasi-static solution (i.e., $ku \cong p_o \sin\omega t$)
$\hookrightarrow u(t) \simeq \frac{p_o}{k} \sin\omega t$

STIFFNESS - CONTROLLED

$r >> 1$
$\omega >> \omega_n$

$R_d \simeq \frac{1}{r^2}$, $u_o \simeq \frac{(u_{st})_o}{r^2}$ force is "rapidly varying"

$\phi \rightarrow 180^\circ$ $\Rightarrow$ displacement is 180° out of phase with force

Inertia forces dominate (i.e., $m\ddot{u} \cong p_o \sin\omega t$)

$$u(t) \simeq -\frac{p_o}{m\omega^2} \sin\omega t = \frac{p_o}{m\omega^2} \sin(\omega t - \pi)$$

MASS - CONTROLLED

$r \simeq 1$
$\omega \simeq \omega_n$

$R_d \simeq \frac{1}{2\xi}$, $u_o \simeq \frac{(u_{st})_o}{2\xi}$

$\phi \rightarrow 90^\circ$; displacement is 90° out of phase with force

Damping forces dominated (i.e, $c\dot{u} \cong p_o \sin\omega t$)

$$u(t) \cong -\frac{p_o}{c\omega} \cos\omega t = \frac{p_o}{c\omega_n} \sin(\omega t - \frac{\pi}{2})$$

DAMPING - CONTROLLED

Response for $\omega = \omega_n$

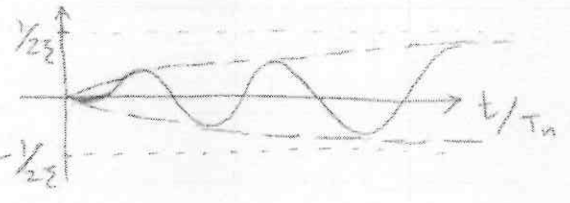
Consider case where $u(0) = \dot{u}(0) = 0$

Include transient and steady state contributions

ICs give: $A = \frac{(u_{st})_0}{2\xi}$, $B = \frac{(u_{st})_0}{2\sqrt{1-\xi^2}}$ in Eqn (II)

OR $$\left.\frac{u(t)}{(u_{st})_0}\right|_{r=1} = \frac{1}{2\xi}\left[\underbrace{e^{-\xi\omega_n t}\left(\cos\omega_D t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D t\right)}_{\text{this terms decreases exponentially with time}} - \cos\omega_n t\right] \quad \text{(A)}$$

$$\Rightarrow \left.\frac{u_0}{(u_{st})_0}\right|_{r=1} = \frac{1}{2\xi}$$ (bounded ; unlike response in undamped case for r = 1)



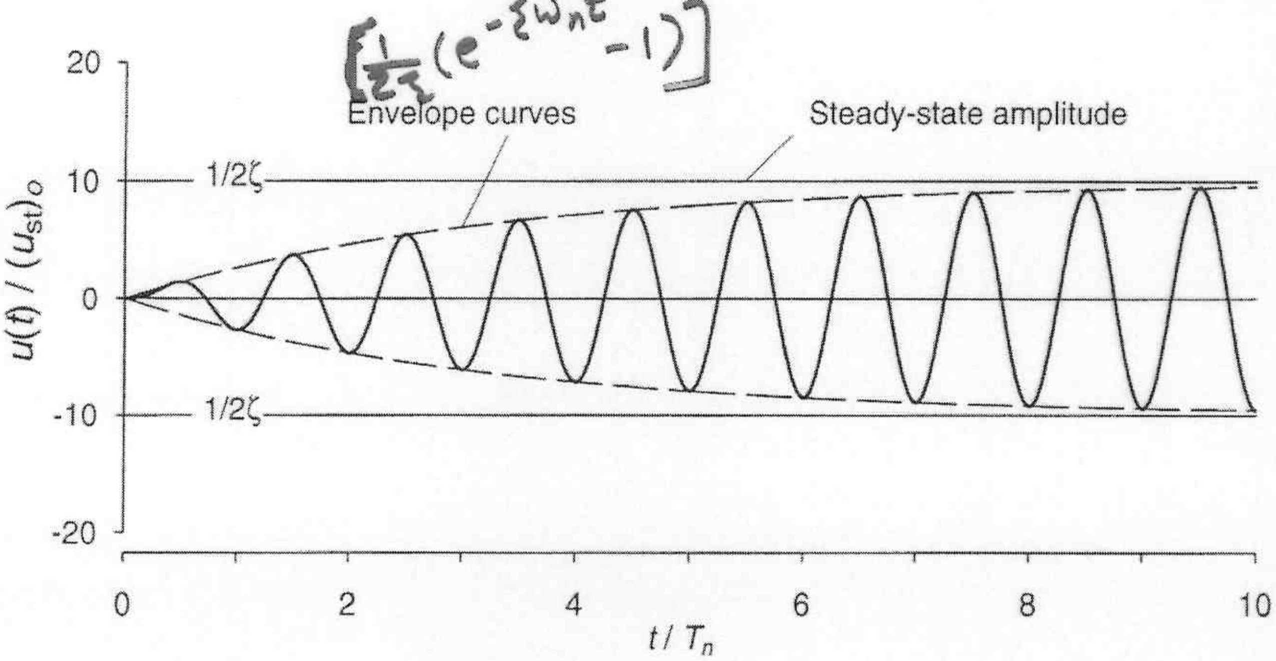
For light damping $\xi << 1$, $$\left.\frac{u(t)}{(u_{st})_0}\right|_{r=1} \simeq \underbrace{\frac{1}{2\xi}\left(e^{-\xi\omega_n t} - 1\right)}_{\text{ENVELOPE}}\underbrace{\cos\omega_n t}_{\text{OSCILLATORY}}$$

j^{th} peak (u_j) occurs at $t = jT_n$

$$\frac{|u_j|}{u_0} = 1 - \exp\left[-2\pi\xi j\right]$$

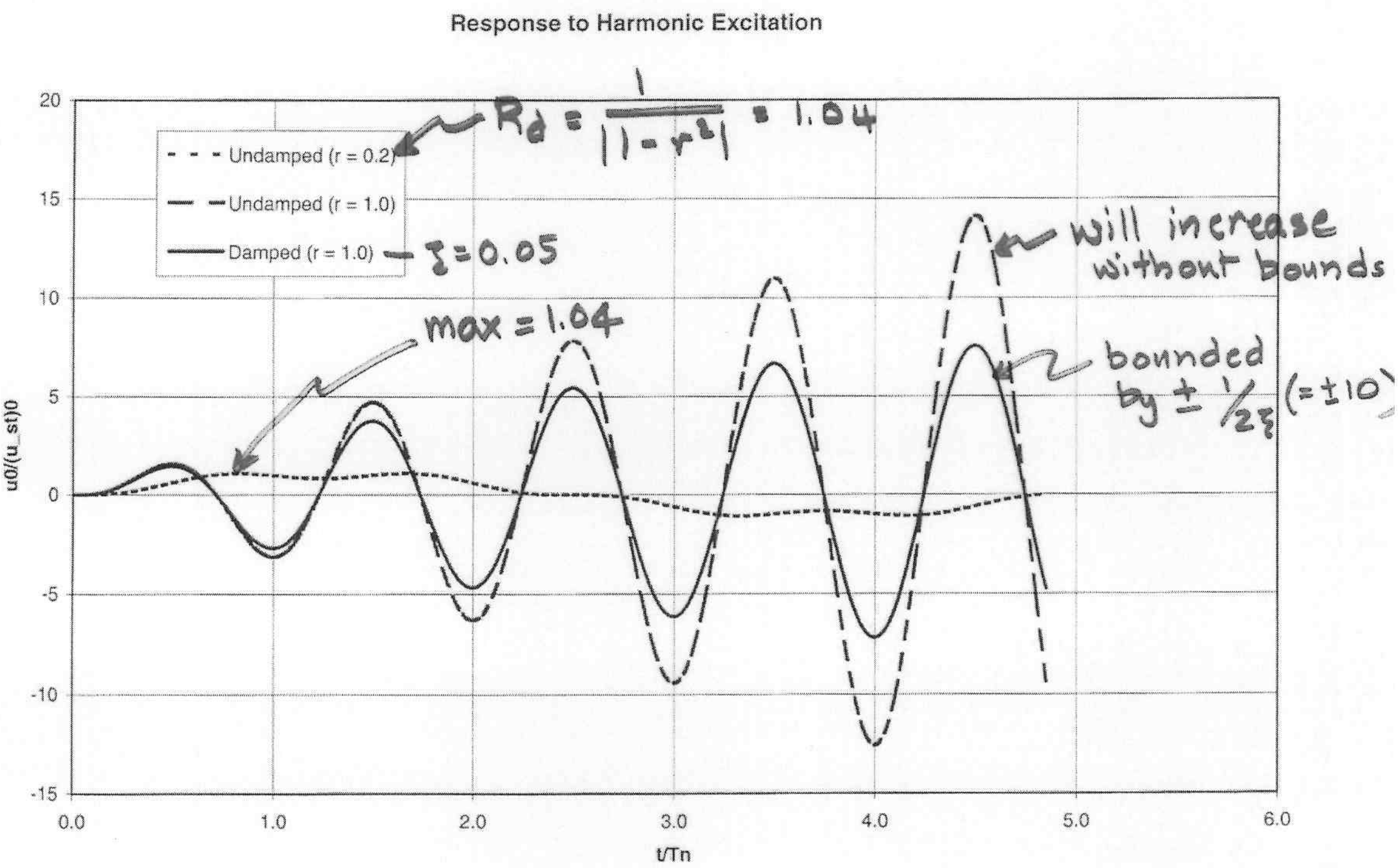
Larger damping ⇒ quicker build-up to u_0

Response to sinusoidal force with $\omega = \omega_n$ ($\zeta = 5\%$)



$\left[\frac{1}{2\zeta}(e^{-\zeta\omega_n t}-1)\right]$

Figure 3.2.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.



$R_d = \frac{1}{|1-r^2|} = 1.04$

$\xi = 0.05$

max = 1.04

will increase without bounds

bounded by $\pm \frac{1}{2\xi}$ (= ±10)

Response to sinusoidal force with $\omega = \omega_n$ (ζ = 1, 5, and 10%)

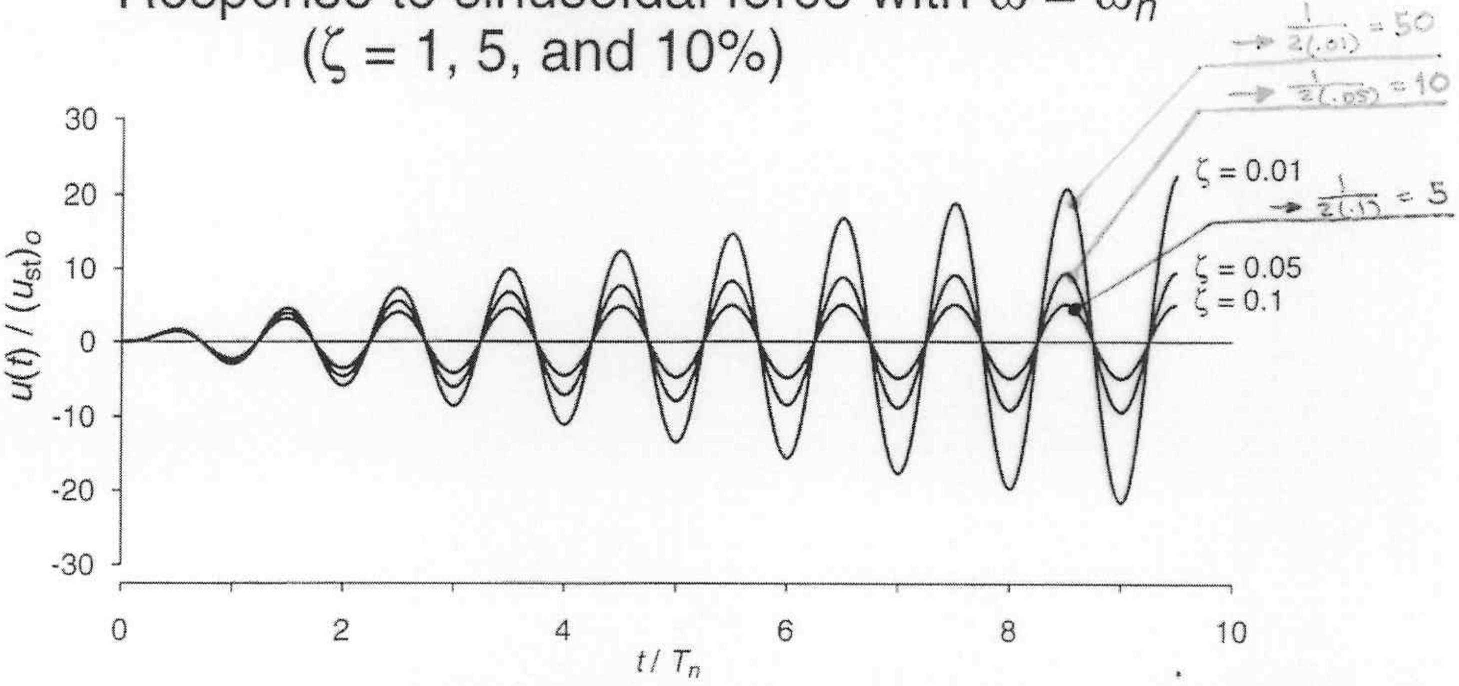
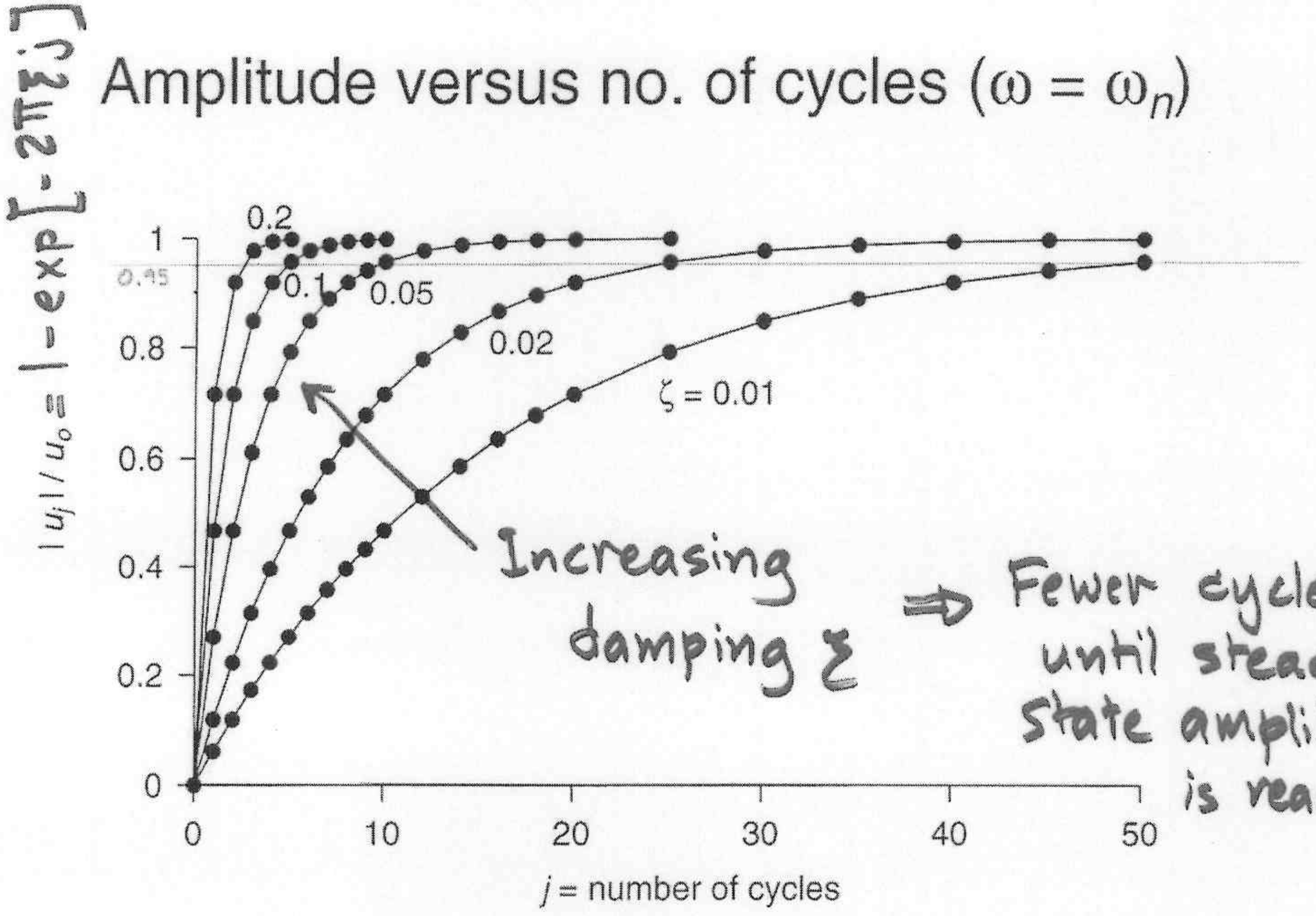


Figure 3.2.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Larger damping ξ ⇒ Smaller Response

⇒ Faster build up to steady state

Amplitude versus no. of cycles ($\omega = \omega_n$)



Increasing damping ξ ⇒ Fewer cycles until steady state amplitude is reached

Figure 3.2.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$$j_{95\%} \simeq \frac{\ln[1-.95]}{(-2\pi\xi)} = \frac{0.477}{\xi}$$

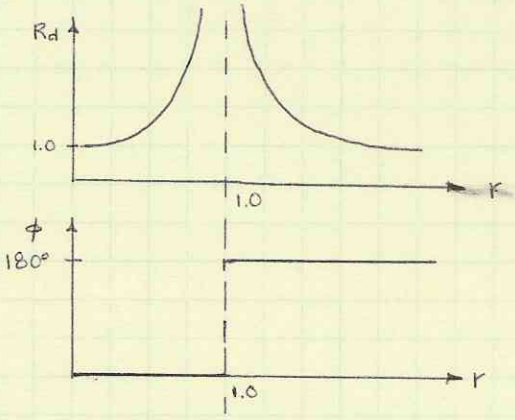
ξ	0.02	0.05	0.10	0.20
$j_{95\%}$	24	10	5	2

↳ no. of cycles till response is 95% of u_o

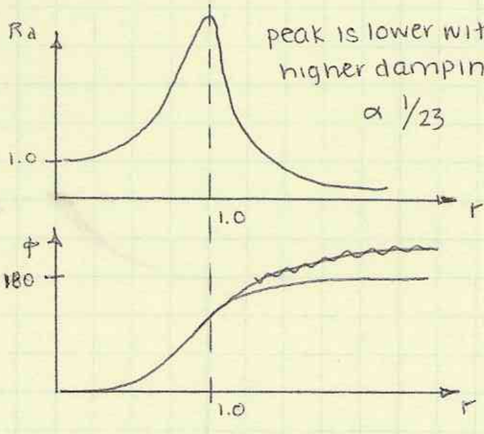
HARMONIC VIBRATION

Steady · State Response

$$u(t) = \underset{\overset{\downarrow}{\frac{p_0}{k}}}{(u_{st})_0} \cdot \underset{\overset{\downarrow}{R_d(r,\zeta)}}{R_d} \cdot \sin(\omega t - \underset{\overset{\downarrow}{\phi(r,\zeta)}}{\phi}) \qquad r = \omega/\omega_n$$



undamped



peak is lower with higher damping $\propto 1/2\zeta$

damped

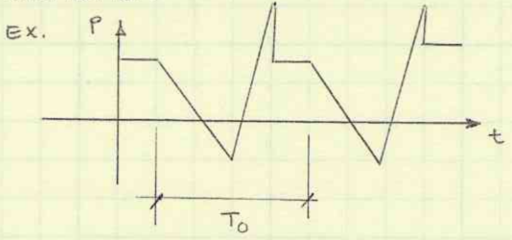
Resonant response

$$u(t) = (u_{st})_0 \frac{1}{2\zeta}\left[e^{-\zeta\omega_n t}\left(\cos\omega_D t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D t\right) - \cos\omega_n t\right]$$

amplitude increases to ∞ in infinite time, t

Periodic Excitation

Ex.



steady · state solution

$C'\sin\omega t + D'\cos\omega t$

$C' = -D, \; D' = C$

u_0, R_d, ϕ are the same as before

$$p(t) = \frac{4p_0}{\pi}\sum_{j=1,3,5}^{\infty}\frac{1}{j}\sin(j\omega_0 t)$$

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PERIODIC EXCITATION

Representing motion, response

$$p(t) = a_o + \Sigma a_{cj} \cos(j\omega_o t) + \Sigma a_{sj} \sin(j\omega_o t)$$

with period T_o, $\omega_o = 2\pi / T_o$

$$u(t) = u_o(t) + \Sigma u_j^c(t) + \Sigma u_j^s(t)$$

$$u_o = a_o / k$$

$$u_j^c(t) = \frac{a_j}{k} \cdot \frac{2\zeta B_j \sin(j\omega_o t) + (1-B^2)\cos(j\omega_o t)}{(1-B_j^2)^2 + (2\zeta B_j)^2}$$

$$u_i^s(t) = \frac{b_j}{k} \frac{(1-B_j^2)\sin(j\omega_o b) - 2\zeta B_j \cos(j\omega_o t)}{(1-B_j^2)^2 + (2\zeta B_j)^2}$$

Follow Manuel's example to do HW in mathcad

functions that are antisymmetric about $t=0$ are called <u>odd-valued functions</u>

cosine term coefficients are zero

only odd-numbered js will yield non-zeros in sine term

Dynamic Response factors

deformation R.F., R_d

$$R_d = \frac{u_o}{u_{st}} \qquad \frac{u(t)}{p_o/k} = R_d \sin(\omega t - \phi) \qquad R_d = \left[(1 - \omega^2/\omega_n^2)^2 + (2\zeta \omega/\omega_n)^2 \right]^{-\frac{1}{2}}$$

velocity response factor, R_v

$$R_v = \frac{\omega}{\omega_n} \cdot R_d, \qquad \dot{u}(t) = \frac{p_o}{\sqrt{km}} R_v \cos(\omega t - \phi)$$

acceleration R.F., R_a

$$R_a = \left(\frac{\omega}{\omega_n}\right)^2 R_d \qquad \ddot{u}(t) = \frac{-p_o}{m} R_a \sin(\omega t - \phi)$$

all RFs depend solely on ω, ω_n, and ζ

$$\frac{R_a}{r} = R_v = r R_d, \quad r = \omega/\omega_n$$

$$\log R_v = \log r + \log R_d$$

- if R_d is constant, $\log R_v$ vs. $\log r$ is a straight line, $m = 1.0$
- if R_a is constant, $\log R_v$ vs. $\log r$ is a straight line, $m = -1.0$
- can be plotted on a 4-way log scale
  - for a given ω/ω_n, only need to find one

Maximum values

R_d at $\omega/\omega_n = (1-2\zeta^2)^{1/2}$, $\max = \left[2\zeta(1-\zeta^2)^{1/2}\right]^{-1}$

R_v at $\omega = \omega_n$, $\max = 1/2\zeta$

R_a at $\omega/\omega_n = (1-2\zeta^2)^{-1/2}$, $\max = \max R_d$

09 Feb 06

Harmonic Force = $p_0 \cos \omega t$

$$m\ddot{u} + c\dot{u} + ku = p_0 \cos\omega t$$

Steady state Solution = $C' \sin\omega t + D' \cos\omega t$ as before

$C' = -D$ and $D' = C$ (C and D were obtained before)

i.e., $$C' = \frac{p_0}{k} \cdot \frac{2\xi r}{[(1-r^2)^2 + (2\xi r)^2]}$$

$$D' = \frac{p_0}{k} \cdot \frac{1-r^2}{[(1-r^2)^2 + (2\xi r)^2]}$$

$$u(t) = (u_{st})_0 \cdot R_d \cdot \cos(\omega t - \phi)$$
$$= u_0 \cdot \cos(\omega t - \phi)$$

u_0, R_d, and ϕ are the same as defined for the sinusoidal excitation ($p_0 \sin\omega t$).

Response to Periodic Excitation

$p(t)$ is periodic with period, T_0

$\Rightarrow \quad p(t + jT_0) = p(t)$ for all integer values of j

Fourier Series expansion of $p(t)$ leads to:

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos(j\omega_0 t) + \sum_{j=1}^{\infty} b_j \sin(j\omega_0 t) \quad \text{(I)}$$

where $\omega_0 = 2\pi / T_0$

and,

Fourier Coefficients

$$a_0 = \frac{1}{T_0}\int_0^{T_0} p(t)\,dt$$

[Average of $p(t)$ over a cycle]

$$a_j = \frac{2}{T_0}\int_0^{T_0} p(t)\cos(j\omega_0 t)\,dt \qquad j = 1, 2, \ldots$$

$$b_j = \frac{2}{T_0}\int_0^{T_0} p(t)\sin(j\omega_0 t)\,dt \qquad j = 1, 2, \ldots$$

For cosine terms and sine terms in (I), we know steady-state solution/response. For constant term, a_0, steady state response $= a_0/k$

$$\Rightarrow u(t) = u_0(t) + \sum_{j=1}^{\infty} u_j^c(t) + \sum_{j=1}^{\infty} u_j^s(t)$$

where $u_0(t) = a_0/k$

$$u_j^c(t) = \frac{a_j}{k} \cdot \frac{2\xi\beta_j \sin(j\omega_0 t) + (1-\beta^2)\cos(j\omega_0 t)}{(1-\beta_j^2)^2 + (2\xi\beta_j)^2}$$

and $$u_j^s(t) = \frac{b_j}{k} \cdot \frac{(1-\beta_j^2)\sin(j\omega_0 t) - 2\xi\beta_j \cos(j\omega_0 t)}{(1-\beta_j^2)^2 + (2\xi\beta_j)^2}$$

where $\beta_j = j\dfrac{\omega_0}{\omega_n}$

a_j, b_j decrease with j usually; $\beta_j > 1$ and increases with j

Both these effects suggest that often a few terms are sufficient.

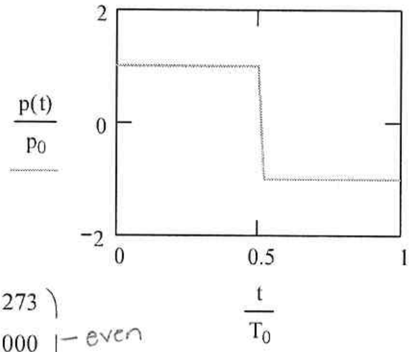
Reference: *Chopra, A.K., Dynamics of Structures - Theory and Applications to Earthquake Engineering, Second Edition, Prentice Hall, 2001*

Example 3.8, page 116

LOAD DESCRIPTION

$p_0 := 1 \qquad T_0 := 1 \qquad \omega_0 := \dfrac{2\cdot\pi}{T_0}$

$$p(t) := \begin{vmatrix} p_0 \ \text{ if } (t \ge 0)\cdot\left(t \le \dfrac{T_0}{2}\right) \\ -p_0 \ \text{ if } \left(t > \dfrac{T_0}{2}\right)\cdot(t \le T_0) \end{vmatrix}$$



FOURIER COEFFICIENTS OF FORCE, p(t)

$a_0 := \dfrac{1}{T_0}\cdot\displaystyle\int_0^{T_0} p(t)\,dt \qquad a_0 = 0$

$j := 1 .. 7$

$a_j := \dfrac{2}{T_0}\cdot\displaystyle\int_0^{T_0} p(t)\cdot\cos(j\cdot\omega_0\cdot t)\,dt$

a_{cj}

$b_j := \dfrac{2}{T_0}\cdot\displaystyle\int_0^{T_0} p(t)\cdot\sin(j\cdot\omega_0\cdot t)\,dt$

a_{sj}

$$a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1.273 \\ 0.000 \\ 0.424 \\ 0.000 \\ 0.255 \\ 0.000 \\ 0.182 \end{pmatrix} \begin{matrix} \\ \text{— even} \\ \text{— odd} \\ \\ \\ \\ \\ \end{matrix}$$

Note: Because *p*(*t*) is an odd-valued function (anti-symmetric about *t* = 0), a_0 and all the Fourier cosine coefficients, a_j, are equal to zero.

2|9

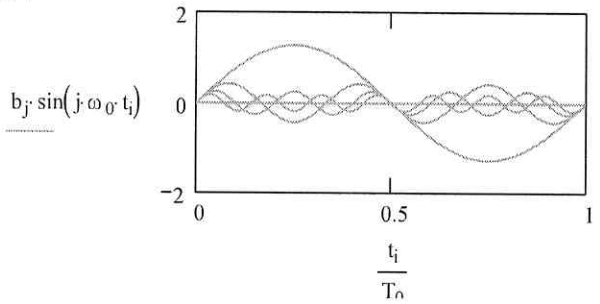
Let's look at the first seven Fourier coefficients or, equivalently, the first four non-zero components here.

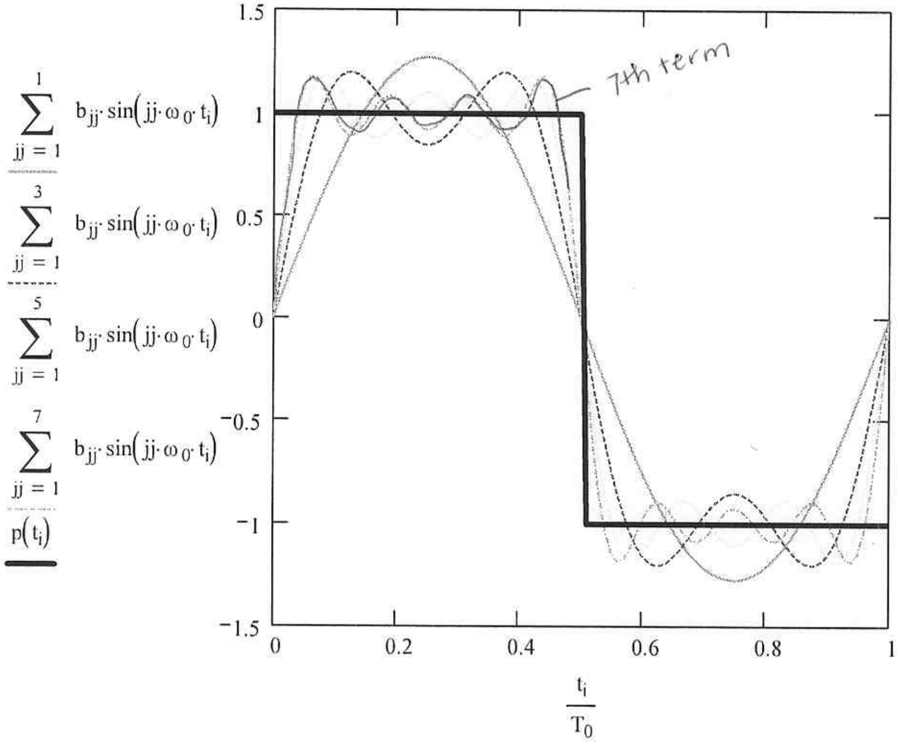
$\Delta t := \frac{T_0}{100}$

$i := 1 .. 101$

$t_i := (i - 1) \cdot \Delta t$

$j := 1... 7$





STRUCTURE PROPERTIES

$\zeta := 0.05 \qquad T_n := \frac{T_0}{4} \qquad m := 1$

$\omega_n := \frac{2\cdot\pi}{T_n} \qquad k := \omega_n^{\,2}\cdot m$

$k = 631.655$

$$R_d(r,\zeta) := \frac{1}{\sqrt{\left(1 - r^2\right)^2 + (2\cdot\zeta\cdot r)^2}}$$

$\beta_j := \frac{j\cdot\omega_0}{\omega_n}$

$$\beta = \begin{pmatrix} 0.250 \\ 0.500 \\ 0.750 \\ 1.000 \\ 1.250 \\ 1.500 \\ 1.750 \end{pmatrix}$$

$$R_d(\beta,\zeta) = \begin{pmatrix} 1.066 \\ 1.330 \\ 2.253 \\ 10.000 \\ 1.735 \\ 0.794 \\ 0.483 \end{pmatrix}$$

Note: These would be the amplifications at each of the first 7 frequencies. It is clear that these amplifications get smaller with increasing index - e.g., the amplification for the 4 sine terms are: 1.066, 2.253, 1.735, and 0.483 corresponding to β values of 0.25, 0.75, 1,25, 1.75, respectively.

$u_{st0} := \frac{p_0}{k}$

$u_{st0} = 1.583 \times 10^{-3}$

2|9

$$\Delta t := \frac{T_0}{100}$$

$$i := 1 .. 101$$

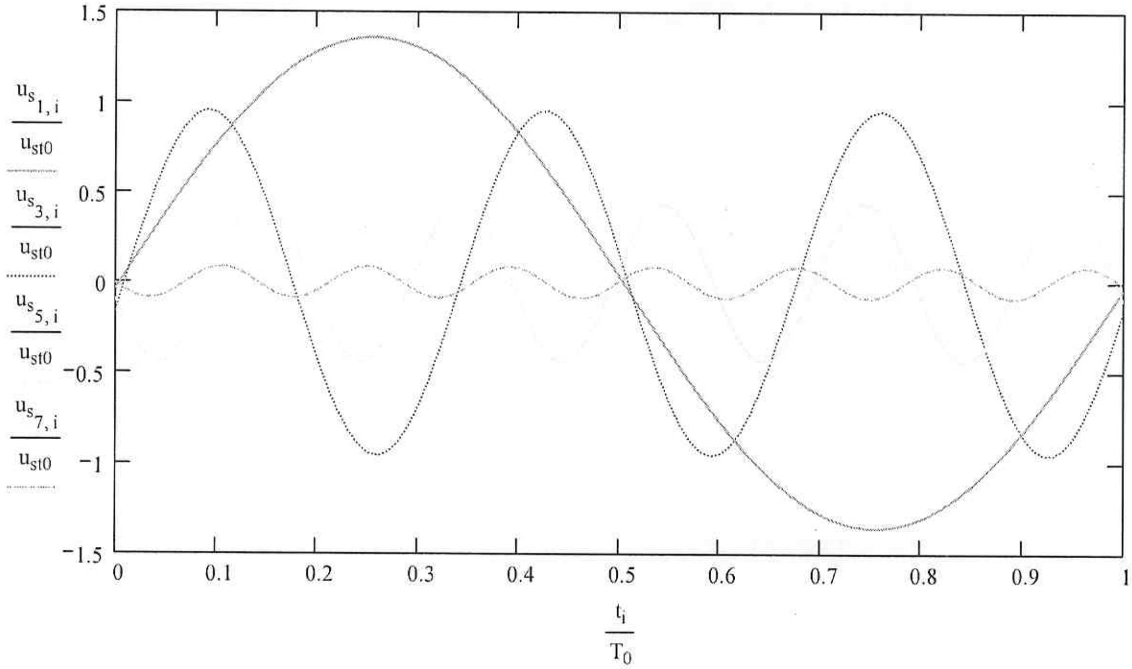
$$t_i := (i-1)\cdot\Delta t$$

$$u_{c_{j,i}} := \frac{a_j}{k}\cdot\frac{2\cdot\zeta\cdot\beta_j\cdot\sin\left(j\cdot\omega_0\cdot t_i\right) + \left[1-\left(\beta_j\right)^2\right]\cdot\cos\left(j\cdot\omega_0\cdot t_i\right)}{\left[1-\left(\beta_j\right)^2\right]^2 + \left(2\cdot\zeta\cdot\beta_j\right)^2}$$

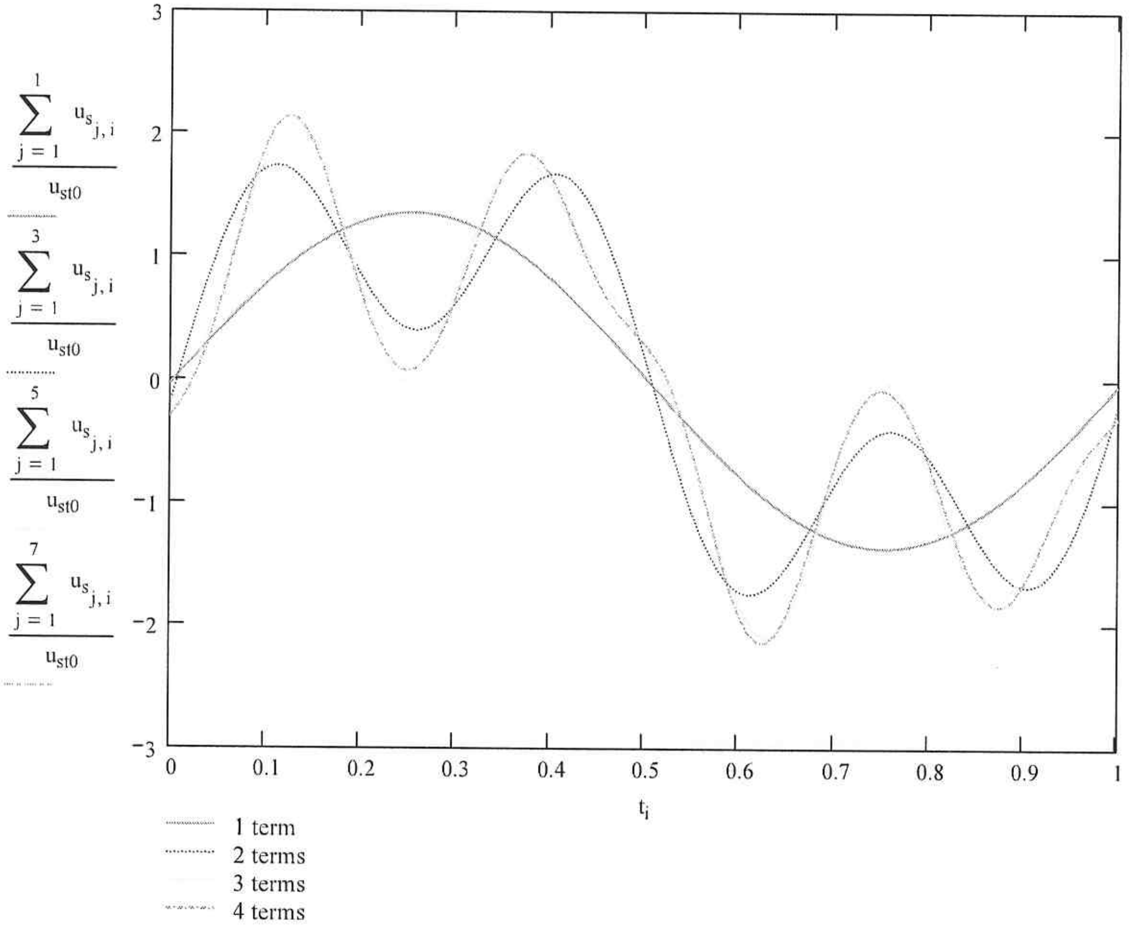
$$u_{s_{j,i}} := \frac{b_j}{k}\cdot\frac{\left[1-\left(\beta_j\right)^2\right]\cdot\sin\left(j\cdot\omega_0\cdot t_i\right) - 2\cdot\zeta\cdot\beta_j\cdot\cos\left(j\cdot\omega_0\cdot t_i\right)}{\left[1-\left(\beta_j\right)^2\right]^2 + \left(2\cdot\zeta\cdot\beta_j\right)^2} = \frac{b_i}{K}\cdot(R_d)_j \sin(j\omega_0 t - \phi_j)$$

$$\phi_j = \tan^{-1}\left(\frac{2\zeta B_j}{1-\beta_j^2}\right)$$

$$\beta_j = \frac{j\omega_0}{\omega_n}$$



Note: u_c is not needed here since all the cosine term coefficients are zero.


$\frac{\sum_{j=1}^{1} u_{s_{j,i}}}{u_{st0}}$
$\frac{\sum_{j=1}^{3} u_{s_{j,i}}}{u_{st0}}$
$\frac{\sum_{j=1}^{5} u_{s_{j,i}}}{u_{st0}}$
$\frac{\sum_{j=1}^{7} u_{s_{j,i}}}{u_{st0}}$
3
2
1
0
−1
−2
−3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t_i
1 term
2 terms
3 terms
4 terms

2/9

Dynamic Response Factors

We defined R_d, the deformation response factor to express the steady-state displacement, $u(t)$

$$\frac{u(t)}{(p_0/k)} = R_d \cdot \sin(\omega t - \phi) \qquad \text{—(I)}$$

R_d = ratio of u_0 (i.e., max. value of $u(t)$) to $(u_{st})_0$ (i.e., static deformation)

Differentiating (I) we get:

$$\frac{\dot{u}(t)}{p_0/\sqrt{km}} = R_v \cdot \cos(\omega t - \phi) \qquad \text{—(II)}$$

where we define R_v, the velocity response factor as

$$R_v = \frac{\omega}{\omega_n} \cdot R_d = r \cdot R_d \qquad \text{—(II')}$$

Differentiating (II) we get:

$$\frac{\ddot{u}(t)}{p_0/m} = -R_a \cdot \sin(\omega t - \phi) \qquad \text{—(III)}$$

where we define R_a, the acceleration response factor as

$$R_a = \left(\frac{\omega}{\omega_n}\right)^2 R_d = r^2 \cdot R_d \qquad \text{—(III')}$$

$$\left[\text{Recall} \quad R_d = \frac{1}{\left[(1-r^2)^2 + (2\xi r)^2\right]^{1/2}} \right]$$

$R_v = r \cdot R_d$; $R_a = r^2 \cdot R_d$

$\Rightarrow$ R_d, R_v, R_a are functions of ξ & r only.

Deformation, velocity and acceleration response factors

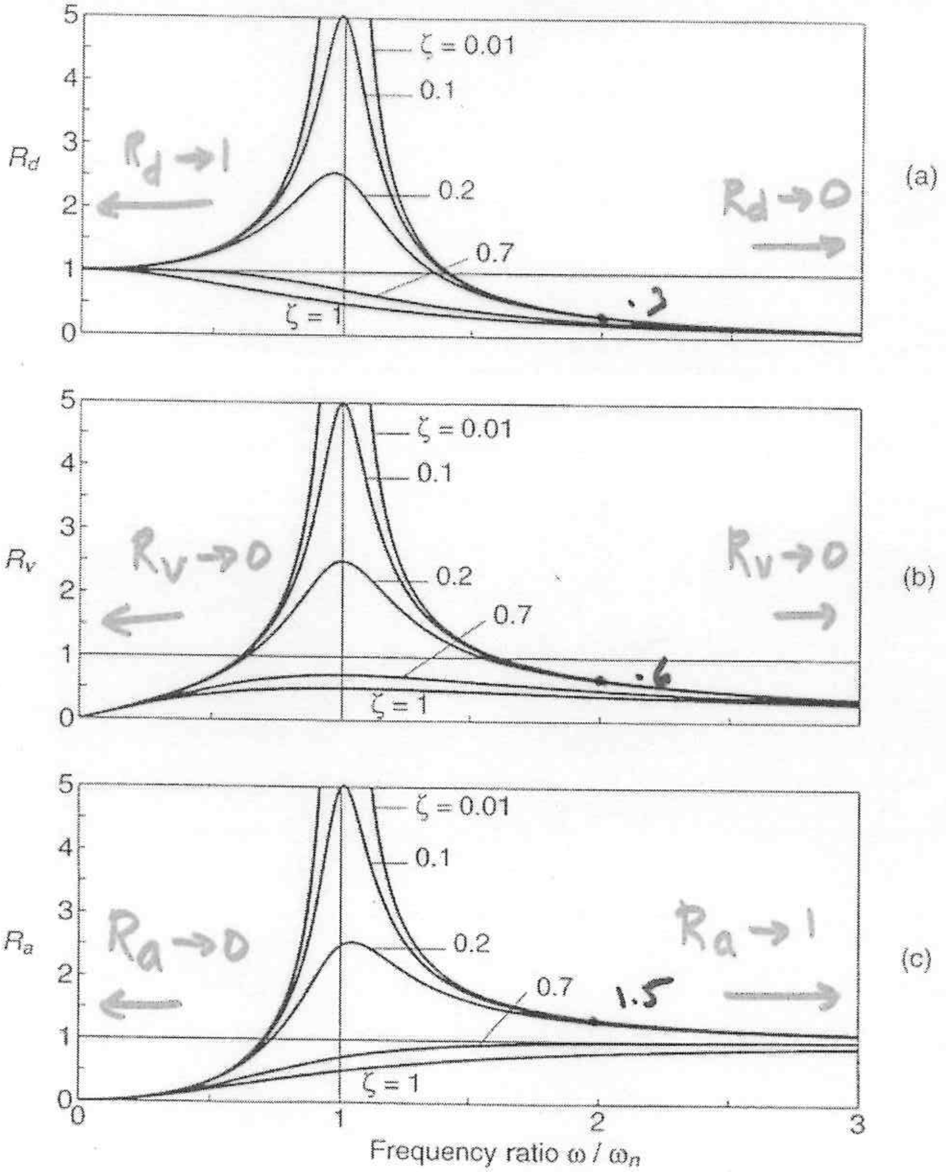


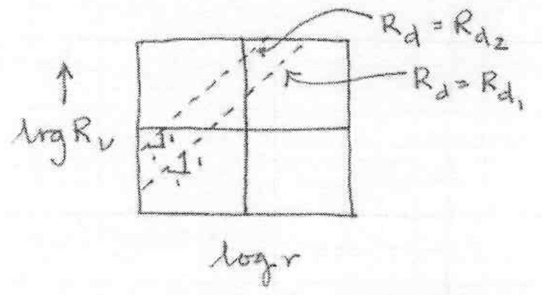
Figure 3.2.7 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Example: Find R_a, R_v, and R_d
for $\omega/\omega_n = 2$, $\zeta = 0.1$

$$\frac{R_a}{r} = R_v = r \cdot R_d \quad \text{——} (\star)$$

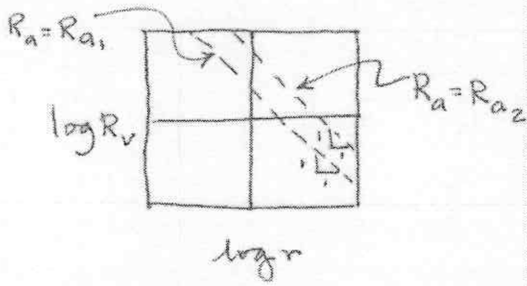
$\Rightarrow \log R_v = \log r + \log R_d$

$\log R_v$ vs. $\log r$ is a straight line with slope = +1 if R_d = const.



$\log R_v = -\log r + \log R_a$

$\log R_v$ vs. $\log r$ is a straight line with slope = −1 if R_a = const.



Combine

Possible to show R_d, R_v, R_a vs. r on one plot using 4-way logarithmic paper

For any r,

Read R_v of vertical scale ↑

Read R_d of one inclined scale ↖

Read R_a of other inclined scale ↗

Note for a given value of r, it is sufficient to read any <u>one</u> of R_a, R_v, or R_d

The other two quantities may be found using Eqn ($\star$).

Four-way logarithmic plot of R_d, R_v, and R_a

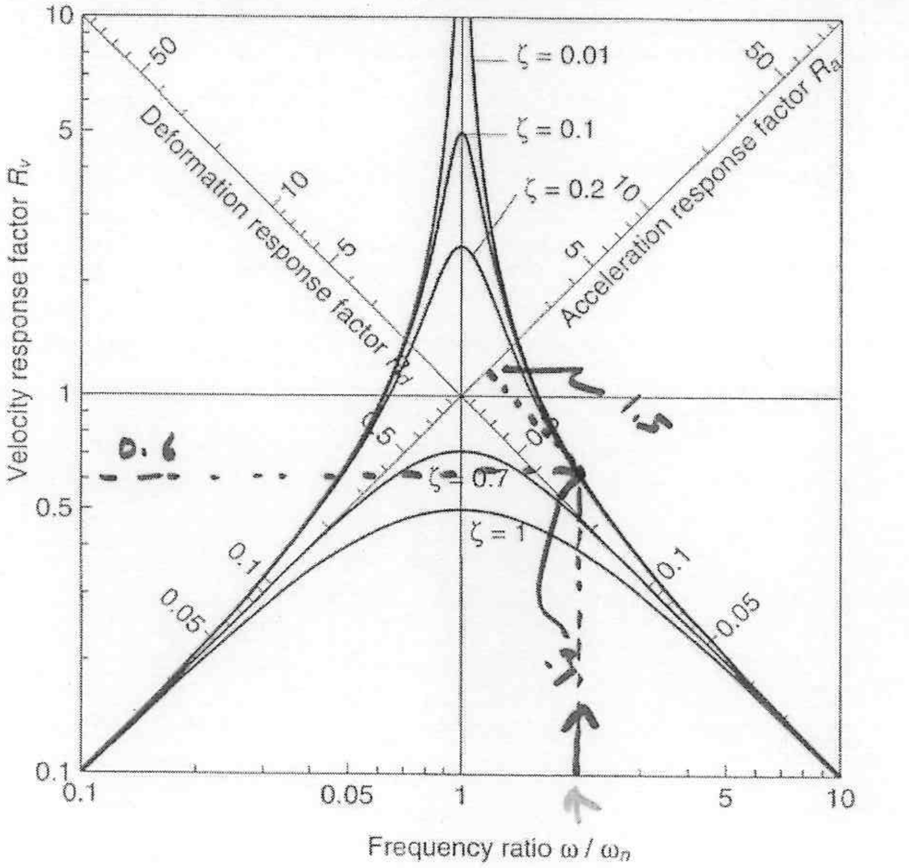


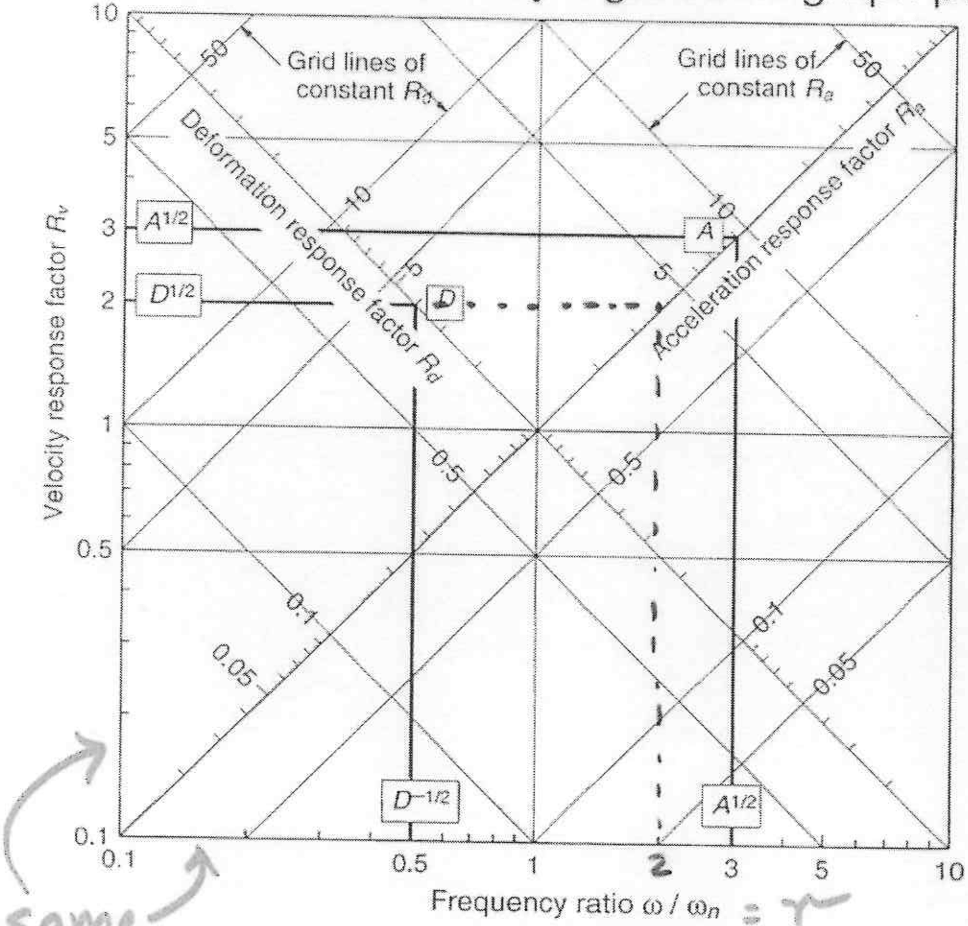
Figure 3.2.8from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Example: Find R_a, R_d, and R_v for $\omega/\omega_n = 2$, $\xi = 0.1$

$$R_V = r \cdot R_d$$

$$\log R_V = \log r + \log R_d \quad (\text{slope} = +1)$$

Construction of four way logarithmic graph paper



same scales

Figure A3.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$$R_V = R_a / r$$

$$\log R_V = -\log r + \log R_a \quad (\text{slope} = -1)$$

28b

Where do peaks of R_d, R_v and R_a occur?

(i.e., at what values of r)

Need to set derivatives to zero

$$R_d = \frac{1}{\left[(1-r^2)^2 + (2\xi r)^2\right]^{1/2}}$$

$$\frac{dR_d}{dr} = -\frac{1}{2}\,\frac{2(1-r^2)(-2r) + (4\xi r)(2\xi)}{\left[(1-r^2)^2 + (2\xi r)^2\right]^{3/2}}$$

Set $\frac{dR_d}{dr} = 0 \Rightarrow 4(1-r^2) = 8\xi^2$

or $r^2 = 1 - 2\xi^2$

$$\Rightarrow \underline{r = (1-2\xi^2)^{1/2}} \quad \text{for max } R_d$$

Note: $r \neq 1$ for maximum R_d as we might have expected

i.e., $\underline{\omega = \omega_n (1-2\xi^2)^{1/2} \quad \text{for max } R_d}$

$\Rightarrow$ Peak of R_d occurs slightly to the left of $r = 1$ or $\omega = \omega_n$

Similarly, $\underline{\omega = \omega_n \quad \text{for max } R_v}$

$\underline{\omega = \omega_n (1-2\xi^2)^{-1/2} \quad \text{for max } R_a}$

Summarizing Maximum Response Values

Response	Frequency at which Max. Occurs	Max. Response
Displacement	$\omega_n (1-2\xi^2)^{1/2}$	$\frac{1}{2\xi\sqrt{1-\xi^2}}$
Velocity	ω_n	$\frac{1}{2\xi}$
Acceleration	$\omega_n (1-2\xi^2)^{-1/2}$	$\frac{1}{2\xi\sqrt{1-\xi^2}}$

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END OF CHAPTER FOUR (3)

Half-Power Bandwidth

estimates damping from forced vibrations

$$R_d(\omega_a) = R_d(\omega_b) = \frac{1}{\sqrt{2}} R_{d\,resonance}$$

└→ aka, the max value of $R_d = \frac{1}{2\xi}\left[1-\xi^2\right]^{-1/2}$

$\omega_a \rightarrow r_a$

$\omega_b \rightarrow r_b$

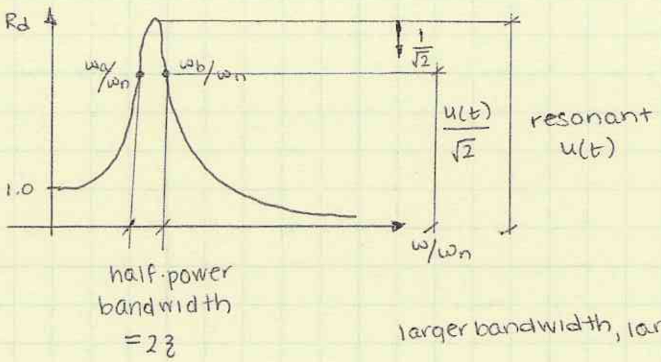
such that...

$$\frac{1}{\sqrt{2}}\left[2\xi\left(1-\xi^2\right)^{1/2}\right]^{-1} = \left[(1-r^2)^2 + (2\xi r)^2\right]^{-1/2}$$

solving for the two roots gives you r_a, r_b

for small damping,

$$r_b - r_a \cong 2\xi, \quad \xi = \frac{1}{2}(r_b - r_a) = \frac{\omega_b - \omega_a}{2\omega_n}$$



larger bandwidth, larger ξ

Estimating natural frequency

(i) Resonance testing

$$\xi = \frac{1}{2}\frac{u_{st}}{u_o \text{ of } \omega=\omega_n}$$

(ii) Frequency response

use half-power bandwidth

Transfer of Excitation

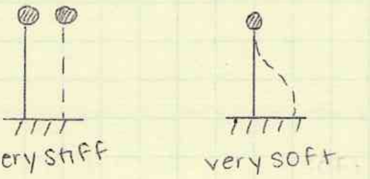
TR = transmissibility ratio

$$TR = \frac{f_{T_o}}{p_o}, \quad f_T = \text{force transmitted to base} = ku(t) + c\dot{u}(t)$$

$$TR = \left[\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}\right]^{1/2}$$

same function for transmission of ground to object or object to ground.

reducing TR ($r > \sqrt{2}$) increases static displacement



remember: as a machine starts up, it must pass through the low ω values.

Half-Power Bandwidth

This quantity is often used to estimate damping from forced vibration tests.

Not necessary to know the applied force only need cyclic frequency of the load.

Obtain plot of R_d vs. r by varying load frequency. Find two frequencies (ω_a and ω_b) on either side of resonant frequency at which the amplitude = $1/\sqrt{2}$ times resonant amplitude

$$R_d(\omega_a) = R_d(\omega_b) = \frac{1}{\sqrt{2}} \underbrace{R_{d\,\text{resonance}}}_{\text{max. value of } R_d}$$

$\omega_a \rightarrow r_a$
$\omega_b \rightarrow r_b$

r_a and r_b satisfy $$\frac{1}{\left[(1-r^2)^2 + (2\xi r)^2\right]^{1/2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{2\xi\sqrt{1-\xi^2}}$$

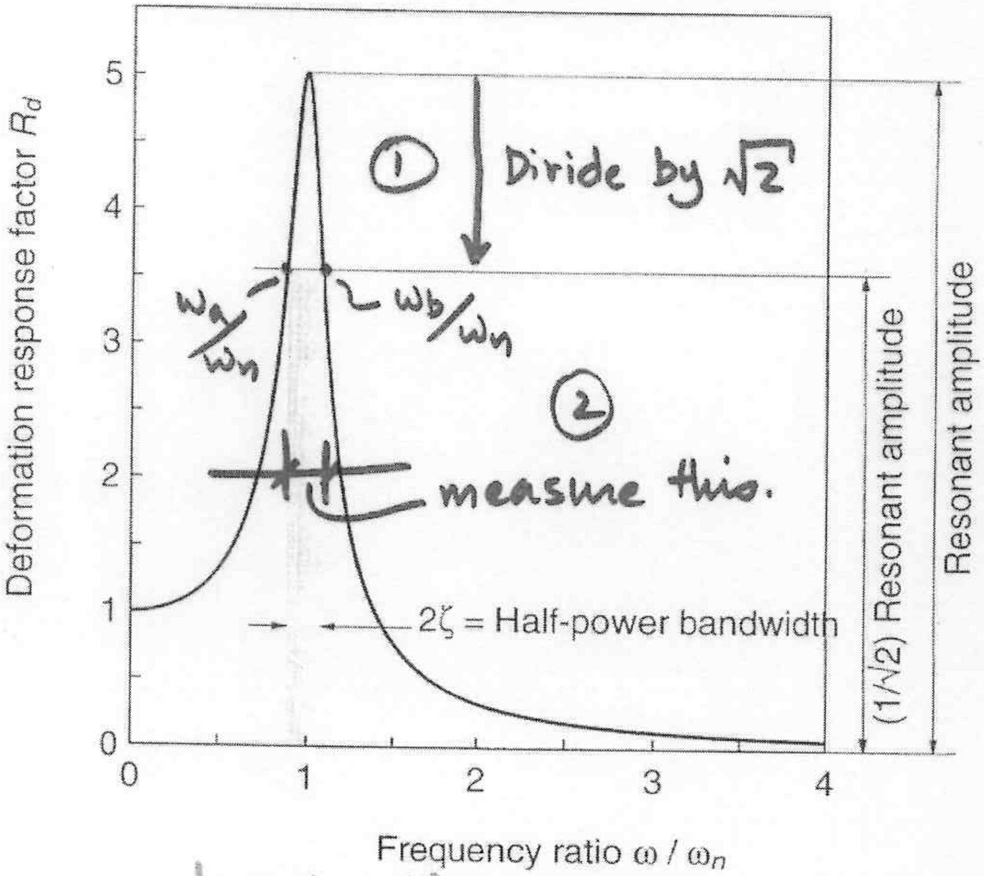
Solve for two roots of r (i.e, r_a and r_b)

For small damping

$$r_b - r_a \simeq 2\xi$$

$$\Rightarrow \quad \xi = \frac{1}{2}\left[r_b - r_a\right] = \frac{\omega_b - \omega_a}{2\omega_n}$$

Half-power band width



Larger bandwidth ⇒ Larger ξ

Figure 3.2.9 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Estimating Natural Frequency and Damping from Harmonic Tests

Experimentally determined Natural frequency = Actual property of structure useful to validate analysis and stiffness/mass assumptions

Experimentally determined damping = Important property usually not well known (esp. for non-standard materials)

(I) Resonance Testing

ω_n : Excite structure at different frequencies, ω

when phase angle = 90°, $\omega = \omega_n$

ξ :

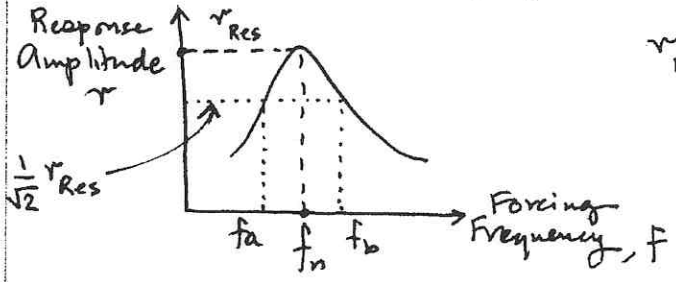
$$\xi = \frac{1}{2} \frac{(u_{st})_0}{(u_0)_{\omega=\omega_n}}$$

Sometimes difficulties arise in obtaining $(u_{st})_0$
because
- difficult to produce forces at low frequencies
- difficult to generate significant static force

Then, $(u_{st})_0$ measured by other tests such as pulling on structure.

(II) Frequency-Response Curve

Measure amplitude of steady-state acceleration at various excitation frequencies.



r_{Res} = max response amplitude occurs at $f = f_n$

Hence, f_n is known (assuming ξ is small)

$$\xi = \frac{f_b - f_a}{2 \cdot f_n}$$

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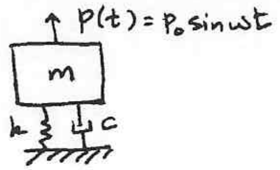
FORCE TRANSMISSION AND VIBRATION ISOLATION



Interest is in ratio of
Force Transmitted to Base, $(f_T)_o$
<u>to</u> Amplitude of applied Force

TR = Transmissibility Ratio

$$= \frac{(f_T)_o}{p_o}$$

f_T = Force transmitted to base

$$= k u(t) + c \dot{u}(t)$$

$$u(t) = (u_{st})_o \cdot R_d \cdot \sin(\omega t - \phi)$$

$$\Rightarrow (f_T)_o = (u_{st})_o \cdot R_d \cdot \sqrt{k^2 + c^2 \omega^2}$$

$$TR = \left[\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2} \right]^{1/2} \Longleftrightarrow$$

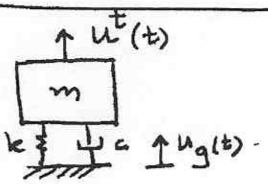
See plot

To keep TR small and less than 1 need small stiffness (so that $r > \sqrt{2}$)

⇒ TRADE-OFF between
Reduced Transmitted Force
<u>and</u> Acceptable Static Displacement

Damping less important BUT need some damping as excitating force frequency builds up through resonance

RESPONSE TO GROUND MOTION AND VIBRATION ISOLATION



$\ddot{u}_g(t) = \ddot{u}_{go} \sin \omega t$

Interest is in ratio of
Total acceleration transmitted
to mass <u>to</u> amplitude of
ground acceleration

TR = Transmissibility Ratio

$$= \frac{\ddot{u}_o^t}{\ddot{u}_{go}}$$

Eqn. of motion:

$$m\ddot{u}^t + c\dot{u} + ku = 0$$

$$TR = \frac{\ddot{u}_o^t}{\ddot{u}_{go}} = \frac{-m\ddot{u}_o^t}{-m\ddot{u}_{go}} = \frac{(ku + c\dot{u})_{max}}{-m\ddot{u}_{go}}$$

$$u(t) = -\frac{m\ddot{u}_{go}}{k} \cdot R_d \cdot \sin(\omega t - \phi)$$

$$TR = \left[\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2} \right]^{1/2}$$

see plot

To keep TR small and less than 1
use small stiffness

For $r >> \sqrt{2}$, TR $\to 0$
and $\ddot{u}_o^t \to 0$

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FORCE TRANSMISSION AND VIBRATION ISOLATION

Transmissibility for harmonic excitation

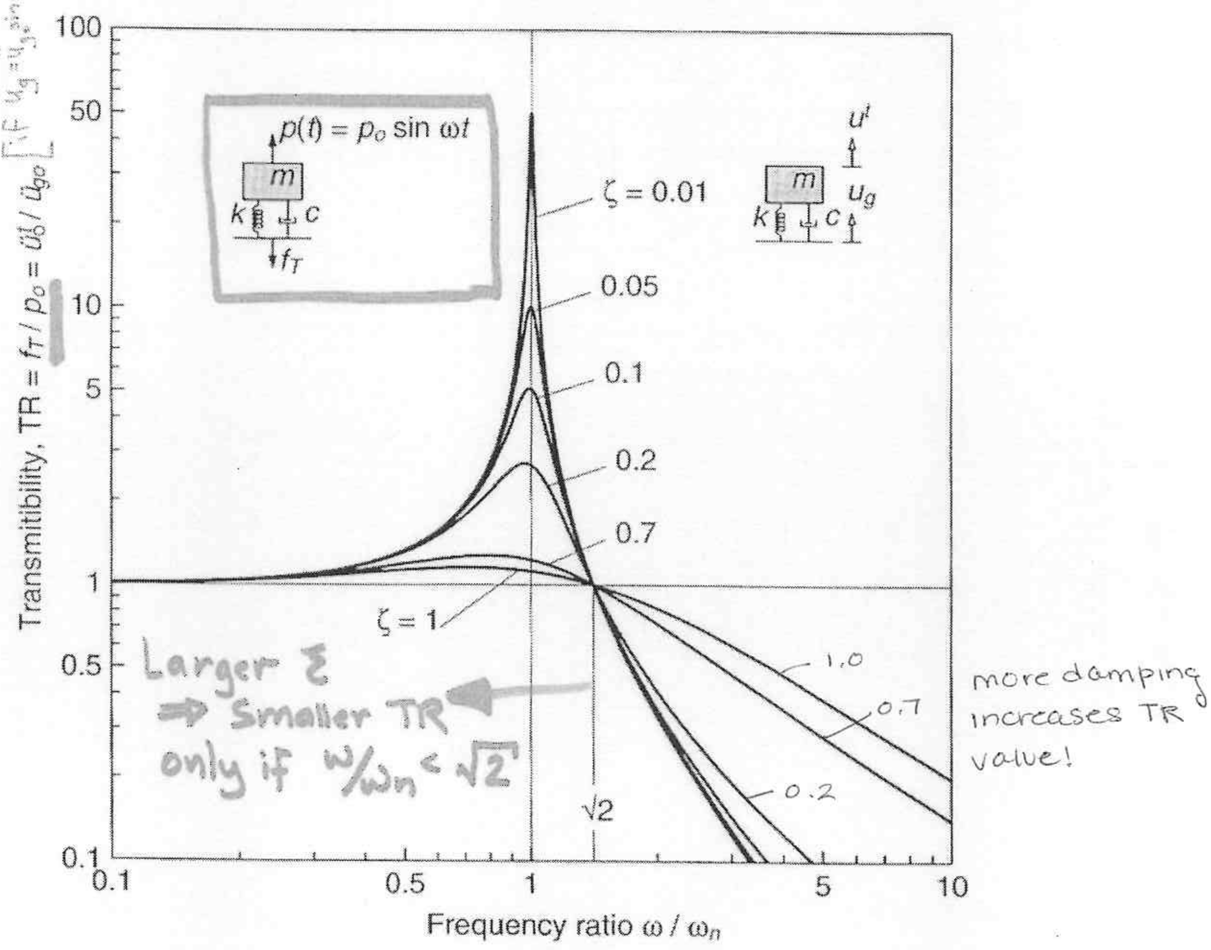


Figure 3.5.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

TR < 1 only if $\omega/\omega_n > \sqrt{2}$
⇒ make ω_n / stiffness small
Note: Damping actually increases TR for $\omega/\omega_n > \sqrt{2}$

soften the system to achieve this – can be bad.

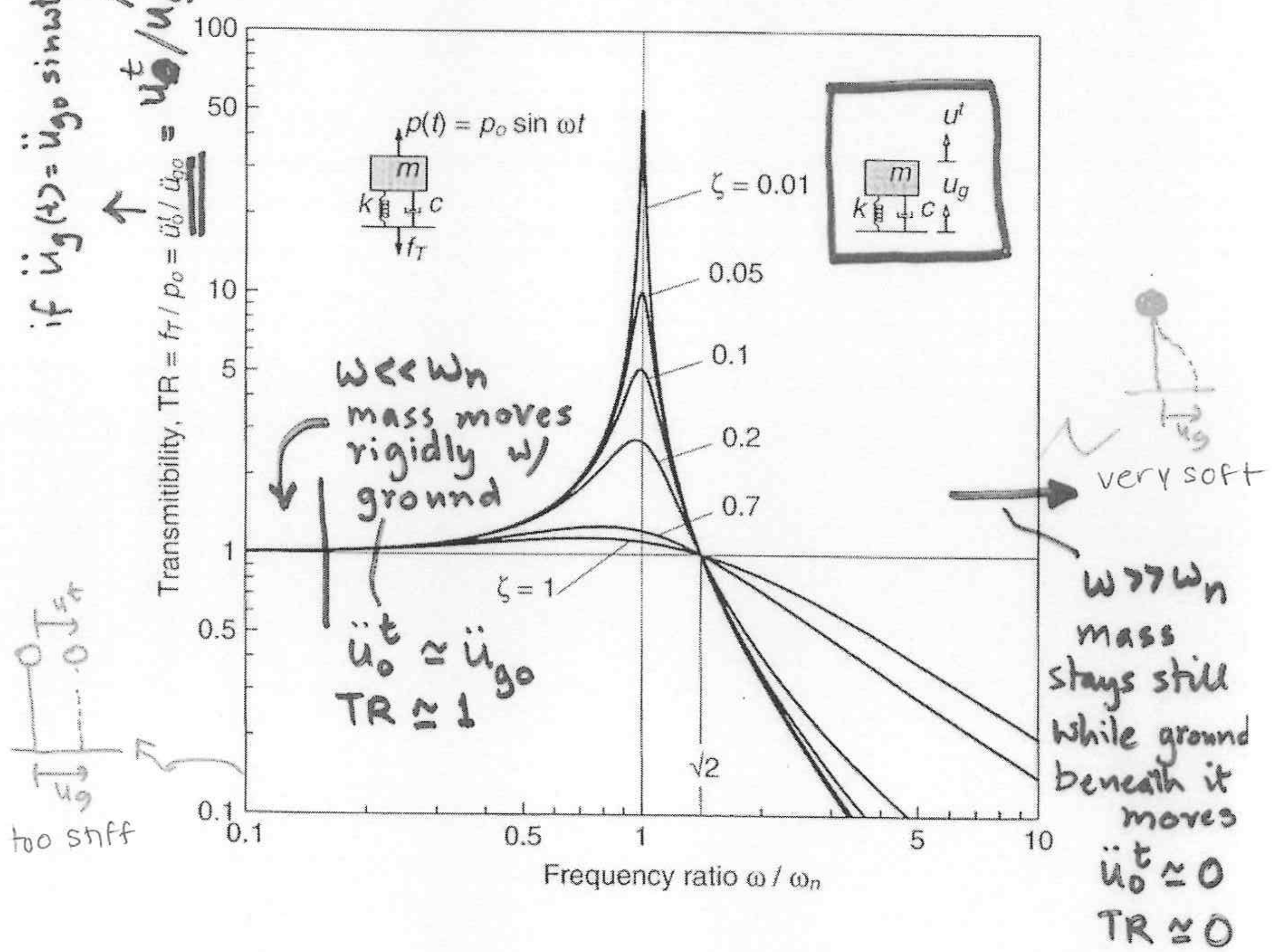
RESPONSE TO GROUND MOTION AND VIBRATION ISOLATION

if $u_g(t) = u_{go}\sin\omega t$

if $\ddot{u}_g(t) = \ddot{u}_{go}\sin\omega t$

$= u_o^t / u_{go}$

Transmissibility for harmonic excitation



$\omega << \omega_n$
mass moves rigidly w/ ground

$\ddot{u}_o^t \simeq \ddot{u}_{go}$
TR ≃ 1

too stiff

very soft

$\omega >> \omega_n$
mass stays still while ground beneath it moves

$\ddot{u}_o^t \simeq 0$
TR ≃ 0

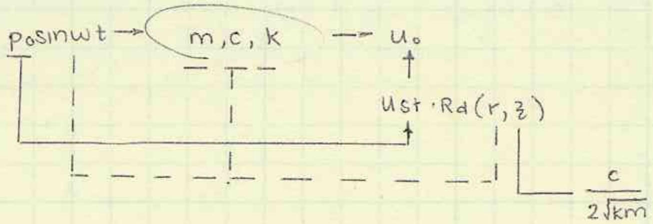
Figure 3.5.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

TR can be made small by using a very flexible support system

k small ⇒ ω_n small ⇒ ω / ω_n large

EXCITATION RESPONSE

Interaction of parts



transmissivity $TR(r, \xi)$, $r = \omega/\omega_n$

Vibration-measuring instruments

$$u(t) = -\left(\frac{1}{\omega_n^2} R_d\right) \ddot{u}_g \left(t - \phi/\omega\right)$$

time lag

$R_d, \phi \propto \omega$

ω_n^2 independent ($\propto m/k$) damn. k/m.

$u(t) = -u_{go} \sin \omega t$

independent of damping

assuming $R_a \sim 1.0$, $\phi = \pi$

accelerations

$\ddot{u}_g(t) \longrightarrow u(t)$

$\ddot{u}_g(t) = \ddot{u}_{go} \sin \omega t$

$P_{eff}(t) = -m\ddot{u}_g(t)$

$u(t) = -\frac{m}{k} \ddot{u}_{go} \cdot R_d \cdot \sin(\omega t - \phi)$

$$u(t) = \frac{-R_d}{\omega_n^2} \cdot \ddot{u}_g(t - \phi/\omega)$$

goal: make $R_d = 1.0$, $\phi = 0$

not really possible

use very high damping – in measurement device, not in a structure.

$R_d \sim 1.0$ with $\xi = 0.70$ from $0 \le \omega/\omega_n \le$ 0.5

ϕ is close enough to linear, as well

... but we don't use these machines anymore.

Chapter Six: Earthquake response

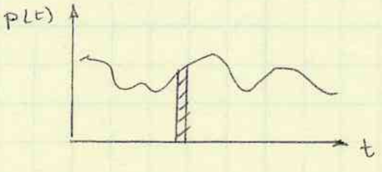
These days, our buildings are lighter (computers, CDs vs books; high quality materials)

* print and read handout 9 article

Most important chapt (with 3); we'll cover it later

Chapter Four: Response to Arbitrary, Step, and Pulse Excitations

Arbitrary excitation



small sections can be thought of as impulses

height = $1/\varepsilon$, for duration ε

as $\varepsilon \to 0$, force is large, time is short

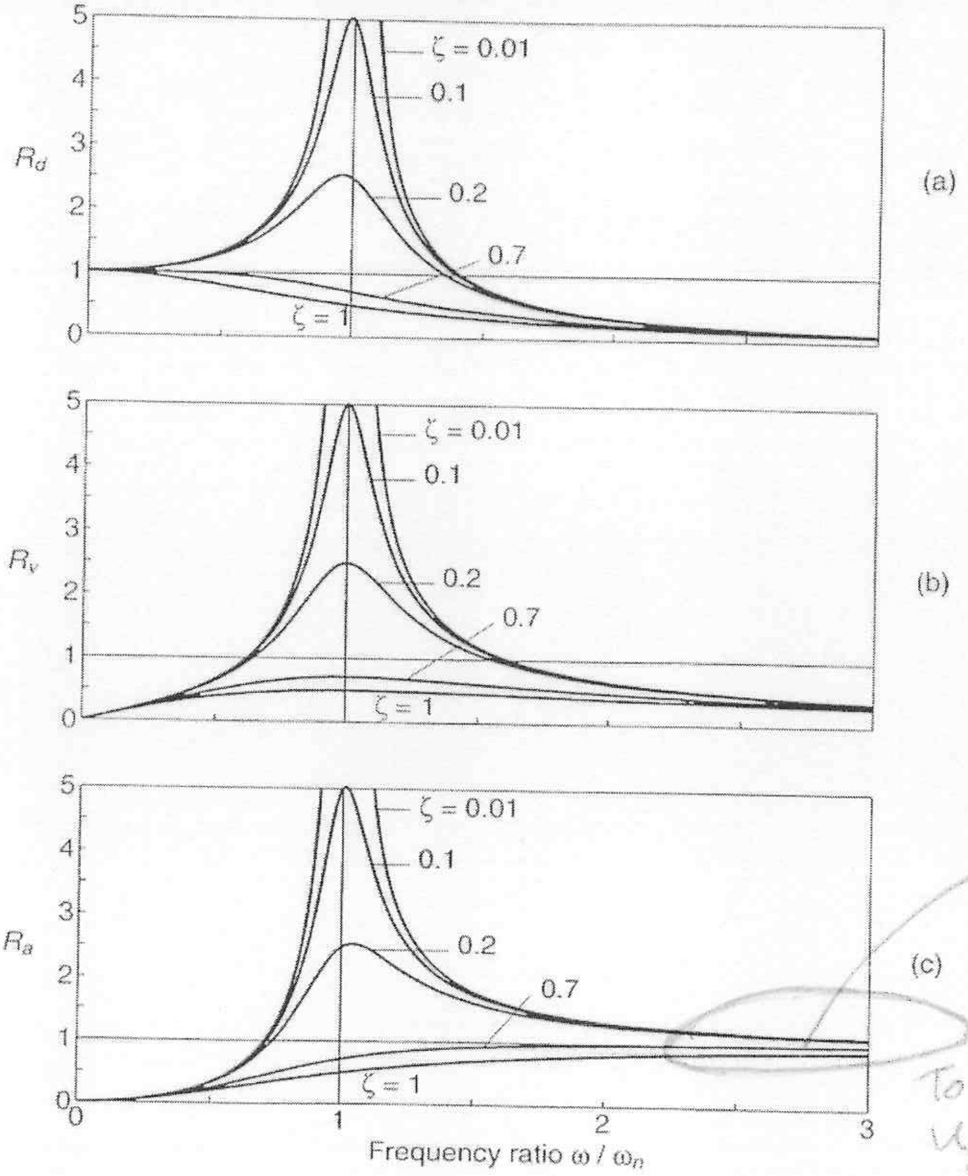
impulsive force

– magnitude = $\int p(t)\,dt = 1.0$

– provides initial conditions

Using $u(t)$ to measure ground displacement, $u_g(t)$

Deformation, velocity and acceleration response factors



$R_a \simeq 1.0$

$\phi \simeq \pi$ or 180°

To measure u_g, use $\omega/\omega_n >> 1$

ξ not important

Figure 3.2.7 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995

$u_g(t) = u_{go} \sin\omega t$

$p_{eff}(t) = m\omega^2 u_{go} \sin\omega t$

$u(t) = \frac{m\omega^2 u_{go}}{k} \cdot R_d \cdot \sin(\omega t - \phi)$

$= R_a \cdot u_{go} \cdot \sin(\omega t - \phi)$

$= -R_a \cdot u_{go} \cdot \sin\omega t = -R_a \cdot u_g(t)$ if $\omega/\omega_n >> 1$

$u(t) = -R_a\, u_g(t)$

Using u(t) to measure ground acceleration, $\ddot{u}_g(t)$

Deformation response factor and phase angle

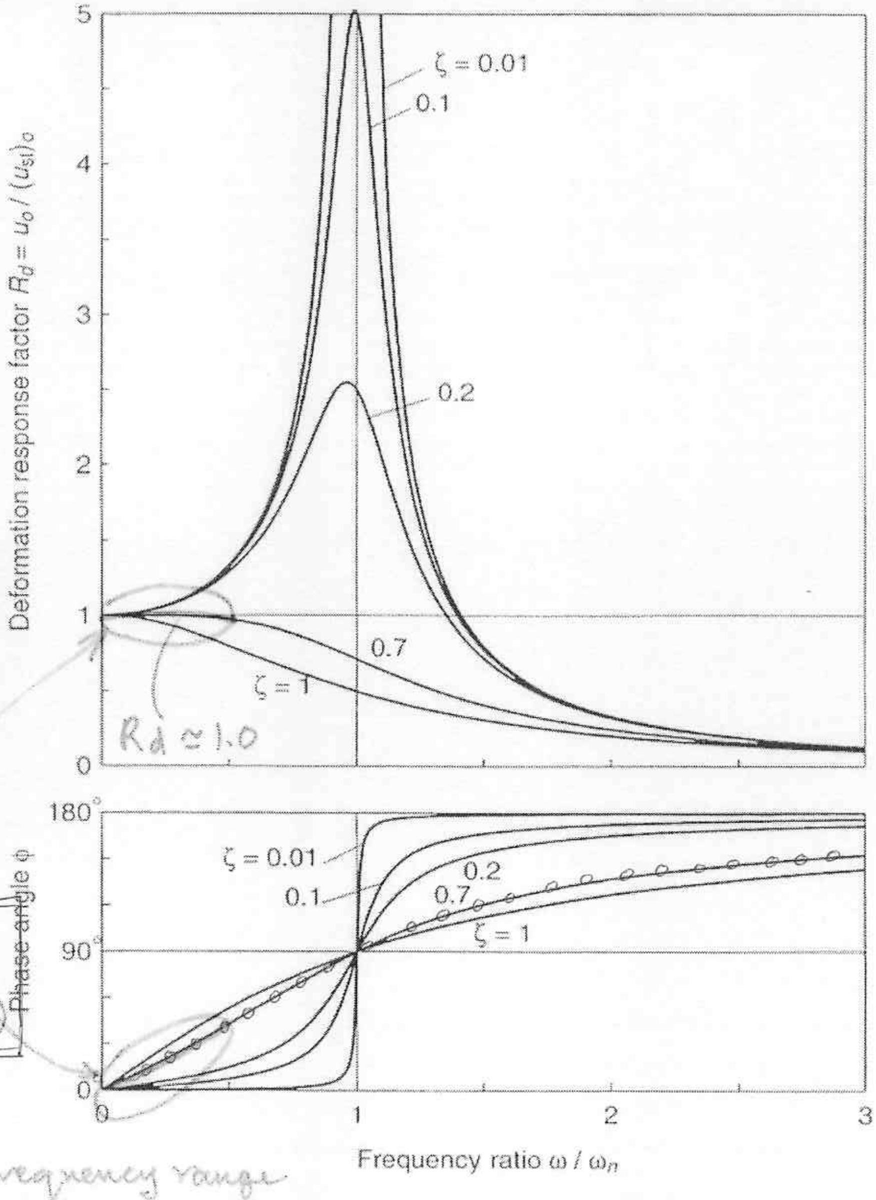


Figure 3.2.6 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

To measure $\ddot{u}_g$, use $\frac{\omega}{\omega_n} \leq 0.5$

$\xi = 0.7$

$$\boxed{u(t) = \left(-\frac{1}{\omega_n^2}\right) \cdot R_d \cdot \ddot{u}_g\left(t - \frac{\phi}{\omega}\right)}$$

make ω_n large to increase frequency range of motions that can be recorded

$$\ddot{u}_g(t) = \ddot{u}_{g_0} \sin \omega t$$

$$P_{eff}(t) = -m\ddot{u}_g(t)$$

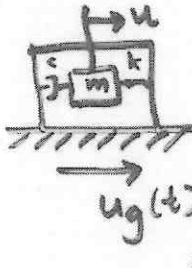
$$u(t) = -\frac{m\ddot{u}_{g_0}}{k} \cdot R_d \cdot \sin(\omega t - \phi)$$

$$= \left(-\frac{1}{\omega_n^2}\right) \cdot R_d \cdot \ddot{u}_g\left(t - \frac{\phi}{\omega}\right)$$

Given $\ddot{u}_g(t) = \ddot{u}_{go}\sin\omega t$, $\ddot{u}_g(t-\phi/\omega) = \ddot{u}_{go}\sin(\omega t-\phi)$

$$u(t) = -\frac{m\ddot{u}_{go}}{k}\cdot R_d\cdot\sin(\omega t-\phi) = -\left[\frac{1}{\omega_n^2}\cdot R_d\right]\ddot{u}_g\left(t-\frac{\phi}{\omega}\right)$$

u(t) can be used to determine $\ddot{u}_g(t)$ if $\frac{1}{\omega_n^2}\cdot R_d$ and $\frac{\phi}{\omega}$ don't vary with forcing frequency, ω



Variation of R_d and ϕ with ω/ω_n

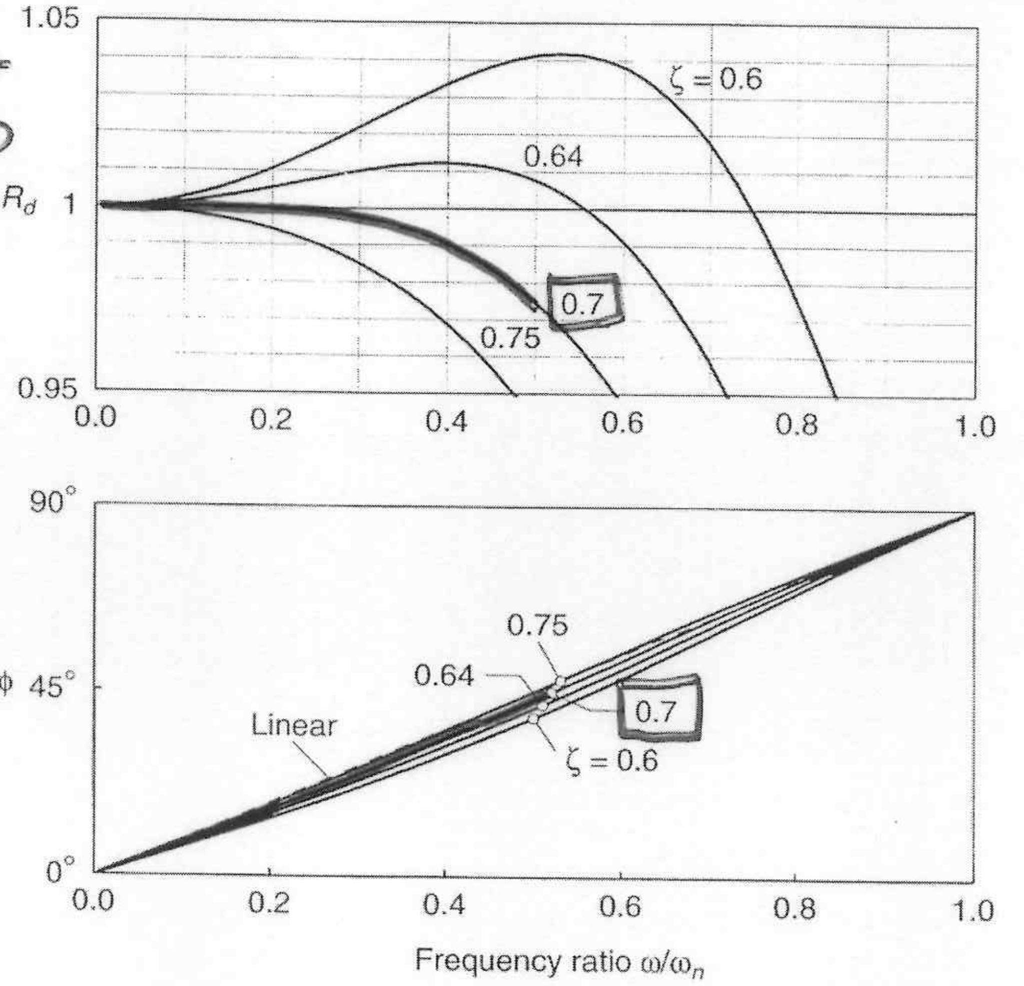


Figure 3.7.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

MEASURING GROUND ACCELERATION

Solution: Choice of $\xi = 0.7$ guarantees that for $\omega/\omega_n < 0.5$; $R_d \simeq 1.0$ and R_d, $\frac{\phi}{\omega}$ don't depend on ω

⇒ To measure motions in 0-25 Hz range, use instrument with $\xi = 0.7$, $f_n \simeq 50$ Hz

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Actual and measured motions
$f_n = 50$ Hz and $\zeta = 0.7$

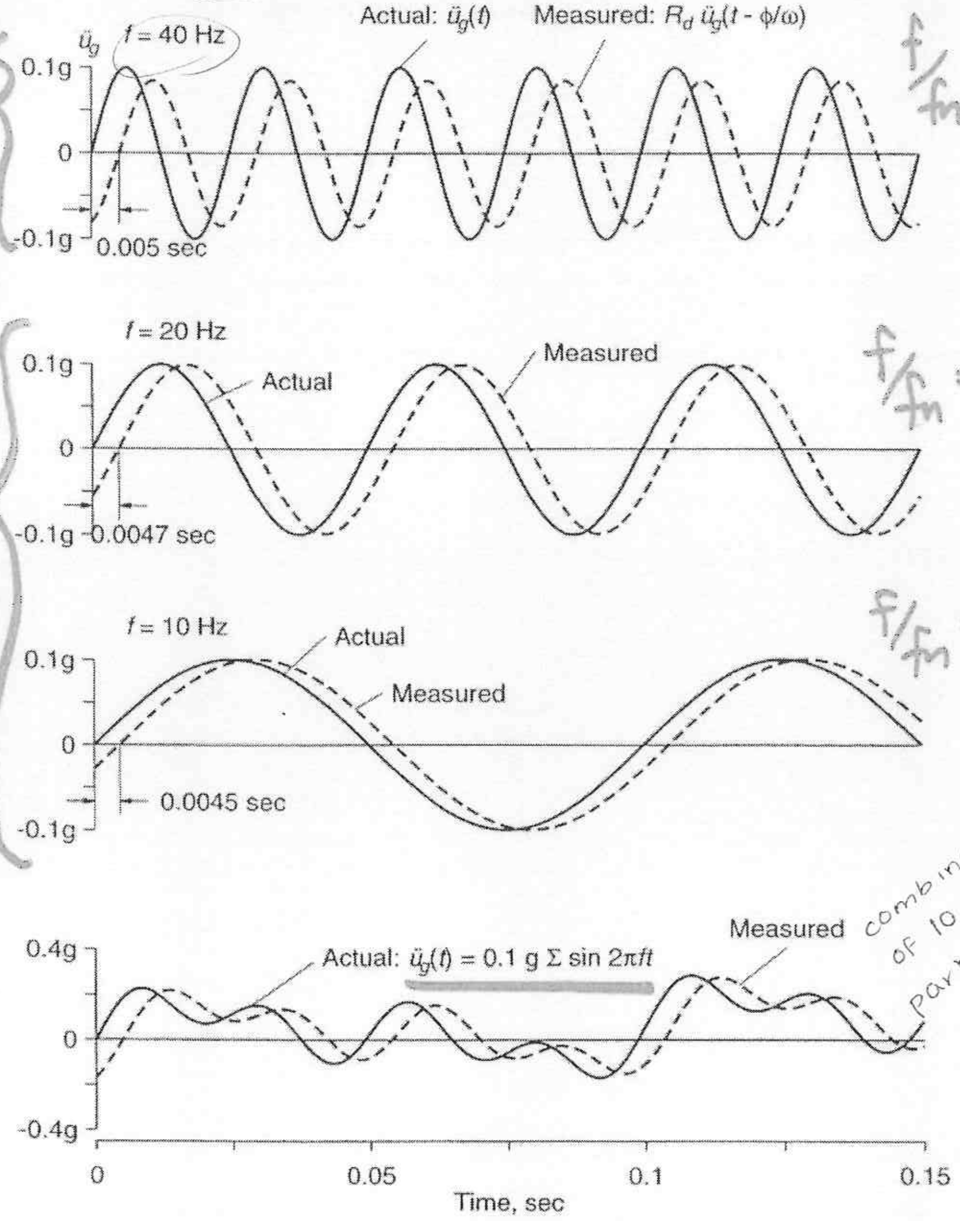

Figure 3.7.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Note smaller measured amplitude because $\frac{f}{f_n} > 0.5$

$f/f_n = 0.8 = \frac{40}{50}$

Measured Motion close to Actual Motion

$f/f_n = 0.4$

$f/f_n = 0.2$

combination of 10, 20, & 40 hz parts of excitation

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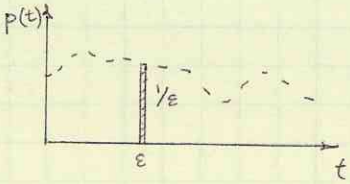
CHAPTER FOUR

Arbitrary forces



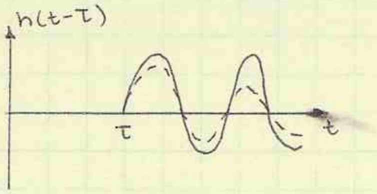
$\varepsilon = \Delta t$ from

$t = \tau$

response to impulse,

$c = 0$:

$c \neq 0$ ---



$$h(t-\tau) \equiv u(t) = \frac{1}{\omega_n m} \sin\left[\omega_n (t-\tau)\right]$$

or, damped, $$u(t) = \frac{e^{-\xi \omega_n (t-\tau)}}{m\omega_D} \sin\left[\omega_D (t-\tau)\right]$$

for $t > \tau$ (time of impulse)

- total force can be modeled with lots of impulses
- response can be computed by adding responses from each

$$u(t) = \int_0^t p(t) h(t-\tau) d\tau$$

convolution integral

∟ unit impulse response function

Duhamel's integral:

$$u(t) = \frac{1}{m\omega_D} \int_0^t p(\tau) e^{-\xi \omega_n (t-\tau)} \sin\left[\omega_D (t-\tau)\right] d\tau$$

for an undamped system,

$$u(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin\left[\omega_n (t-\tau)\right] d\tau$$

- assumed initial conditions, $u(0) = 0$, $\dot{u}(0) = 0$

more general equations on 5th page of Handout 10

Step and Ramp Forces

$$u(t) = u_{st}\left(1 - \cos\frac{2\pi t}{T_n}\right)$$

resulting max deflection is twice that of the same load applied slowly

damped: $$u(t) = u_{st}\left[1 - e^{-\xi \omega_n t}\left(\cos \omega_D t + \frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_D t\right)\right]$$

Response to Arbitrary, Step, and Pulse Excitations

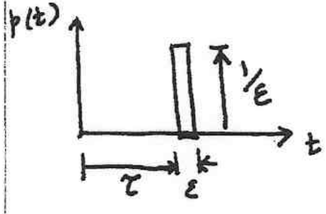
- Arbitrary Excitation

Equation of motion: $m\ddot{u} + c\dot{u} + ku = p(t)$

ICs : $u(0) = 0$; $\dot{u}(0) = 0$

Suppose $p(t) = 1/\varepsilon$ which lasts for duration ε (at $t = \tau$)

If $\varepsilon \to 0$, we have a very large force that acts for a very short time



This is an Impulsive Force

Magnitude of Impulse $= \int p(t)\,dt = 1$

since $p(t) = \delta(t-\tau)$; Dirac delta function

Newton's 2d law for a force p acting on a mass, m $\rightarrow$ $p = \frac{d}{dt}(m\dot{u}) = m\ddot{u}$

and $\int p\,dt = m\Delta\dot{u}$

This is also true for our mass-spring-damper system if the force acts for so short a time that the spring and damper have no time to respond.

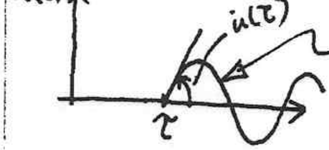
Since $\varepsilon \to 0$, this is the case here.

Since we have $\int p\,dt = 1$

$$\dot{u}(\tau) = \frac{1}{m}$$

also $u(\tau) = 0$

No force after $t = \tau$ $\Rightarrow$ Free vibration following $t = \tau$



Free vibration solution for $t > \tau$ with ICs $u(\tau)$ and $\dot{u}(\tau)$ known

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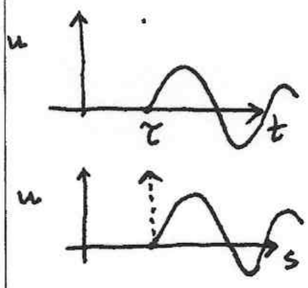
Solution: $u(s) = u(s=0)\cos\omega_n s + \frac{\dot{u}(s=0)}{\omega_n}\sin\omega_n s \qquad s\geq 0$

if $\xi=0$



$[s = t-\tau]$

$u(s=0) = u(t=\tau) = 0$

$\dot{u}(s=0) = \dot{u}(t=\tau) = \frac{1}{m}$

$$\Rightarrow u(t) = 0\cdot\cos\omega_n(t-\tau) + \frac{1}{m\omega_n}\sin\omega_n(t-\tau)$$

$$\boxed{u(t) = \frac{1}{m\omega_n}\cdot\sin\omega_n(t-\tau) \qquad t\geq\tau}$$

This is called the Unit Impulse Response Function $h(t-\tau)$

$$\boxed{h(t-\tau) = \frac{1}{m\omega_n}\sin\omega_n(t-\tau) \quad ; \quad t\geq\tau}$$

$\xi=0$

For $\xi\neq 0$

$$\boxed{h(t-\tau) = \frac{1}{m\omega_D}e^{-\xi\omega_n(t-\tau)}\sin\omega_D(t-\tau) \quad ; \quad t\geq\tau}$$

Use superposition of non-unit impulsive forces $p(\tau)d\tau$ times $h(t-\tau)$ to get total response $u(t)$

$$u(t) = \int_0^t p(\tau)h(t-\tau)d\tau$$

OR

$$\boxed{u(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)e^{-\xi\omega_n(t-\tau)}\sin\omega_D(t-\tau)\,d\tau}$$

DUHAMEL'S INTEGRAL

$$\boxed{u(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\sin\omega_n(t-\tau)d\tau \quad (\xi=0 \text{ case})}$$

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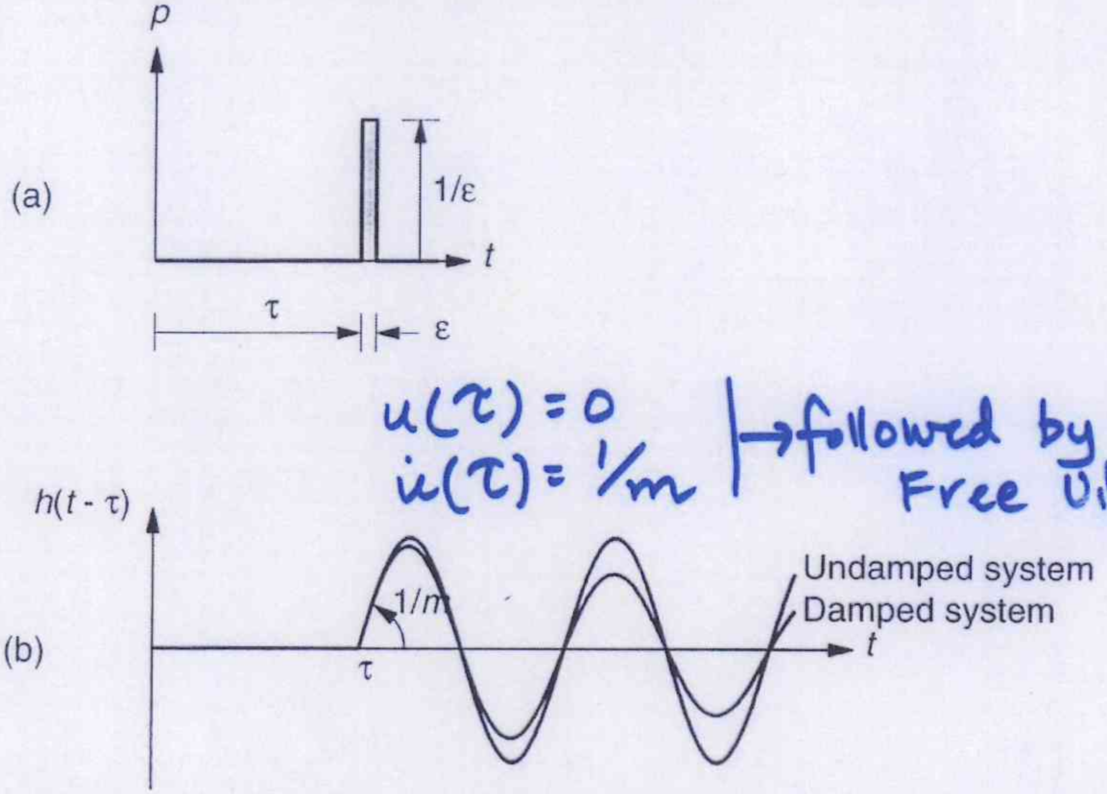
Response to unit impulse



$u(\tau) = 0$
$\dot{u}(\tau) = 1/m$ → followed by Free Vibrations

Figure 4.1.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

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Convolution integral concept

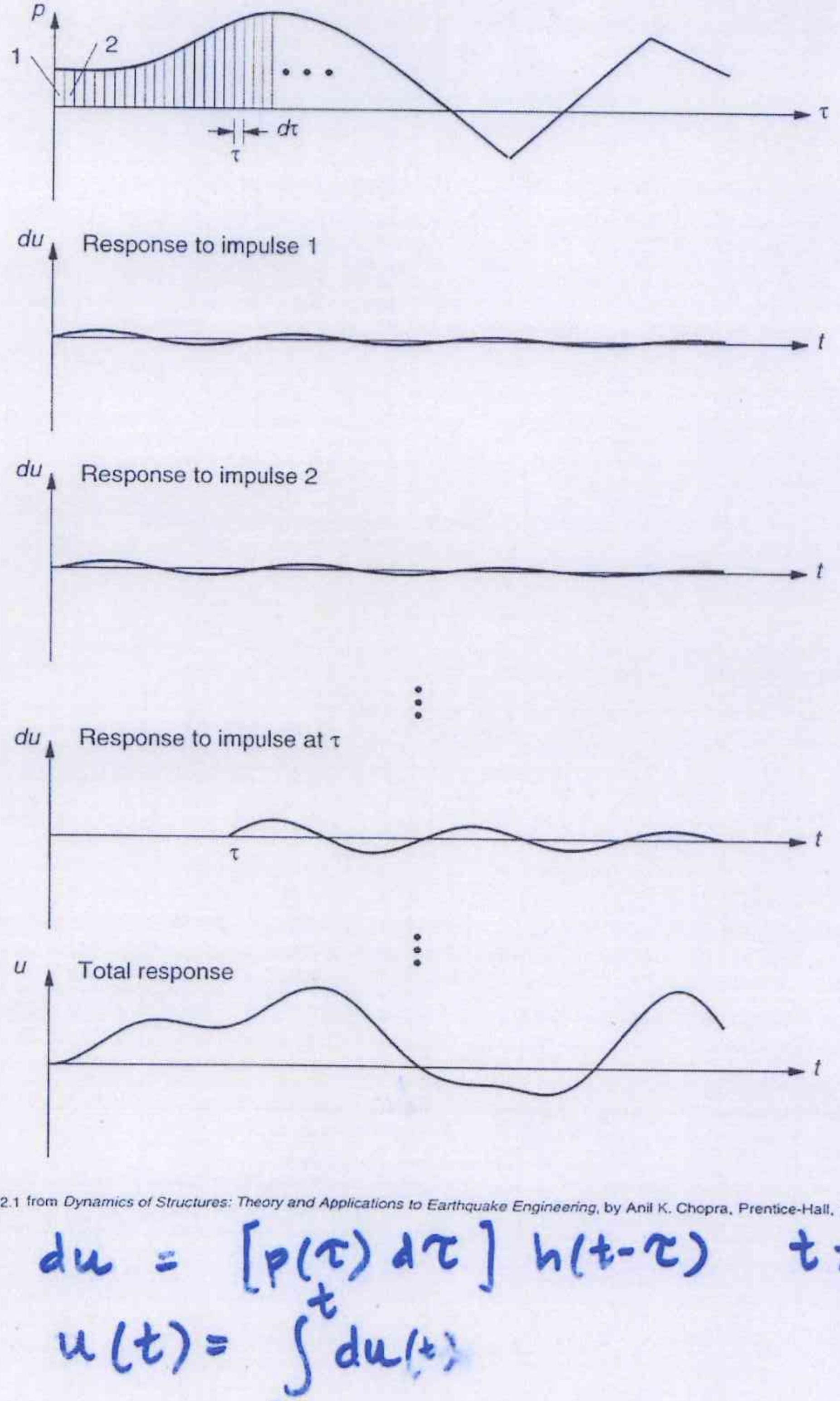

Figure 4.2.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$$du = [p(\tau)\, d\tau]\ h(t-\tau) \qquad t > \tau$$

$$u(t) = \int_0^t du(t)$$

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Force, $p(t)$ = Sequence of infinitesimally short impulses

Response to <u>each</u> impulse = magnitude of impulse × Unit Impulse Response Function

e.g., Response to impulse at time τ = $p(\tau)\,d\tau$ × $h(t-\tau)$ for $t > \tau$

TOTAL RESPONSE = $\displaystyle\int_0^t p(\tau)\,h(t-\tau)\,d\tau$

if $u(0), \dot{u}(0) = 0$

Generally,

$$u(t) = e^{-\xi\omega_n t}\left[u(0)\cos\omega_D t + \frac{\dot{u}(0) + \xi\omega_n u(0)}{\omega_D}\sin\omega_D t\right]$$

$$+ \int_0^t p(\tau)\,h(t-\tau)\,d\tau \qquad \text{for } t > 0$$

This (Duhamel's Integral) approach is

- applicable only to linear systems
- an alternative to classical method of solution (for simple analytical forms of $p(t)$, closed-form evaluation is possible)
- For complicated $p(t)$, integral can be evaluated numerically but this is inefficient relative to other methods we'll study.

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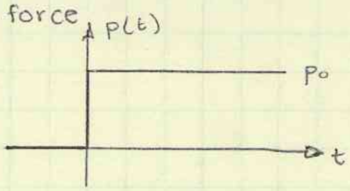
RESPONSE TO EXCITATIONS

Arbitrary excitations

$$u(t) = \int_0^t \frac{p(\tau)\,d\tau}{\omega_n m} \cdot \sin \omega_n (t-\tau)$$

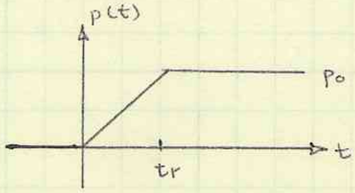
— step force



$$u(t) = u_{ST}\left[1 - e^{-\xi\omega_n t}\left(\cos\omega_D t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D t\right)\right]$$

oscillates around u_{st} with amplitude u_{st} if $\xi = 0$; else, damping occurs in time.

— ramp force



$$u(t) = u_{st}(t) - u_{st_0} \cdot \frac{\sin(2\pi t/T_n)}{2\pi\, t_r/T_n}$$

system oscillates about static solution with a period T_n

ramped in finite time:

$$p(t) = \begin{vmatrix} p_0 (t/t_r) & t \le t_r & (1) \\ p_0 & t > t_r & (11) \end{vmatrix}$$

for (1): $$u(t) = u_{st}\left[t/t_r - \frac{\sin\omega_n t}{\omega_n t_r}\right]$$

for (11): $$u(t) = u_{st}\left[1 - \frac{1}{2\pi t_r/T_n}\left[\sin\frac{2\pi t}{T_n} - \sin 2\pi\left(t/T_n - t_r/T_n\right)\right]\right]$$

maximum value:

$$u(t) = u_{st}\left[1 + \frac{|\sin(\pi t_r/T_n)|}{\pi t_r/T_n}\right]$$

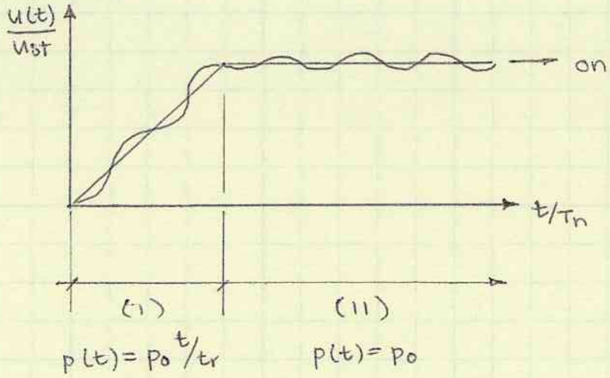
depends only on t_r/T_n or how long the ramping is compared to the natural period, T_n.

General observations:

1. in (1), oscillations with period T_n occur around static solution.
2. in (11), oscillations with period T_n also occur around static solution.
3. if t_r is small, $u(t)$ is like the step solution
4. if t_r is large, $u(t)$ is like the static solution
5. if $\omega_n t_r = 2\pi n$, $u(t_r) = u_{st}$, $\dot{u}(t_r) = 0$, and there are NO oscillations in (11).

EXCITATION RESPONSE

Ramped functions

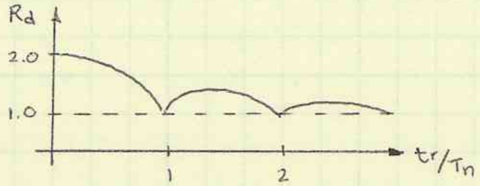


on and onand on.

(in case of no damping)

back to maximum

$$R_d = 1 + \frac{1}{\omega_n t_r}\sqrt{(1-\cos\omega_n t_r)^2 + (\sin\omega_n t_r)^2}$$



R_d moves towards 1.0

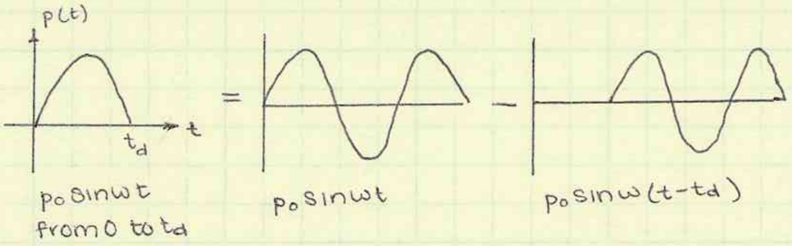
more observations:

- $t_r < T_n/4$ (short rise), $R_d = 2$, sudden force
- $t_r > 3T_n$ (slow rise), $R_d = 1$, static force
- $t_r = n$, $R_d = 1$, no oscillations in (II)

Pulse Excitations (handout 11)

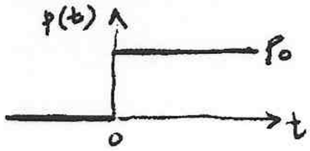
Solution methods

- classical
- Duhamel
- superposition →



Response to Step and Ramp Forces

Step Force



$$p(t) = p_0 \qquad t \geq 0$$

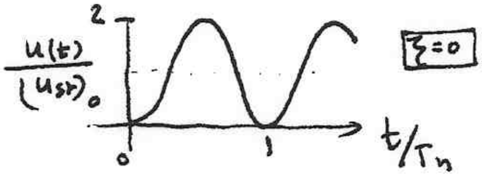
(i) Assume $\underline{c=0}$ first

$$u(t) = \int_0^t p_0 \cdot h(t-\tau)\, d\tau = \frac{p_0}{m\omega_n} \int_0^t \sin \omega_n (t-\tau)\, d\tau$$

$$= \frac{p_0}{m\omega_n^2} \cos \omega_n (t-\tau) \Big|_0^t = \frac{p_0}{k}\left[1 - \cos \omega_n t\right]$$

$$\Rightarrow \quad \frac{u(t)}{(u_{st})_0} = \left[1 - \cos\left(2\pi \frac{t}{T_n}\right)\right]$$

Oscillations between 0 and $2(u_{st})_0$ with period $= T_n$



Max. value of $u(t) = u_0 = 2(u_{st})_0$

$$\frac{du}{dt} = (u_{st})_0 \cdot \omega_n (\sin \omega_n t_m) = 0$$

$$\Rightarrow \quad \sin \omega_n t_m = 0 \quad \text{or} \quad \omega_n t_m = j\pi$$

j = integer

$$t_m = j \frac{T}{2}$$

If j is even, we get minima $\quad u = 0$

If j is odd, we get maxima $\quad u = 2(u_{st})_0$

(ii) $\underline{c \neq 0}$

$$u(t) = \int_0^t p_0 \cdot \frac{e^{-\xi\omega_n(t-\tau)}}{m\omega_D} \sin \omega_D (t-\tau)\, d\tau$$

$$= \frac{p_0}{m\omega_D} \cdot \int_0^t \sin \omega_D (t-\tau) \frac{d}{d\tau}\left[\frac{e^{-\xi\omega_n(t-\tau)}}{\xi\omega_n}\right] d\tau$$

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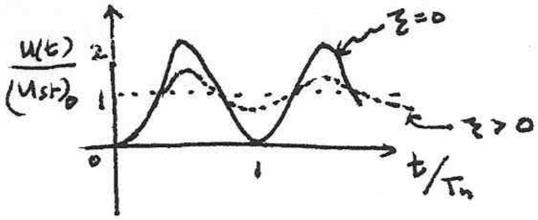
Integration by parts leads to:

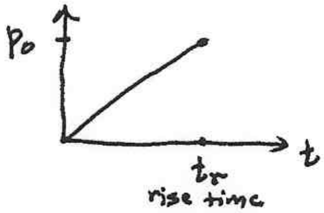
$$\frac{u(t)}{(u_{st})_0} = 1 - e^{-\xi\omega_n t}\left(\cos\omega_D t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D t\right)$$



Oscillations about $(u_{st})_0$ are with amplitude $(u_{st})_0$ if $c=0$

Oscillations about $(u_{st})_0$ are with amplitude less than $(u_{st})_0$ (when $c\neq 0$) and oscillations decay with time.

Ramp Force



$$p(t) = p_0 \cdot \frac{t}{t_r} \qquad t \geq 0$$

p_0 is small enough that the spring force is within elastic limit

Consider Undamped Case

$$u(t) = \int_0^t p_0 \cdot \frac{\tau}{t_r} \cdot \frac{1}{m\omega_n} \sin \omega_n (t-\tau)\, d\tau$$

$$= \frac{p_0}{m\omega_n t_r} \int_0^t \tau \cdot \frac{d}{d\tau}\left[\frac{\cos\omega_n(t-\tau)}{\omega_n}\right] d\tau$$

Integration by parts leads to:

$$\frac{u(t)}{(u_{st})_0} = \frac{t}{t_r} - \frac{\sin\omega_n t}{\omega_n t_r} \qquad \left[(u_{st})_0 = \frac{p_0}{k}\right]$$

OR $$\frac{u(t)}{(u_{st})_0} = \frac{(t/T_n)}{(t_r/T_n)} - \frac{\sin(2\pi t/T_n)}{2\pi t_r/T_n}$$

IF $u_{st}(t) = \frac{p(t)}{k} = (u_{st})_0 \frac{t}{t_r} = (u_{st})_0 \cdot \frac{t/T_n}{t_r/T_n}$;

$$u(t) = u_{st}(t) - (u_{st})_0 \cdot \frac{\sin(2\pi t/T_n)}{2\pi t_r/T_n}$$

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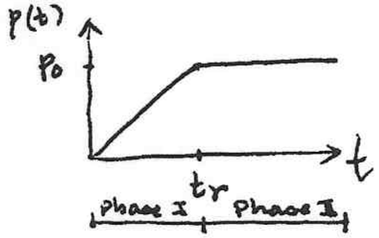
$\Rightarrow$ system oscillates about static solution with period equal to T_n.

Step Force with Finite Rise Time



$$p(t) = \begin{cases} p_0 \dfrac{t}{t_r} & t \le t_r \quad \text{(RAMP)} \\ p_0 & t > t_r \quad \text{(CONSTANT)} \end{cases}$$

Consider Undamped case $(c=0)$

Phase I : $t \le t_r$

we know $u(t) = (u_{st})_0 \left(\dfrac{t}{t_r} - \dfrac{\sin \omega_n t}{\omega_n t_r} \right)$

also $\dot{u}(t) = (u_{st})_0 \left(\dfrac{1}{t_r} - \dfrac{\cos \omega_n t}{t_r} \right)$

Phase II :

Solve using Duhamel's integral OR using results from step force solution together with free vibration resulting from conditions at end of phase I.

alternative 1 : $u(t) = \displaystyle\int_0^{t_r} p_0 \frac{t}{t_r} \cdot h(t-\tau)\,d\tau + \int_{t_r}^{t} p_0 \cdot h(t-\tau)\,d\tau$

alternative 2 : $u(t) = \overbrace{u(t_r) \cos \omega_n (t-t_r) + \dfrac{\dot{u}(t_r)}{\omega_n} \sin \omega_n (t-t_r)}^{\text{Free Vibration at end of Phase I}}$

$$+ \underbrace{(u_{st})_0 \left[1 - \cos \omega_n (t-t_r) \right]}_{\text{Step Force at } t=t_r}$$

where from phase I,

$u(t_r) = (u_{st})_0 \left[1 - \dfrac{\sin \omega_n t_r}{\omega_n t_r} \right]$

and $\dot{u}(t_r) = (u_{st})_0 \left[\dfrac{1}{t_r} - \dfrac{\cos \omega_n t_r}{t_r} \right]$

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alternative 2 is easier here

$$u(t) = (u_{st})_0 \left\{ 1 + \frac{1}{\omega_n t_r} \left[(1 - \cos\omega_n t_r) \sin \omega_n (t - t_r) - \sin \omega_n t_r \cdot \cos \omega_n (t - t_r) \right] \right\}$$

$t > t_r$

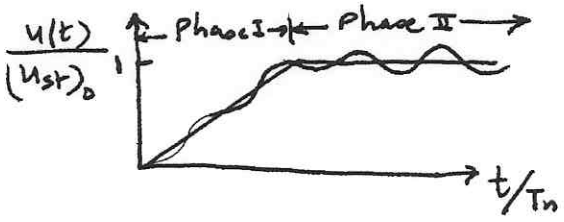
OR $$u(t) = (u_{st})_0 \left\{ 1 - \frac{1}{\omega_n t_r} \left[\sin \omega_n t - \sin \omega_n (t - t_r) \right] \right\}$$

SOLUTION:

Phase I
$t \leq t_r$
$$\frac{u(t)}{(u_{st})_0} = \frac{t/T_n}{t_r/T_n} - \frac{\sin 2\pi t/T_n}{2\pi\, t_r/T_n}$$

Phase II
$t > t_r$
$$\frac{u(t)}{(u_{st})_0} = 1 - \frac{1}{2\pi t_r/T_n} \cdot \left[\sin 2\pi t/T_n - \sin 2\pi \left(t/T_n - t_r/T_n \right) \right]$$

Note: $u(t)$ depends on ratio of t_r/T_n not on t_r or T_n ~~alone~~ separately.



Observations

1. In Phase I, oscillations with period T_n occur around static solution
2. In Phase II, oscillations with period T_n also occur around static solution
3. If t_r is SMALL, quick rise to p_0 ⇒ $u(t)$ is like step solution
4. If t_r is LARGE, slow rise to p_0 ⇒ $u(t)$ is like static solution
5. If $\omega_n t_r = 2n\pi$ (n, integer); $u(t_r) = (u_{st})_0$, $\dot{u}(t_r) = 0$ and there are NO oscillations in Phase II.

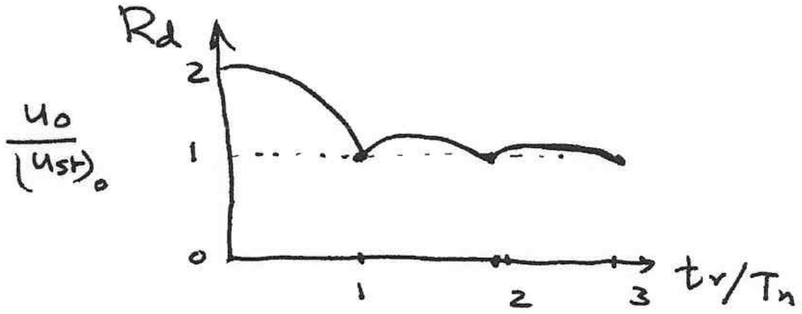
Maximum Value of $u(t)$ occurs in Phase II

$$\text{Max } u(t) = u_o = (u_{st})_o \left[1 + \frac{1}{\omega_n t_r} \sqrt{(1-\cos\omega_n t_r)^2 + (\sin\omega_n t_r)^2} \right.$$

$$R_d = \frac{u_o}{(u_{st})_o} = 1 + \frac{\left|\sin(\pi t_r/T_n)\right|}{\pi t_r/T_n}$$

Deformation Response Factor, R_d

depends on ratio t_r/T_n ; not on t_r and T_n separately.



Observations

1. If $t_r < T_n/4$ SHORT RISE TIME

   $R_d \simeq 2$ ⇒ sudden force

2. If $t_r > 3T_n$ SLOW RISE TIME

   $R_d \simeq 1$ ⇒ excitation like a static force

3. If $t_r/T_n = n$ (n, integer)

   $R_d = 1$ and no oscillations in Phase II

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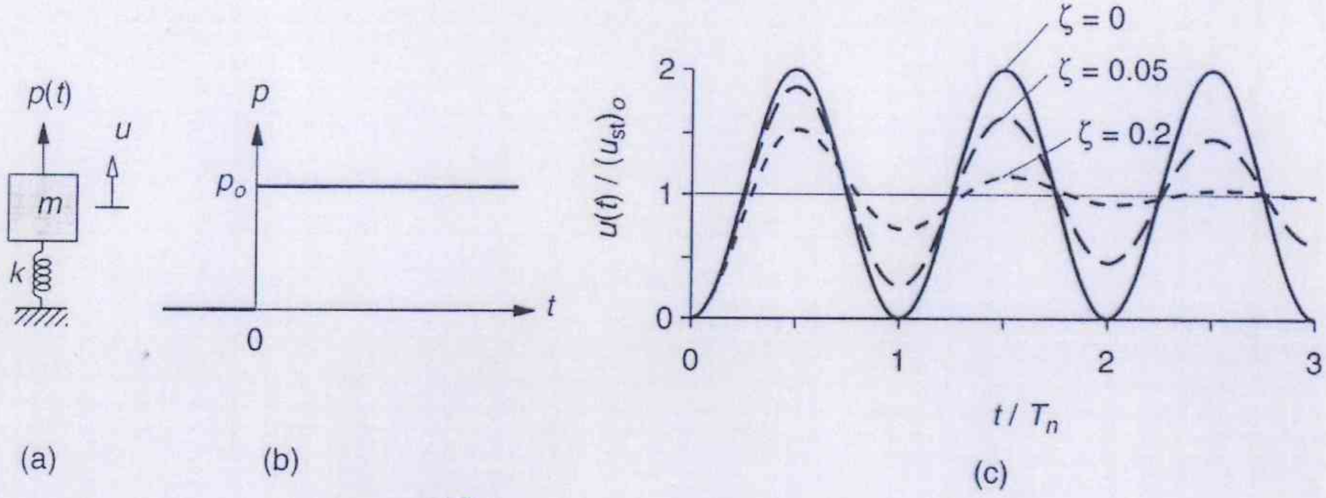
Response to step force



Undamped: $\frac{u(t)}{(u_{st})_0} = 1 - \cos 2\pi t/T_n$ $[\text{maximum} = 2.0]$

Damped: $= 1 - e^{-\xi\omega_n t}\left[\cos\omega_D t + \frac{\xi}{\sqrt{1-\xi^2}} \sin\omega_D t\right]$

Figure 4.3.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$[\text{maximum} < 2.0]$
(solution/response
decays with time)

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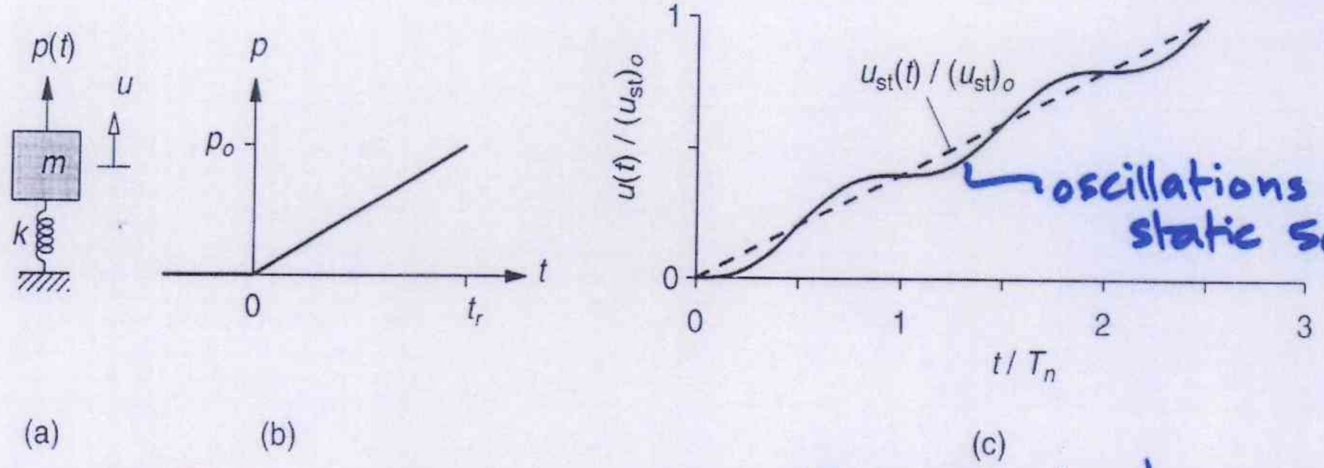
10-11

Response to ramp force

$\boxed{t_r/T_n : 2.5}$



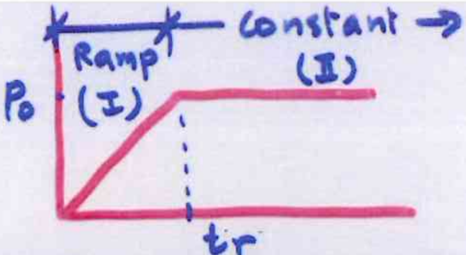
oscillations about static solution

2.5 cycles in t_r seconds

DYNAMIC STATIC

$$\frac{u(t)}{(u_{st})_o} = \frac{u_{st}(t)}{(u_{st})_o} - \frac{\sin(2\pi t/T_n)}{2\pi t_r/T_n}$$

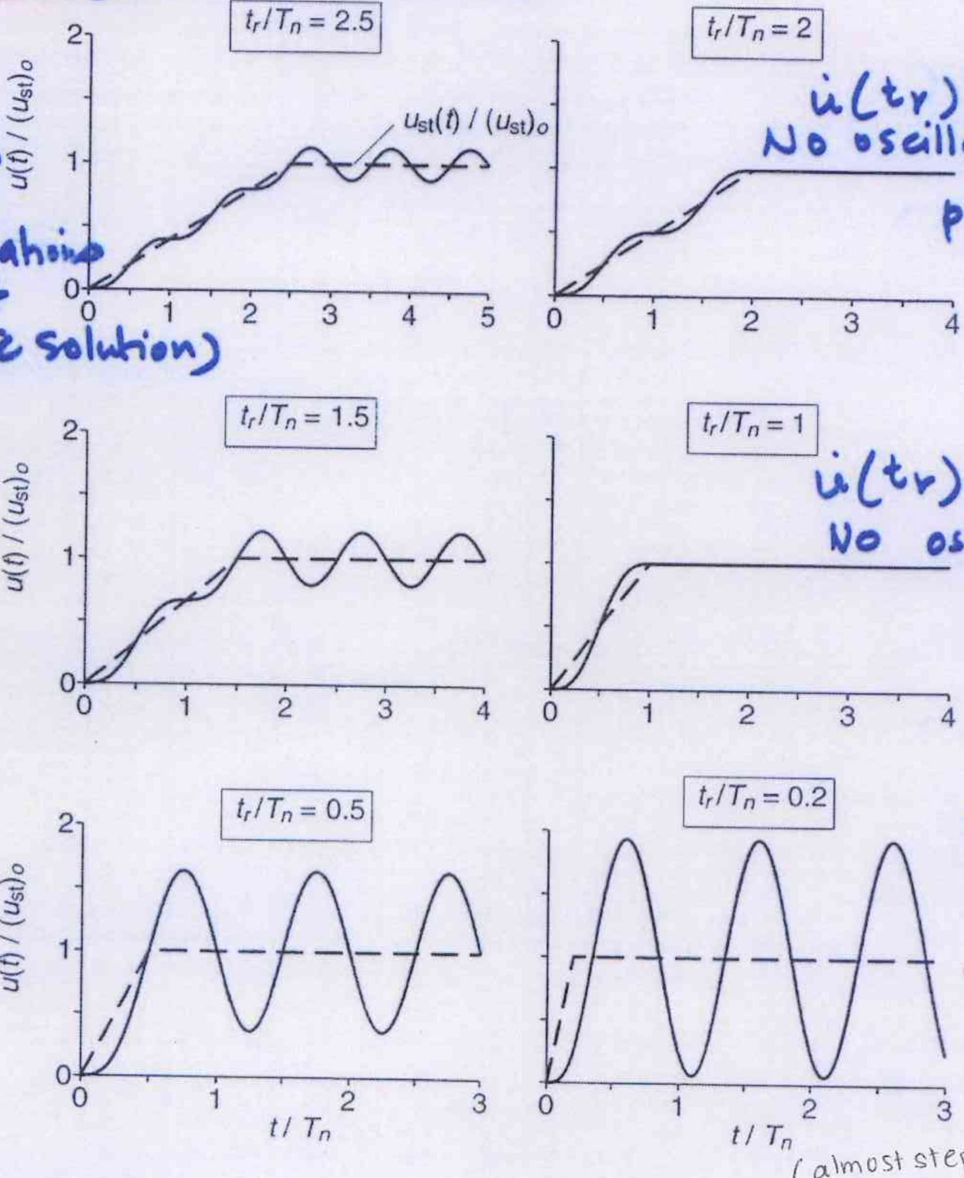
Figure 4.4.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.



Response to step force with rise time

tr large (slow rise to Po)

like static solution (small oscillations about static solution)



$\dot{u}(t_r) = 0$ No oscillations in phase II

$\dot{u}(t_r) = 0$ No oscillations in phase II

tr small quick rise to Po ⇒ Solution is like STEP solution

(almost step solution)

Figure 4.5.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

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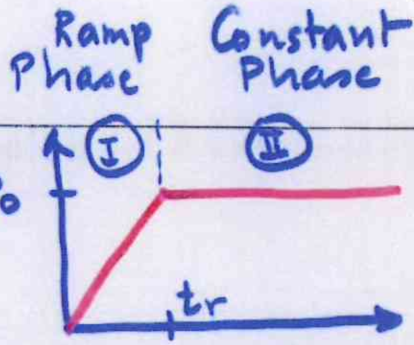
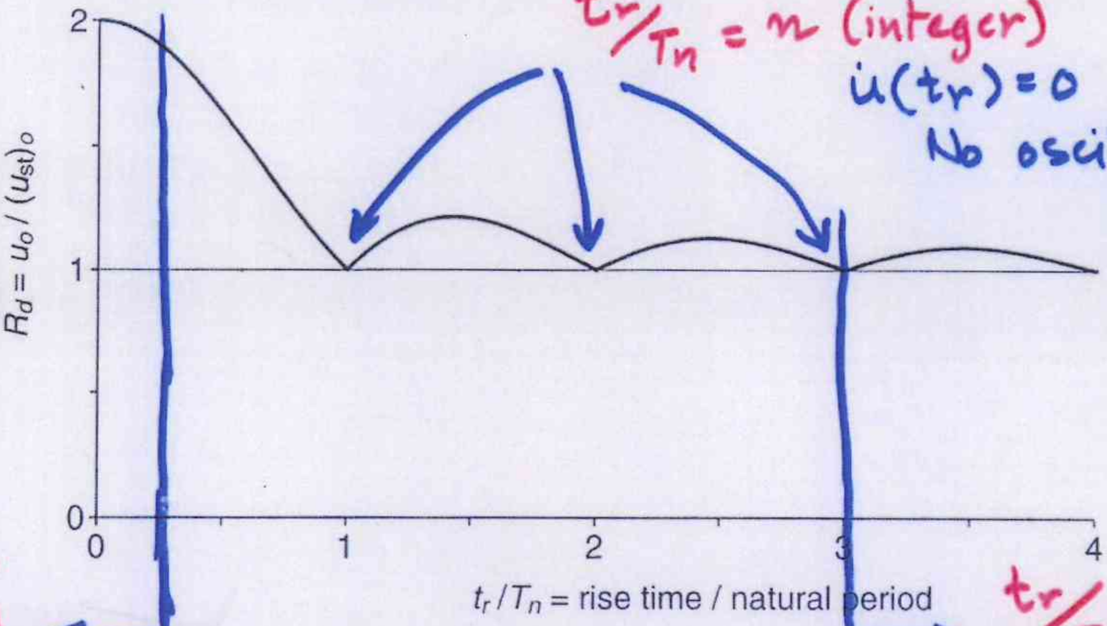


Response spectrum for step force with rise time



$t_r/T_n = n$ (integer)

$\dot{u}(t_r) = 0$

No oscillations in constant phase (Ⅱ)

$t_r/T_n < 1/4$

short Rise Time

like STEP solution.

$u_0 \doteq 2 (u_{st})_0$

$t_r/T_n > 3$

Long rise time

Excitation is like static force

$u_o \simeq (u_{st})_o$

Figure 4.5.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

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Response to Pulse Excitations

Pulse excitation : short-duration loadings

e.g., air pressure from blasts, explosions

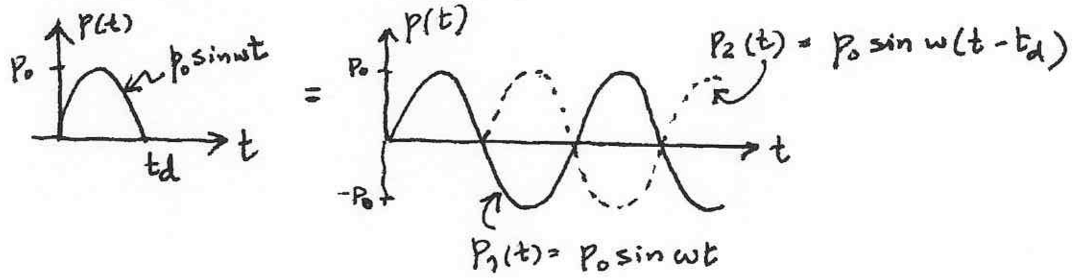
Solution methods

1. Classical method
2. Duhamel's Integral
3. Superposition of known simpler solutions

We've seen how to employ 1 and 2.

Examples of 3 (Superposition)

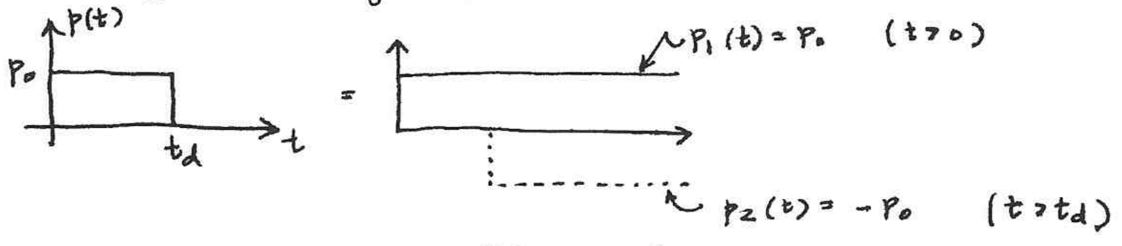
Sine Pulse of duration t_d



$$p(t) = p_1(t) + p_2(t)$$

Use harmonic excitation solution results to get $u(t)$ for pulse

Rectangular Pulse of duration t_d



$$p(t) = p_1(t) + p_2(t)$$

Use step loading solution results to get $u(t)$ for pulse

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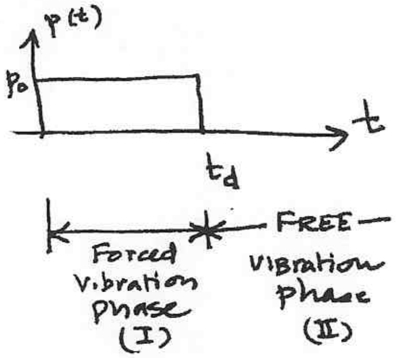
Rectangular Pulse Force (Undamped Solution)

Classical Method of solution

$$m\ddot{u} + ku = p(t) \quad ; \quad u(0) = \dot{u}(0) = 0$$

where
$$p(t) = \begin{cases} p_o & t \le t_d \\ 0 & t > t_d \end{cases}$$



In forced vibration phase (I)
we saw from step load solution

$$\underline{\underline{u(t) = (u_{st})_o \left[1 - \cos\left(2\pi \frac{t}{T_n}\right)\right]}} \quad ; \quad 0 \le t \le t_d \qquad (1)$$

At $t = t_d$, the force is removed

The system will undergo free vibration with displacement and velocity at $t = t_d$ providing "initial" conditions

From (1) $\quad \dot{u}(t) = (u_{st})_o \frac{2\pi}{T_n} \cdot \sin\left(2\pi \frac{t}{T_n}\right)$

$$\Rightarrow \boxed{\begin{aligned} u(t_d) &= (u_{st})_o \left[1 - \cos\left(2\pi \frac{t_d}{T_n}\right)\right] \\ \dot{u}(t_d) &= (u_{st})_o \cdot \left(\frac{2\pi}{T_n}\right) \cdot \sin\left(2\pi \frac{t_d}{T_n}\right) \end{aligned}} \qquad (2)$$

In Free vibration phase (II) $(t > t_d)$

$$u(t) = u(t_d) \cdot \sin \omega_n (t - t_d) + \frac{\dot{u}(t_d)}{\omega_n} \cos \omega_n (t - t_d)$$

where $u(t_d)$ & $\dot{u}(t_d)$ are given by (2).

$$\Rightarrow \left\| \begin{aligned} \frac{u(t)}{(u_{st})_o} &= (1 - \cos \omega_n t_d) \cos \omega_n (t - t_d) + \sin \omega_n t_d \cdot \sin \omega_n (t - t_d) \; ; \quad t > t_d \\ &= \cos \omega_n (t - t_d) - \cos \omega_n t \\ &= 2 \cdot \sin\left(\pi \frac{t_d}{T_n}\right) \cdot \sin\left[2\pi\left(\frac{t}{T_n} - \frac{1}{2}\frac{t_d}{T_n}\right)\right] \; ; \quad t > t_d \end{aligned} \right. \qquad (3)$$

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In Phases I and II, the normalized response $\frac{u(t)}{(u_{st})_0}$ when plotted versus $\frac{t}{T_n}$ is seen to depend on the ratio, $\frac{t_d}{T_n}$ not separately on t_d or T_n

Plots of normalized response show:

- In Phase I, oscillations at period T_n about static solution, $(u_{st})_0$ (See Eq. (1))
- In Phase II, oscillations at period T_n about original equilibrium, $u=0$ (See Eq. (3))
- because force is applied suddenly, dynamic response is usually larger than static response
- If $\frac{t_d}{T_n} = 1,2,3....$ $u(t_d) = 0$; $\dot{u}(t_d) = 0$ (See Eq. (2))
  and $u(t) = 0$ in Phase II
  i.e. no motion after force is removed
  if $t_d = j\,T_n$ (where $j = 1,2,3,....$)
- In Phase I, response can rise to a maximum value of $(u_{st})_0 \times 2$ if $t/T_n = 1/2$ is possible sometime before force is removed (See Eq. (1))
  If $t_d/T_n \geq 1/2$, $\max = 2(u_{st})_0$
  If $t_d/T_n < 1/2$, $\max = u(t_d)$
  $= (u_{st})_0\left[1 - \cos\frac{2\pi t_d}{T_n}\right]$
- In Phase II,
  $\max = 2 \cdot \sin\left(\frac{\pi t_d}{T_n}\right)$ (see Eq. (3))
  For any t_d value

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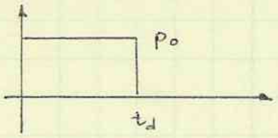
ARBITRARY LOADS

Test 3/2, 9:30am
- chapters 1-4
- bring one page notes

Short-term loads



Phase I – step force solution
Phase II – free vibration solution

(I) $u(t) = u_{st}\left[1-\cos\left(2\pi t/T_n\right)\right]$, $t \leq t_d$

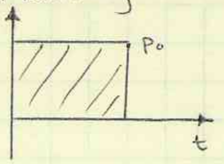
(II) $u(t) = u_{st}\left[2\sin\left(\pi t_d/T_n\right)\sin\left(2\pi\left(t/T_n - t_d/2T_n\right)\right)\right]$, $t > t_d$

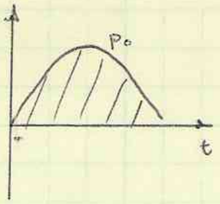
where is max value? in (I) or (II)?
depends on t_d, T_n; specifically, t_d/T_n
if $t_d < T_n/2$, forced vibration will not reach a peak (at $\omega_n T_n/2 = \pi$)

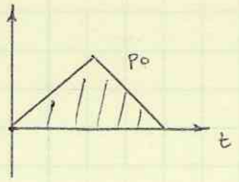
- if $t_d/T_n \geq 1/2$, max = u_{st} } in Phase I
- if $t_d/T_n < 1/2$, max = $u(t_d)$ } in Phase I
- at $t > t_d$, max = $2\sin(\pi t_d/T_n)$ in Phase II for any t_d

Short pulse – look at free vibration
Long pulse – look during pulse

Types of loading







rectangle is the worst, as load gets to p_o in zero time, then doesn't lessen.

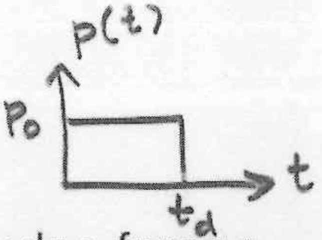
$R_d = \frac{u}{u_{st}} = 2.0$ for □

Effect of damping
- just makes math messier
- where harmonic excitation drops significantly, pulse excitation doesn't change much
- neglecting damping is conservative

Phase

I $t \leq t_d$

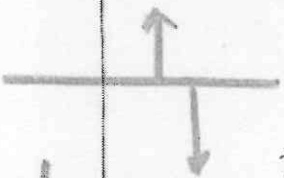
II $t > t_d$



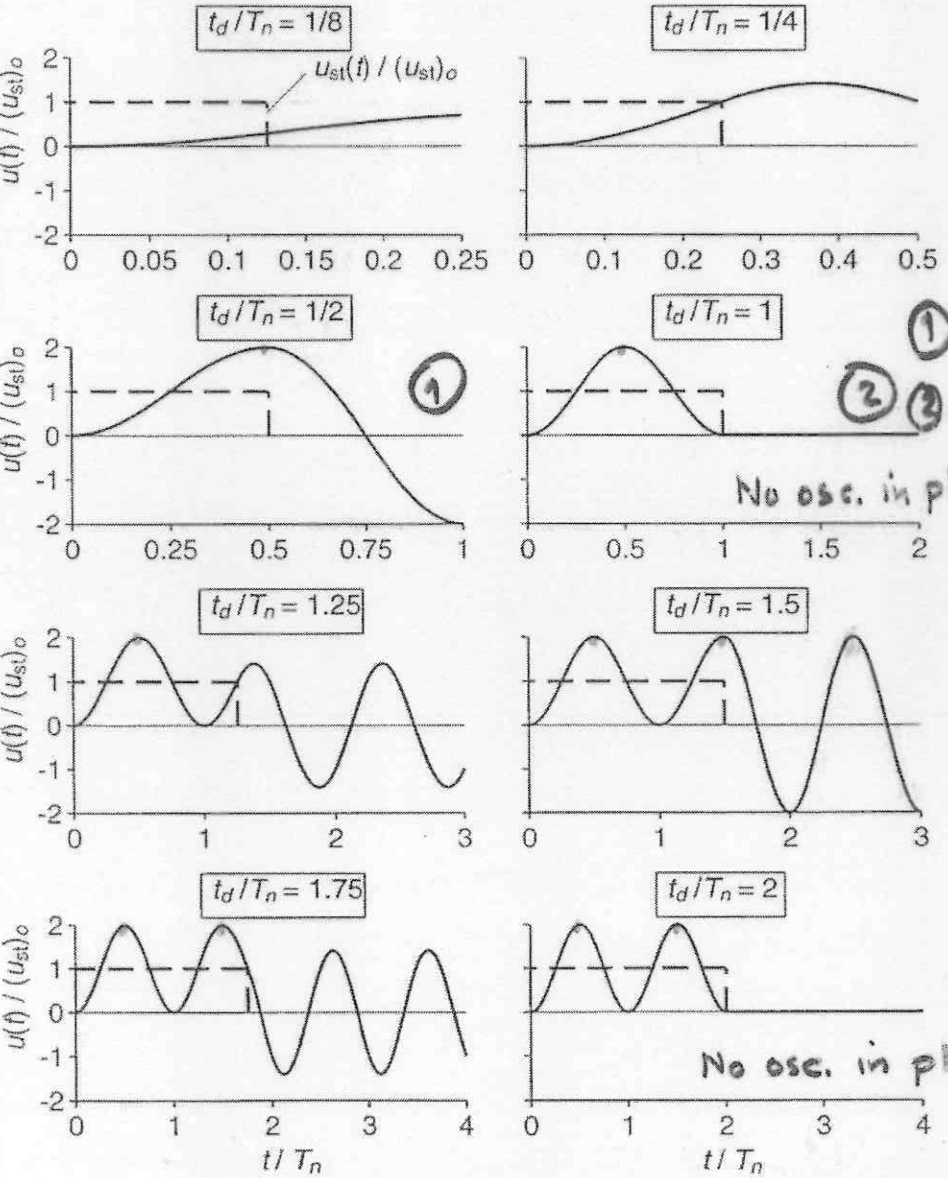
Response to rectangular pulse forces

Maximum

$t_d < T_n/2$



$t_d \geq T_n/2$



Phase II

① At $t = t_d$ and

② ③ in Phase I

No osc. in phase II

Phase II

Phase I

No osc. in phase II

Figure 4.7.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

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Phase Response

I $u(t) = (u_{st})_o\left[1 - \cos\frac{2\pi t}{T_n}\right]$

II $u(t) = (u_{st})_o\left[2\sin\frac{\pi t_d}{T_n}\right]\sin\left[2\pi\left(\frac{t}{T_n} - \frac{1}{2}\frac{t_d}{T_n}\right)\right]$

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Summarizing Maxima (u_o)

	$t_d < T_n/2$	$t_d \geq T_n/2$
Phase I max (Forced vibration)	$(u_{st})_o \left[1 - \cos 2\pi \frac{t_d}{T_n}\right]$	$2(u_{st})_o$
Phase II max (Free vibration	$2 \sin \pi \frac{t_d}{T_n} \cdot (u_{st})_o$	$2 \sin \pi \frac{t_d}{T_n} \cdot (u_{st})_o$
OVERALL MAXIMUM = max{Phase I max, Phase II max}	$2 \sin \pi \frac{t_d}{T_n} \cdot (u_{st})_o$ [Short Pulse ⇒ Max in Phase II]	$2(u_{st})_o$ [Long Pulse ⇒ Max in Phase I]

R_d = Dynamic Response Factor $= \dfrac{u_o}{(u_{st})_o}$

$$R_d = \begin{cases} 2\sin\left(\pi \frac{t_d}{T_n}\right) & ; \quad t_d/T_n < 1/2 \\ 2 & ; \quad t_d/T_n \geq 1/2 \end{cases}$$

R_d versus t_d/T_n ≡ shock spectrum can be employed for any rectangular pulse loading with any amplitude and duration.

Maximum Displacement, $u_o = (u_{st})_o \cdot R_d = \dfrac{p_o}{k} \cdot R_d$

To get internal forces and stresses, first find maximum equivalent static force $f_{s_o} = k u_o = p_o \cdot R_d$

Then, perform static analysis with force $= f_{s_o}$

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Shock spectrum for rectangular pulse force

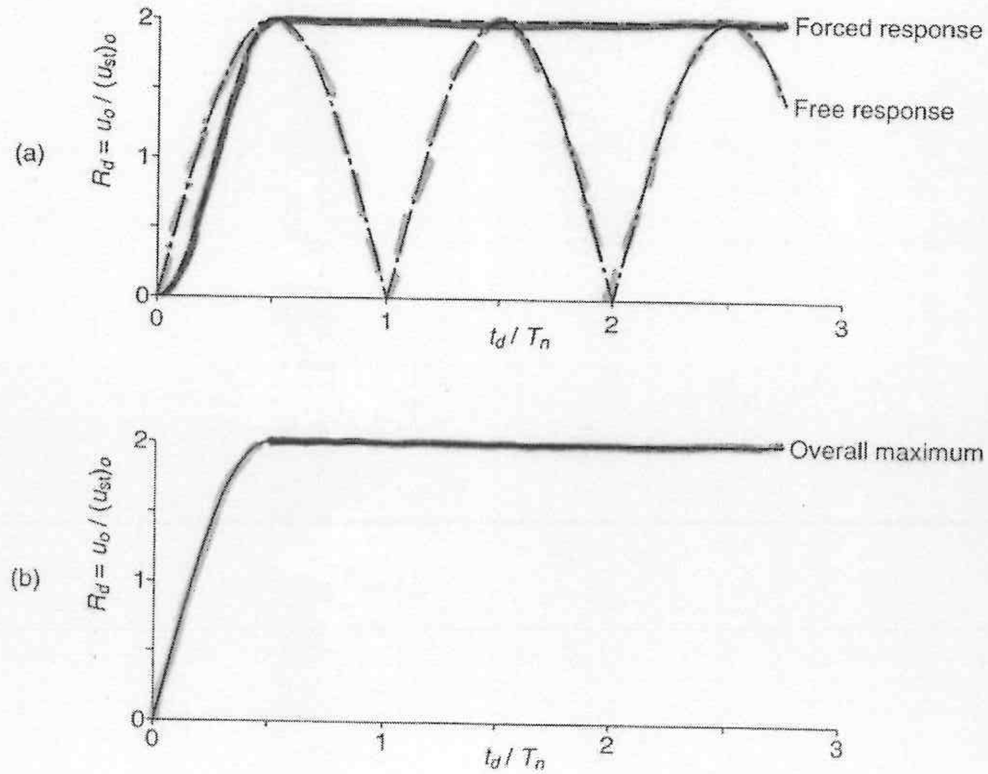

Figure 4.7.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

[redacted]		R_d (force phase)	R_d (free vib phase)	R_d (overall)
$\frac{t_d}{T_n} \le \frac{1}{2}$	Ⓐ	$1-\cos\left(2\pi \frac{t_d}{T_n}\right)$	$\Big\} \; 2\left\|\sin \frac{\pi t_d}{T_n}\right\|$	$2\sin \frac{\pi t_d}{T_n}$
$\frac{t_d}{T_n} > \frac{1}{2}$	Ⓑ	2		2

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Example 4.1

A one-story building, idealized as a 12-ft-high frame with two columns hinged at the base and a rigid beam, has a natural period of 0.5 sec. Each column is an American standard wide-flange steel section W8 × 18. Its properties for bending about its major axis are $I_x = 61.9$ in^4, $S = I_x/c = 15.2$ in^3; $E = 30{,}000$ ksi. Neglecting damping, determine the maximum response of this frame due to a rectangular pulse force of amplitude 4 kips and duration $t_d = 0.2$ sec. The response quantities of interest are displacement at the top of the frame and maximum bending stress in the columns.

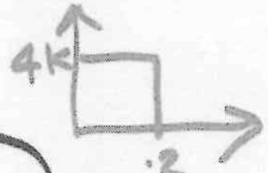
Solution

1. *Determine R_d.*

$$\frac{t_d}{T_n} = \frac{0.2}{0.5} = 0.4 < \frac{1}{2}$$

$$R_d = \frac{u_o}{(u_{st})_o} = 2\sin\frac{\pi t_d}{T_n} = 2\sin(0.4\pi) = 1.902$$

$P_o = 4K$



OR use Fig 4.7.3

2. *Determine the lateral stiffness of the frame.*

$$k_{col} = \frac{3EI}{L^3} = \frac{3(30{,}000)61.9}{(12\times 12)^3} = 1.865 \text{ kips/in.}$$

$$k = 2 \times 1.865 = 3.73 \text{ kips/in.}$$

3. *Determine $(u_{st})_o$.*

$$(u_{st})_o = \frac{p_o}{k} = \frac{4}{3.73} = 1.07 \text{ in.} \checkmark$$

4. *Determine the maximum dynamic deformation.*

$$u_o = (u_{st})_o R_d = (1.07)(1.902) = 2.04 \text{ in.} \checkmark$$

5. *Determine the bending stress.* The resulting bending moments in each column are shown in Fig. E4.1c. At the top of the column the bending moment is largest and is given by

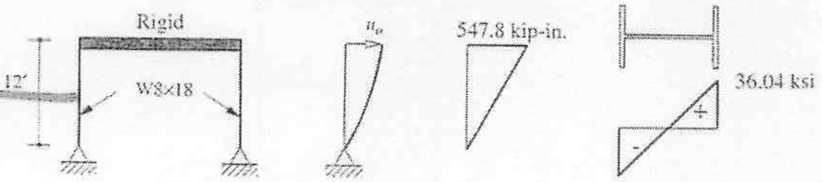
$$M = \frac{3EI}{L^2}u_o = \left[\frac{3(30{,}000)61.9}{(12\times 12)^2}\right]2.04 = 547.8 \text{ kip-in.}$$

Alternatively, we can find the bending moment from the equivalent static force:

$$f_{So} = p_o R_d = 4(1.902) = 7.61 \text{ kips}$$

$$= k u_o$$

$$\frac{3EI}{L^3}$$



Because both columns are identical in cross section and length, the force f_{So} will be shared equally. The bending moment at the top of the column is

$$M = \frac{f_{So}}{2}h = \left(\frac{7.61}{2}\right)12 \times 12 = 547.8 \text{ kip-in.}$$

The bending stress is largest at the outside of the flanges at the top of the columns:

$$\sigma = \frac{M}{S} = \frac{547.8}{15.2} = 36.04 \text{ ksi}$$

The stress distribution is shown in Fig. E4.1d.

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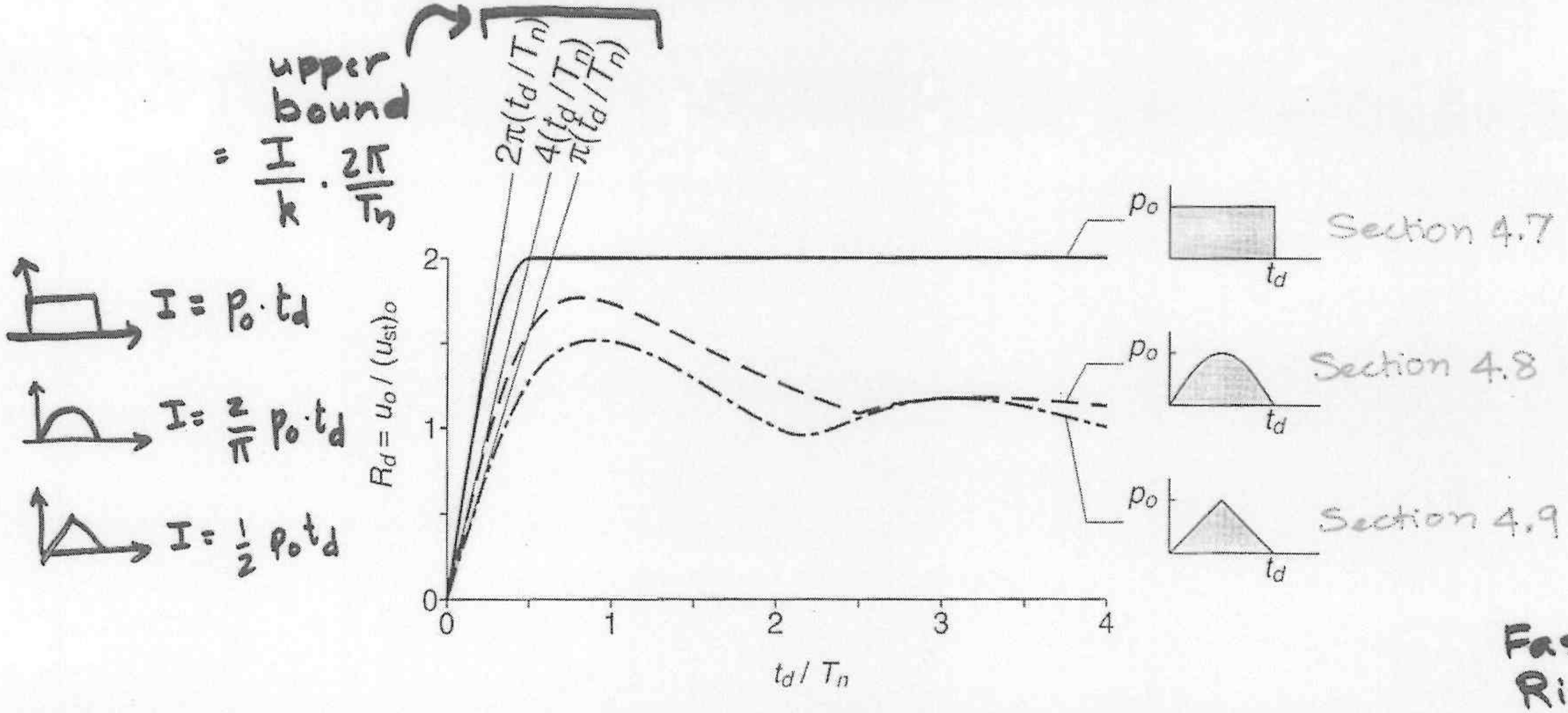
Shock spectra for force pulses of equal amplitude



$I = p_o \cdot t_d$

$I = \frac{2}{\pi} p_o \cdot t_d$

$I = \frac{1}{2} p_o t_d$

Figure 4.10.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Faster Rise to Po

→ Larger Response

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Pulse Shape Effects

Faster rise to p_o ⇒ Larger Dynamic Response

Rectangular Pulse (response) > Sine Pulse resp. > Triangular Pulse resp.

When t_d is very small

e.g. $t_d < T_n/2$

overall maximum occurs in free vibration phase

i.e., after load is removed

⇒ Response is controlled by time integral of the pulse

Limiting Case: $\dfrac{t_d}{T_n} \to 0$

Pulse becomes an IMPULSE

$$\text{Magnitude of Impulse} = I = \int_0^{t_d} p(t)\,dt$$

$$\Rightarrow \text{Response} = I \times \text{Unit Impulse Response}$$

$$u(t) = I \times \frac{1}{m\omega_n} \cdot \sin \omega_n t$$

$$\text{Maximum } u_o = \frac{I}{m\omega_n} = \frac{I}{k} \cdot \frac{2\pi}{T_n}$$

Rectangular Pulse $I = p_o t_d$

Sine Pulse $I = (2/\pi)\, p_o t_d$

Triangular Pulse $I = (1/2)\, p_o t_d$

↓ Decreasing I

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Approximate Analysis for Short Pulses

When $\frac{t_d}{T_n} < \frac{1}{4}$,

Maximum deformation is controlled by the Pulse Area (NOT by the Pulse shape)

Consider

Rectangular Pulse of amplitude $p_0/2$

Sine (half-cycle) pulse of amplitude $\pi/4\, p_0$

Triangular pulse of amplitude p_0

All have same area $= \int_0^{t_d} p(t)dt = \frac{1}{2} p_0 t_d$

Replotting Shock spectra for these pulses shows:

VERY SHORT PULSES | For $t_d/T_n < 1/4$, $\frac{u_0}{(p_0/k)}$ for all 3 shapes is the same.

LONG PULSES | Note for $\frac{t_d}{T_n} > 1/2$, our "impulse" approximation doesn't work since the maximum occurs during the pulse nor after the pulse is removed

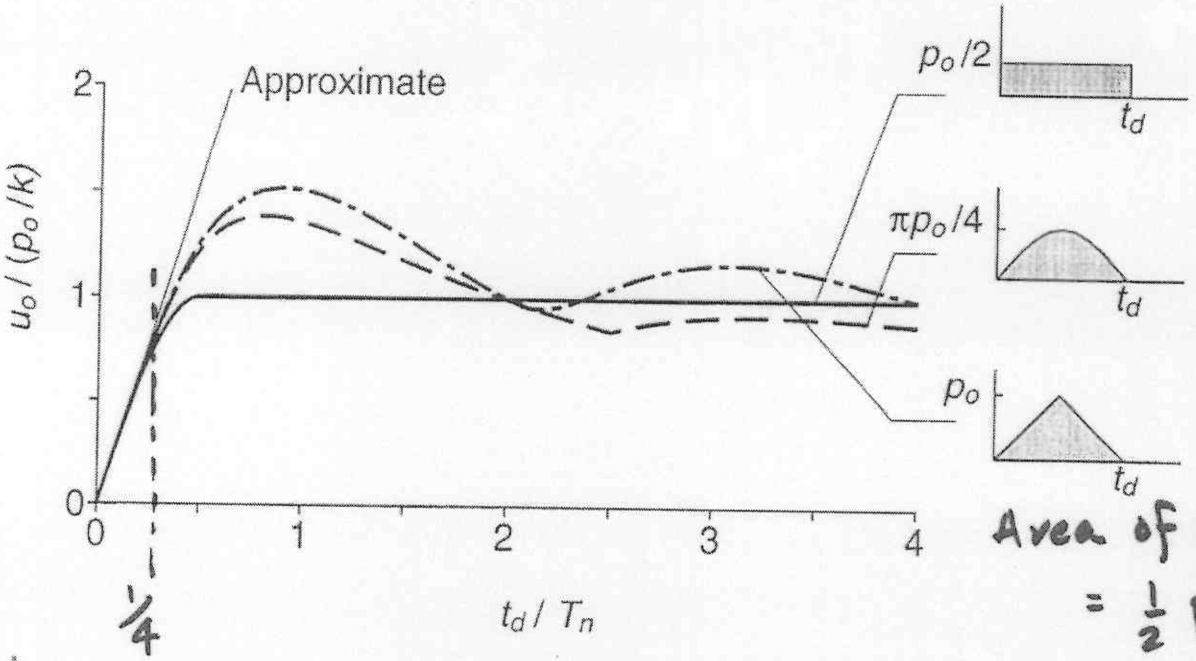
The "impulse" approximation is for situations where the maximum occurs in free vibration (when pulse is removed) and hence is invalid for long pulses

Summary:

For $t_d/T_n < 1/4$, maximum response depends on Pulse Area
Approximate Solution can be used

For $t_d/T_n > 1/4$, maximum response depends on Pulse Shape
Approximate Solution can not be used
Use Shock Spectra instead

Shock spectra for force pulses of equal area



Area of each pulse
$= \frac{1}{2} p_o t_d$

For $t_d/T_n < 1/4$, approximate solution is good
max. independent of pulse shape. Only pulse area important.

Figure 4.10.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

For $t_d/T_n > 1/4$, pulse shape is important

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Figu

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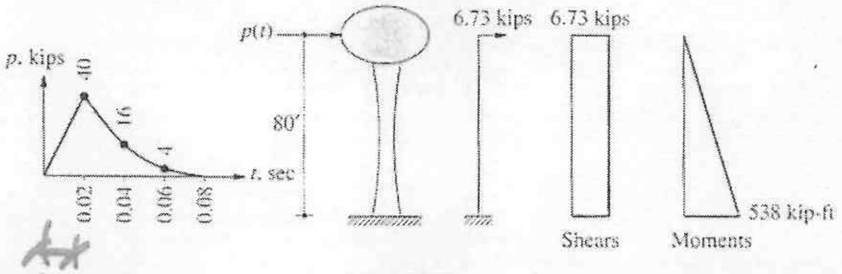
Example 4.2

The 80-ft-high full water tank of Example 2.6 is subjected to the force $p(t)$ shown in Fig. E4.2, caused by an aboveground explosion. Determine the maximum base shear and bending moment at the base of the tower supporting the tank.

$t_d = .08$

$T_n = 1.12$



Δt

'02

Figure E4.2

t_d/T_n

Solution For this water tank, weight $w = 100.03$ kips, $k = 8.2$ kips/in, $T_n = 1.12$ sec, and $\zeta = 1.23\%$. The ratio $t_d/T_n = 0.08/1.12 = 0.071$. Because $t_d/T_n < 0.25$, the forcing function may be treated as a pure impulse of magnitude

$$\mathcal{I} = \int_0^{0.08} p(t)\,dt = \frac{0.02}{2}[0 + 2(40) + 2(16) + 2(4) + 0] = 1.2 \text{ kip-sec}$$

.071

where the integral is calculated by the trapezoidal rule. Neglecting the effect of damping, the maximum displacement is

$$\frac{\mathcal{I}}{m\omega_n} \leftarrow \qquad u_o = \frac{\mathcal{I}}{k}\frac{2\pi}{T_n} = \frac{(1.2)2\pi}{(8.2)(1.12)} = 0.821 \text{ in.}$$

The equivalent static force f_{So} associated with this displacement is [from Eq. (1.8.1)]

$$f_{So} = ku_o = (8.2)0.821 = 6.73 \text{ kips}$$

The resulting shearing forces and bending moments over the height of the tower are shown in Fig. E4.2. The base shear and moment are

$$V_b = 6.73 \text{ kips} \qquad M_b = 538 \text{ kip-ft} \Rightarrow 6.73 \times 80$$

$\frac{t_d}{T_n} < \frac{1}{4}$ ok to use Impulse solution

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Effects of Viscous Damping

For single pulse excitations, damping is usually <u>not</u> important

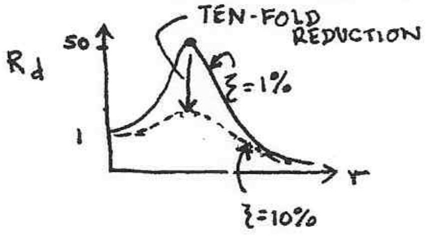
This is in contrast with harmonic excitation (esp. near resonance)

e.g., Increase in Damping from 1% to 10%

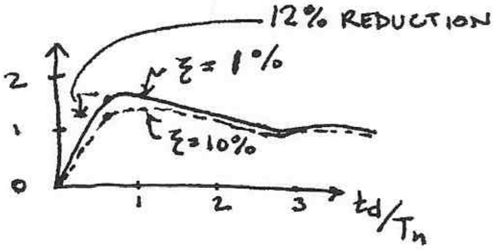
Causes decrease in response by factor of 10 for harmonic excitation near resonance (for $r \approx 1$)

Only decreases response by approximately 12% for half-cycle sine pulse with $t_d/T_n = 1/2$

Damping is important for harmonic excitation because of the cumulative energy dissipated in all of the vibration cycles prior to steady state. (Sect 3.2)
In contrast, for pulse, very little energy is dissipated before maximum is reached.



Harmonic Excitation



Pulse Excitation
(Half-cycle sine)

Practical Significance:

- Neglecting Damping for Pulse-Type Excitations is conservative but not overly conservative
- Shock Spectra for Undamped Systems may be used without concern for inaccuracy / over-prediction of response.

Response to Ground Motion

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g \qquad -(1)$$

$$\Rightarrow \quad p_{eff}(t) = -m\ddot{u}_g(t) \quad ; \quad (u_{st})_0 = \frac{(p_{eff})_0}{k} = \frac{\ddot{u}_{g0}}{\omega_n^2}$$

$$R_d = \frac{u_0}{(u_{st})_0} \quad \text{in chapters 3 \& 4}$$

$$\Rightarrow \quad R_d = \frac{u_0}{\ddot{u}_{g0}/\omega_n^2} = \frac{\omega_n^2 u_0}{\ddot{u}_{g0}} \qquad -(2)$$

This means that we can use all R_d plots of chapters 3 and 4 to find maximum response to ground motion. The only thing we need to do is interpret

$$R_d \quad \text{as} \quad \frac{\omega_n^2 u_0}{\ddot{u}_{g0}} \quad \text{not as} \quad \frac{u_0}{(p_0/k)}$$

IF $\xi = 0$ [i.e., no damping]

Eq (1) becomes $\quad m\ddot{u} + ku = -m\ddot{u}_g$

OR $\quad m\ddot{u}^t = -ku$

or $\quad \ddot{u}^t = -\omega_n^2 u \quad \& \quad \ddot{u}_0^t = \omega_n^2 u_0$

$$\Rightarrow \quad R_d = \frac{\ddot{u}_0^t}{\ddot{u}_{g0}} \qquad \text{from Eqn. (2)}$$

So, for undamped systems, R_d plots of chapters 3 & 4 show $\dfrac{\ddot{u}_0^t}{\ddot{u}_{g0}}$ due to ground motion

Summary: R_d v/s r plots of Chapter 3 and R_d v/s t_d/T_n plots (Shock Spectra) of Chapter 4 can be used for ground motion [where $R_d = \omega_n^2 u_0 / \ddot{u}_{g0}$]

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Example 4.3

Consider the SDF model of an automobile described in Example 3.4 running over the speed bump shown in Fig. E4.3 at velocity v. Determine the maximum force developed in the suspension spring and the maximum vertical acceleration of the mass if **(a)** $v = 5$ mph, and **(b)** $v = 10$ mph.

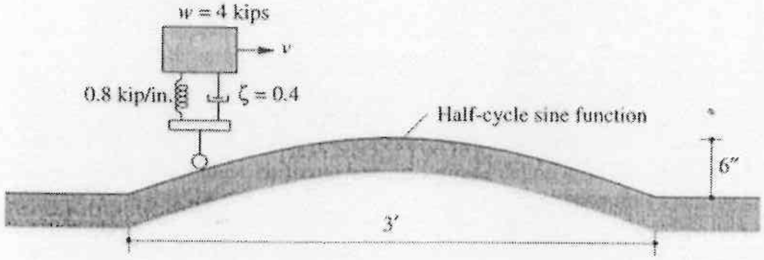


Figure E4.3

Solution

1. *Determine the system and excitation parameters.*

$$m = \frac{4000}{386} = 10.363 \text{ lb-sec}^2/\text{in.}$$

$$k = 800 \text{ lb/in.}$$

$$\omega_n = 8.786 \text{ rad/sec} \qquad T_n = 0.715 \text{ sec}$$

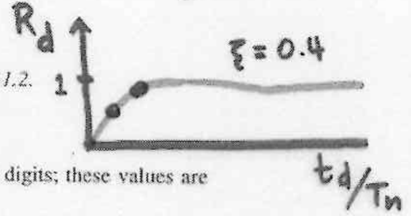
Faster Speed ↓

$$v = 5 \text{ mph} = 7.333 \text{ ft/sec} \qquad t_d = \frac{3}{7.333} = 0.4091 \text{ sec} \qquad \frac{t_d}{T_n} = 0.572$$

$$v = 10 \text{ mph} = 14.666 \text{ ft/sec} \qquad t_d = \frac{3}{14.666} = 0.2046 \text{ sec} \qquad \frac{t_d}{T_n} = 0.286$$

↓ Shorter Pulse

$$u_g(t) = 6 \sin \frac{\pi t}{t_d} \qquad \ddot{u}_{go} = \frac{6\pi^2}{t_d^2}$$



2. *Determine R_d for the t_d/T_n values above from Fig. 4.11.2.*

$$R_d = \frac{\omega_n^2 u_o}{\ddot{u}_{go}} \longleftarrow R_d = \begin{cases} 1.015 & v = 5 \text{ mph} \\ 0.639 & v = 10 \text{ mph} \end{cases}$$

Obviously, R_d cannot be read accurately to three or four significant digits; these values are from the numerical data used in plotting Fig. 4.11.2.

3. *Determine the maximum force, f_{So}.*

$$\longrightarrow u_o = \frac{\ddot{u}_{go}}{\omega_n^2} R_d = 1.5 \left(\frac{T_n}{t_d}\right)^2 R_d$$

$$u_o = \begin{cases} 1.5\left(\dfrac{1}{0.572}\right)^2 1.015 = 4.65 \text{ in.} & v = 5 \text{ mph} \\ 1.5\left(\dfrac{1}{0.286}\right)^2 0.639 = 11.7 \text{ in.} & v = 10 \text{ mph} \end{cases}$$

↓ Easy on those speed bumps!

$$f_{So} = k u_o = 0.8 u_o = \begin{cases} 3.72 \text{ kips} & v = 5 \text{ mph} \\ 9.37 \text{ kips} & v = 10 \text{ mph} \end{cases}$$

Observe that the force in the suspension is much larger at the higher speed. The large deformation of the suspension suggests that it may deform beyond its linearly elastic limit.

4. *Determine the maximum acceleration, $\ddot{u}_o^t$.* Equation (4.12.3) provides a relation between $\ddot{u}_o^t$ and R_d that is exact for systems without damping but is approximate for damped systems. These approximate results can readily be obtained for this problem:

If $\xi = 0$,

$$R_d = \frac{\ddot{u}_o^t}{\ddot{u}_{go}} \longrightarrow \ddot{u}_o^t = \ddot{u}_{go} R_d = \frac{6\pi^2}{t_d^2} R_d$$

$$\ddot{u}_o^t = \begin{cases} \left[\dfrac{6\pi^2}{(0.4091)^2}\right] 1.015 = 359.1 \text{ in./sec}^2 & v = 5 \text{ mph} \\ \left[\dfrac{6\pi^2}{(0.2046)^2}\right] 0.639 = 903.7 \text{ in./sec}^2 & v = 10 \text{ mph} \end{cases}$$

assuming $\xi = 0$

Observe that the acceleration of the mass is much larger at the higher speed; in fact, it exceeds 1 g, indicating that the SDF model would lift off from the road.

To evaluate the error in the approximate solution for $\ddot{u}_o^t$, a numerical solution of the equation of motion was carried out, leading to the "exact" value of $\ddot{u}_o^t = 422.7$ in./sec^2 for $v = 5$ mph.

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NUMERICAL METHODS

Dynamic Response Analysis

General EQs:

$m\ddot{u} + c\dot{u} + f_s(u, \dot{u}) = p(t)$

$m\ddot{u} + c\dot{u} + f_s(u, \dot{u}) = -m\ddot{u}_g(t)$ for earthquake

- arbitrary forces
- still single DOF

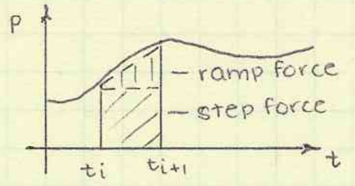
Methods of Solving

1. interpolation of excitation function
2. Finite difference schemes
3. Assumed variation of Acceleration

Concerns (more on this later)

- convergence as Δt decreases
- stability of solution, despite approximation
- accuracy against the exact solution

(1) Interpolation of Excitation Function



$p(\tau) = p_i + \frac{\Delta p_i}{\Delta t_i} \cdot \tau \quad , \quad 0 < \tau < \Delta t_i$

allows boundary (initial) conditions of zero; move origin to $t = t_i$

Recurrence Formulas:

$u_{i+1} = Au_i + B\dot{u}_i + Cp_i + Dp_{i+1}$

$\dot{u}_{i+1} = A'u_i + B'\dot{u}_i + C'p_i + D'p_{i+1}$

$\xi = 0$, coefficients $\propto \omega_n, \Delta t, k$

$\xi \neq 0$, $\omega_n, \Delta t, k, \xi, \omega_D$

tabulated in 5.2.1

* can only be applied to <u>linear</u> systems

Summary:

1. Discretize excitation $p(t)$, $-m\ddot{u}_g(t)$
2. Given properties of structure, time interval of discretation, determine A, B, C, D & A', B', C', D'
3. Use equations above to calculate $u_{i+1}, \dot{u}_{i+1}$; initial ~~eq~~conditions $u_0, \dot{u}_0$

Make note:

- Δt small → more accurate solution
- Method is great for earthquake loads, due to how load data is gathered and presented
- coefficient equations – put them in mathcad and SAVE. (or in Excel)

Numerical Methods in Dynamic Response Analysis

For arbitrarily specified excitation (e.g., earthquakes)
OR for nonlinear systems,
we need numerical time-stepping methods
for integration of the differential equations.

General Equation of Motion:

$$m\ddot{u} + c\dot{u} + f_S(u,\dot{u}) = p(t)$$

OR $m\ddot{u} + c\dot{u} + f_s(u,\dot{u}) = -m\ddot{u}_g(t)$ for Earthquake

Given $u(0) = u_0$; $\dot{u}(0) = \dot{u}_0$

Assume $p(t)$ is discretized at time intervals Δt_i

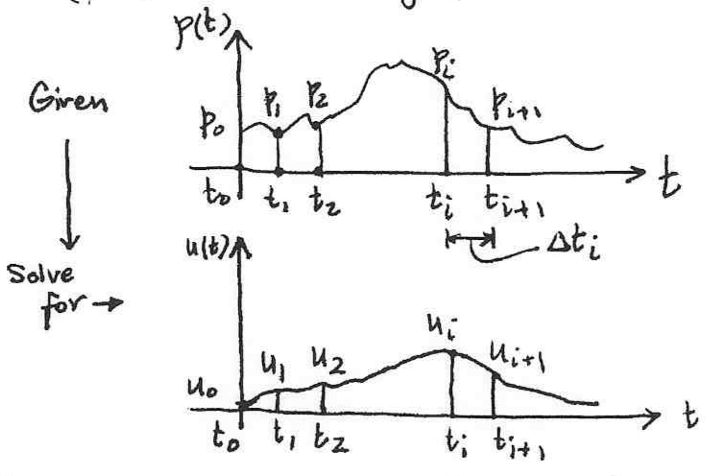
$$p_i = p(t_i) \qquad i = 0, 1, \ldots N$$

$$\Delta t_i = t_{i+1} - t_i \qquad \text{(usually } \Delta t_i = \Delta t = \text{constant)}$$

$u_i, \dot{u}_i, \ddot{u}_i \equiv$ displacement, velocity, acceleration at t_i

$$\Rightarrow \quad m\ddot{u}_i + c\dot{u}_i + (f_s)_i = p_i \qquad (\text{or} = -m\ddot{u}_{gi})$$

$(f_s)_i \equiv$ resisting force at time t_i [$= ku_i$ for linear elastic systems]



Time-stepping starting w/ ICs $\Rightarrow \quad m\ddot{u}_{i+1} + c\dot{u}_{i+1} + (f_s)_{i+1} = p_{i+1}$

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$$\underset{\substack{\uparrow}}{i \longrightarrow i+1}$$

<u>Note</u>: Time stepping procedures are usually not exact

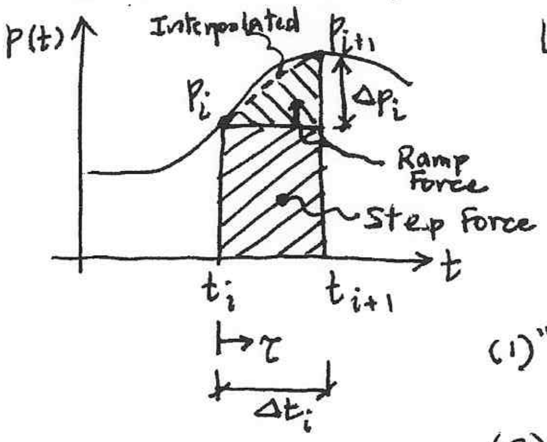
Issues:

1. Convergence - Method should yield exact solution (or approach it) as Δt is made smaller
2. Stability - Method should be stable despite numerical round-off errors
3. Accuracy - Method should yield results that are close enough to the exact solution.

Three types of methods:

1. Interpolation of Excitation Function
2. Finite Difference Schemes
3. Assumed Variation of Acceleration

Method 1: Interpolation of Excitation Function



Linear interpolation of load over the time interval, Δt_i

$$\Rightarrow \quad p(\tau) = p_i + \frac{\Delta p_i}{\Delta t_i} \cdot \tau \quad (0 < \tau < \Delta t_i)$$

Response at time, τ is due to

(1) "Initial" conditions at $t = t_i$ (or $\tau = 0$) Free Vibration

(2) Step Force p_i with zero ICs

(3) Ramp Force $\frac{\Delta p_i}{\Delta t_i} \cdot \tau$ with zero ICs

$$p_{i+1} - p_i = \Delta p_i$$

Consider undamped case $(c = 0)$

$$\Rightarrow u_{i+1} = u_i \cos(\omega_n \Delta t_i) + \frac{\dot{u}_i}{\omega_n} \sin(\omega_n \Delta t_i) \qquad \left[\begin{matrix} (1) \\ \text{Eq. } 2.1.3 \end{matrix}\right] \text{Free Vib.}$$

$$+ \frac{p_i}{k}\left[1 - \cos(\omega_n \Delta t_i)\right] \qquad \left[\begin{matrix} (2) \\ \text{Eq. } 4.3.2 \end{matrix}\right] \text{Step}$$

$$+ \frac{\Delta p_i}{k} \cdot \frac{1}{\omega_n \Delta t_i}\left[\omega_n \Delta t_i - \sin(\omega_n \Delta t_i)\right] \qquad \left[\begin{matrix} (3) \\ \text{Eq. } 4.4.2 \end{matrix}\right] \text{Ramp} \quad \text{—}(1)$$

also, $$\dot{u}_{i+1} = -\omega_n u_i \sin(\omega_n \Delta t_i) + \dot{u}_i \cos(\omega_n \Delta t_i)$$

$$+ \omega_n \frac{p_i}{k} \sin(\omega_n \Delta t_i)$$

$$+ \frac{\Delta p_i}{k} \cdot \frac{1}{\Delta t_i}\left[1 - \cos(\omega_n \Delta t_i)\right] \quad \text{—}(2)$$

In Equations (1) & (2), group terms containing u_i, $\dot{u}_i$, p_i, and p_{i+1} on the right hand sides.

$$\Rightarrow \text{RECURRENCE FORMULAS} \quad \boxed{\begin{aligned} u_{i+1} &= A u_i + B \dot{u}_i + C p_i + D p_{i+1} \\ \dot{u}_{i+1} &= A' u_i + B' \dot{u}_i + C' p_i + D' p_{i+1} \end{aligned}} \text{—}(3)$$

$$\text{For } \xi = 0, \quad \left.\begin{matrix} A, B, C, D \\ A', B', C', D' \end{matrix}\right\} \rightarrow f(\omega_n, \Delta t, k)$$

$$\text{For } \xi \neq 0, \quad \left.\begin{matrix} A, B, C, D \\ A', B', C', D' \end{matrix}\right\} \rightarrow f(\omega_n, \Delta t, k, \xi, \omega_D)$$

See Table 5.2.1 for A, B, C, D & A', B', C', D'

Note $A, B, \ldots$ are computed <u>once</u>. Eq. 3 allows direct solution

<u>Note</u>: Can only be applied for LINEAR systems.

12-4

Steps:

1. Discretize excitation $p(t)$ or $-m\ddot{u}_g(t)$
2. Given structure's properties and time interval for force discretization, determine A, B, C, D & A', B', C', D' using Table 5.2.1
3. Use Eqs. 3 to obtain u_{i+1}, $\dot{u}_{i+1}$ starting with initial conditions u_0 and $\dot{u}_0$

Note: 1. Approximation of Exact Solution is improved if Δt is made small.

2. Method is useful for earthquake excitation where ground acceleration is at closely spaced intervals.

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Method based on Interpolation of Excitation

(only for linear systems)

TABLE 5.2.1 COEFFICIENTS IN RECURRENCE FORMULAS ($\zeta < 1$)

$$A = e^{-\zeta\omega_n\,\Delta t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D\,\Delta t + \cos\omega_D\,\Delta t\right)$$

$$B = e^{-\zeta\omega_n\,\Delta t}\left(\frac{1}{\omega_D}\sin\omega_D\,\Delta t\right)$$

$$C = \frac{1}{k}\left\{\frac{2\zeta}{\omega_n\,\Delta t} + e^{-\zeta\omega_n\,\Delta t}\left[\left(\frac{1-2\zeta^2}{\omega_D\,\Delta t} - \frac{\zeta}{\sqrt{1-\zeta^2}}\right)\sin\omega_D\,\Delta t - \left(1+\frac{2\zeta}{\omega_n\,\Delta t}\right)\cos\omega_D\,\Delta t\right]\right\}$$

$$D = \frac{1}{k}\left[1 - \frac{2\zeta}{\omega_n\,\Delta t} + e^{-\zeta\omega_n\,\Delta t}\left(\frac{2\zeta^2-1}{\omega_D\,\Delta t}\sin\omega_D\,\Delta t + \frac{2\zeta}{\omega_n\,\Delta t}\cos\omega_D\,\Delta t\right)\right]$$

$$A' = -e^{-\zeta\omega_n\,\Delta t}\left(\frac{\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_D\,\Delta t\right)$$

$$B' = e^{-\zeta\omega_n\,\Delta t}\left(\cos\omega_D\,\Delta t - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D\,\Delta t\right)$$

$$C' = \frac{1}{k}\left\{-\frac{1}{\Delta t} + e^{-\zeta\omega_n\,\Delta t}\left[\left(\frac{\omega_n}{\sqrt{1-\zeta^2}} + \frac{\zeta}{\Delta t\sqrt{1-\zeta^2}}\right)\sin\omega_D\,\Delta t + \frac{1}{\Delta t}\cos\omega_D\,\Delta t\right]\right\}$$

$$D' = \frac{1}{k\,\Delta t}\left[1 - e^{-\zeta\omega_n\,\Delta t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D\,\Delta t + \cos\omega_D\,\Delta t\right)\right]$$

Initial Calculations (computed only once)

Recurrence Formulas:

$$u_{i+1} = Au_i + B\dot{u}_i + Cp_i + Dp_{i+1}$$

$$\dot{u}_{i+1} = A'u_i + B'\dot{u}_i + C'p_i + D'p_{i+1}$$

$m\ddot{u}_i + c\dot{u}_i + ku_i = p_i$ (to get $\ddot{u}$, if needed)

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NUMERICAL METHODS

Central Difference Method

(example of a finite difference scheme)

General formulas

$$\dot{u}_i = \frac{1}{2\Delta t}\left(u_{i+1} - u_{i-1}\right)$$

$$\ddot{u}_i = \frac{1}{\Delta t}\left(\dot{u}_{i+1/2} - \dot{u}_{i-1/2}\right) = \frac{1}{\Delta t^2}\left(u_{i+1} - 2u_i + u_{i-1}\right)$$

$$\hat{k} = \frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t}$$ (independent of k!)

$$\hat{p}_i = p_i - \left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t}\right]u_{i-1} - \left[k - \frac{2m}{(\Delta t)^2}\right]u_i$$

$\hat{k}\, u_{i+1} = \hat{p}_i$

for linear elastic systems

If $i = 0$,

$$u_{-1} = u_0 - \dot{u}_0 \Delta t + \ddot{u}_0 \frac{(\Delta t)^2}{2}$$

$$\ddot{u}_0 = \frac{1}{m}\left(p_0 - c\dot{u}_0 - ku_0\right)$$

Stability Requirement:

$$\frac{\Delta t}{T_n} < \frac{1}{\pi} = 0.318$$

for earthquake response, $\Delta t \sim 0.015\text{s}$

so, if $T_n > 0.06\text{s}$, we'll be good

Equations summarized in
Table 5.3.1 (notes 12-8).
Again, save in MathCAD/Excel

Central Difference Method

Example of a Finite Difference scheme

Central Difference formulas

for velocity : $$\dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\Delta t} \quad -(1)$$

for acceleration : $$\ddot{u}_i = \frac{\dot{u}_{i+1/2} - \dot{u}_{i-1/2}}{\Delta t}$$

$$= \frac{\left(\frac{u_{i+1} - u_i}{\Delta t}\right) - \left(\frac{u_i - u_{i-1}}{\Delta t}\right)}{\Delta t}$$

$$= \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2} \quad -(2)$$

Substitute (1) & (2) into Equation of Motion

For linear elastic systems,

At time t_i ; $$m\left[\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2}\right] + c\left[\frac{u_{i+1} - u_{i-1}}{2\Delta t}\right] + ku_i = p_i$$

OR $$\underbrace{\left[\frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t}\right]}_{\Downarrow \text{ Define } \hat{k}} u_{i+1} = \underbrace{p_i - \left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t}\right]u_{i-1} - \left[k - \frac{2m}{(\Delta t)^2}\right]u_i}_{\Downarrow \text{ Define } \hat{p}_i}$$

$$\boxed{\hat{k}\; u_{i+1} = \hat{p}_i} \quad -(3)$$

$$\hat{k} = \frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t} \equiv \text{function of } c, m, \Delta t$$

$$\hat{p}_i = p_i - \left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t}\right]u_{i-1} - \left[k - \frac{2m}{(\Delta t)^2}\right]u_i = p_i - a u_{i-1} - b u_i$$

$$\equiv \text{function of } m, c, k, \Delta t, p_i, u_{i-1}, u_i$$

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Can solve Eq. (3) for u_{i+1} if we knew $\hat{k}$, $\hat{p}_i$

Note: $\hat{p}_i$ depends on displacements at times i and $i-1$

To get u_1, we'd set $i=0$ in Eq (3), hence we need $\hat{p}_0$ (which depends on u_0 & u_{-1})

What is u_{-1}?

To find u_{-1}, use central difference formulas (1) & (2) with $i=0$.

Eq (1) gives: $\dot{u}_0 = \dfrac{u_1 - u_{-1}}{2\Delta t}$

Eq (2) gives: $\ddot{u}_0 = \dfrac{u_1 - 2u_0 + u_{-1}}{(\Delta t)^2}$

Eliminate u_1 in expressions above and express u_{-1} in terms of u_0, $\dot{u}_0$, $\ddot{u}_0$

$$\boxed{u_{-1} = u_0 - \dot{u}_0 \Delta t + \ddot{u}_0 \frac{(\Delta t)^2}{2}} \quad (4)$$

We know u_0 & $\dot{u}_0$ from the ICs given

To get $\ddot{u}_0$, use Equation of Motion,

$$m\ddot{u}_0 + c\dot{u}_0 + ku_0 = p_0$$

$$\Rightarrow \boxed{\ddot{u}_0 = \frac{p_0 - c\dot{u}_0 - ku_0}{m}} \quad (5)$$

Stability Requirement: $\dfrac{\Delta t}{T_n} < \dfrac{1}{\pi} = 0.318$

Usually we have $\Delta t = 0.01$ to 0.02 sec (esp. for earthquake response). This means as long as our structure's natural period is greater than approx. 0.06 sec., our solution will be stable with this method.

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TABLE 5.3.1 CENTRAL DIFFERENCE METHOD

1.0 *Initial calculations* (done once only)

1.1 $\ddot{u}_0 = \dfrac{p_0 - c\dot{u}_0 - ku_0}{m}$.

1.2 $u_{-1} = u_0 - \Delta t \dot{u}_0 + \dfrac{(\Delta t)^2}{2}\ddot{u}_0$.

1.3 $\hat{k} = \dfrac{m}{(\Delta t)^2} + \dfrac{c}{2\Delta t}$.

1.4 $a = \dfrac{m}{(\Delta t)^2} - \dfrac{c}{2\Delta t}$.

1.5 $b = k - \dfrac{2m}{(\Delta t)^2}$.

LOOP

2.0 *Calculations for time step i*

2.1 $\hat{p}_i = p_i - au_{i-1} - bu_i$.

2.2 $u_{i+1} = \dfrac{\hat{p}_i}{\hat{k}}$.

2.3 If required: $\dot{u}_i = \dfrac{u_{i+1} - u_{i-1}}{2\Delta t}$, $\ddot{u}_i = \dfrac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2}$.

3.0 *Repetition for the next time step*

Replace i by $i + 1$ and repeat steps 2.1, 2.2, and 2.3 for the next time step.

Method is unstable if $\dfrac{\Delta t}{T_n} > \dfrac{1}{\pi} = 0.318$

NEWMARK'S METHOD

Homework
- print out!
- follow problem in text with some different values
- from chapter five
- use Excel to plot

Chapter Six: Earthquake Response

record ground acceleration at site

get response spectrum

get max. displacement for system in question

Newmark's Method

Example of a Method based on an assumed variation of acceleration in a time step, Δt

Parameters : β, γ → define how acceleration varies over a time step
→ determine accuracy/stability characteristics

(a) $\dot{u}_{i+1} = \dot{u}_i + [(1-\gamma)\Delta t]\ddot{u}_i + (\gamma \Delta t)\ddot{u}_{i+1}$

(b) $u_{i+1} = u_i + (\Delta t)\dot{u}_i + [(0.5-\beta)(\Delta t)^2]\ddot{u}_i + [\beta(\Delta t)^2]\ddot{u}_{i+1}$

(c) $m\ddot{u}_{i+1} + c\dot{u}_{i+1} + (f_s)_{i+1} = p_{i+1}$

(a), (b), and (c) above enable computation of

$$\left.\begin{array}{r} u_{i+1} \\ \dot{u}_{i+1} \\ \text{and } \ddot{u}_{i+1} \end{array}\right\} \text{ from } \begin{array}{l} u_i \\ \dot{u}_i \\ \ddot{u}_i \end{array}$$

The procedure is ITERATIVE in general (i.e., for nonlinear systems)

For linear systems (when $(f_s)_{i+1} = k \cdot u_{i+1}$),
Solution is NON-ITERATIVE

It is convenient to rewrite equations (a), (b), and (c) in incremental form.

$\Delta u_i = u_{i+1} - u_i$; $\Delta \dot{u}_i = \dot{u}_{i+1} - \dot{u}_i$; $\Delta \ddot{u}_i = \ddot{u}_{i+1} - \ddot{u}_i$
& $\Delta p_i = p_{i+1} - p_i$

$$(a') \qquad \Delta \dot{u}_i = (\Delta t)\ddot{u}_i + (\gamma \Delta t)\Delta \ddot{u}_i$$

$$(b') \qquad \Delta u_i = (\Delta t)\dot{u}_i + \frac{(\Delta t)^2}{2}\ddot{u}_i + \left[\beta(\Delta t)^2\right]\Delta \ddot{u}_i$$

$$(c') \qquad m\Delta \ddot{u}_i + c\Delta \dot{u}_i + k\Delta u_i = \Delta p_i$$

Write $\Delta \dot{u}_i$ in terms of $\dot{u}_i$, $\ddot{u}_i$, and Δu_i using (a') and (b') and eliminating $\Delta \ddot{u}_i$

$$\Rightarrow \quad \Delta \dot{u}_i = \frac{\gamma}{\beta(\Delta t)}\Delta u_i - \frac{\gamma}{\beta}\dot{u}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i \quad \text{—(I)}$$

also from (b')
$$\Delta \ddot{u}_i = \frac{1}{\beta(\Delta t)^2}\Delta u_i - \frac{1}{\beta \Delta t}\dot{u}_i - \frac{1}{2\beta}\ddot{u}_i \quad \text{—(II)}$$

Substitute (I) & (II) in (c') to get:

$$\hat{k}\,\Delta u_i = \Delta \hat{p}_i$$

where
$$\hat{k} = k + \frac{\gamma}{\beta \Delta t}\cdot c + \frac{1}{\beta(\Delta t)^2}\cdot m$$

$$\Delta \hat{p}_i = \Delta p_i + \left(\frac{1}{\beta \Delta t}m + \frac{\gamma}{\beta}\cdot c\right)\dot{u}_i + \left[\frac{1}{2\beta}\cdot m + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)\cdot c\right]\ddot{u}_i$$

$\hat{k} \equiv$ function of m, c, k; Δt; β, γ

$\Delta \hat{p}_i \equiv$ function of m, c, k; Δt; β, γ; Δp_i; $\dot{u}_i, \ddot{u}_i$

$$\Delta u_i = \frac{\Delta \hat{p}_i}{\hat{k}} \quad \text{— (III)}$$

Obtain Δu_i using (III)
Obtain $\Delta \dot{u}_i$ using (I) and Δu_i
Obtain $\Delta \ddot{u}_i$ using (II) and Δu_i

$u_{i+1} = u_i + \Delta u_i$; $\dot{u}_{i+1} = \dot{u}_i + \Delta u_i$; $\ddot{u}_{i+1} = \ddot{u}_i + \Delta \ddot{u}_i$

alternatively, for $\ddot{u}_{i+1}$, can use:

$$\ddot{u}_{i+1} = \frac{p_{i+1} - c\dot{u}_{i+1} - ku_{i+1}}{m}$$

↑ also to obtain $\ddot{u}_0$ at start.

Implicit Methods (like Newmark's Method)

→ Obtain solution at time $i+1$ using equilibrium (equation of motion) at time $i+1$

Explicit Methods (like Central Difference Method)

→ Obtain solution at time $i+1$ <u>without</u> using equilibrium (equation of motion) at time $i+1$

Newmark's Method Special Cases

1. Average Acceleration

   Acceleration within a time step is assumed to be the average of acceleration values at beginning and end of the time step.

$$\ddot{u}(\tau) = \frac{1}{2}\left[\ddot{u}_{i+1} + \ddot{u}_i\right] ; \quad 0 < \tau < \Delta t$$

2. Linear Acceleration

   Acceleration within a time step is assumed to vary linearly over the time step.

$$\ddot{u}(\tau) = \ddot{u}_i + \frac{\tau}{\Delta t}\left(\ddot{u}_{i+1} - \ddot{u}_i\right) ; \quad 0 < \tau < \Delta t$$

★ Average Acceleration : Newmark with $\gamma = 1/2$, $\beta = 1/4$
Linear Acceleration : Newmark with $\gamma = 1/2$, $\beta = 1/6$

TABLE 5.4.1 AVERAGE ACCELERATION AND LINEAR ACCELERATION METHODS

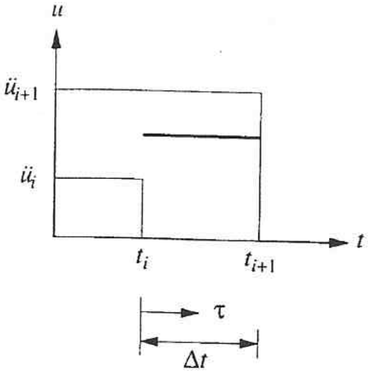
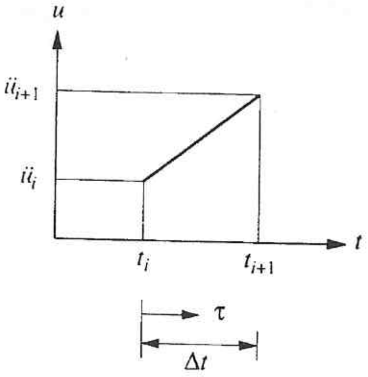
	Average Acceleration	Linear Acceleration	
			
Integrate	$\ddot u(\tau) = \frac{1}{2}(\ddot u_{i+1} + \ddot u_i)$	$\ddot u(\tau) = \ddot u_i + \frac{\tau}{\Delta t}(\ddot u_{i+1} - \ddot u_i)$	(5.4.2)
	$\dot u(\tau) = \dot u_i + \frac{\tau}{2}(\ddot u_{i+1} + \ddot u_i)$	$\dot u(\tau) = \dot u_i + \ddot u_i \tau + \frac{\tau^2}{2\Delta t}(\ddot u_{i+1} - \ddot u_i)$	(5.4.3)
Integrate	$\dot u_{i+1} = \dot u_i + \frac{\Delta t}{2}(\ddot u_{i+1} + \ddot u_i)$	$\dot u_{i+1} = \dot u_i + \frac{\Delta t}{2}(\ddot u_{i+1} + \ddot u_i)$	(5.4.4)
	$u(\tau) = u_i + \dot u_i \tau + \frac{\tau^2}{4}(\ddot u_{i+1} + \ddot u_i)$	$u(\tau) = u_i + \dot u_i \tau + \ddot u_i \frac{\tau^2}{2} + \frac{\tau^3}{6\Delta t}(\ddot u_{i+1} - \ddot u_i)$	(5.4.5)
	$u_{i+1} = u_i + \dot u_i \, \Delta t + \frac{(\Delta t)^2}{4}(\ddot u_{i+1} + \ddot u_i)$	$u_{i+1} = u_i + \dot u_i \, \Delta t + (\Delta t)^2 \left(\frac{1}{6}\ddot u_{i+1} + \frac{1}{3}\ddot u_i\right)$	(5.4.6)
	$\gamma = 1/2, \ \beta = 1/4$	$\gamma = 1/2, \ \beta = 1/6$	

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TABLE 5.4.2 NEWMARK'S METHOD: LINEAR SYSTEMS

Special cases

(1) Average acceleration method ($\gamma = \frac{1}{2}, \beta = \frac{1}{4}$) → Unconditionally stable

(2) Linear acceleration method ($\gamma = \frac{1}{2}, \beta = \frac{1}{6}$) → Stable if $\frac{\Delta t}{T_n} \leq 0.551$

1.0 *Initial calculations*

1.1 $\ddot{u}_0 = \dfrac{p_0 - c\dot{u}_0 - ku_0}{m}$.

1.2 Select Δt.

1.3 $\hat{k} = k + \dfrac{\gamma}{\beta\,\Delta t}c + \dfrac{1}{\beta(\Delta t)^2}m$.

1.4 $a = \dfrac{1}{\beta\,\Delta t}m + \dfrac{\gamma}{\beta}c$; and $b = \dfrac{1}{2\beta}m + \Delta t\left(\dfrac{\gamma}{2\beta} - 1\right)c$.

2.0 *Calculations for each time step, i*

2.1 $\Delta\hat{p}_i = \Delta p_i + a\dot{u}_i + b\ddot{u}_i$.

2.2 $\Delta u_i = \dfrac{\Delta\hat{p}_i}{\hat{k}}$.

2.3 $\Delta\dot{u}_i = \dfrac{\gamma}{\beta\,\Delta t}\Delta u_i - \dfrac{\gamma}{\beta}\dot{u}_i + \Delta t\left(1 - \dfrac{\gamma}{2\beta}\right)\ddot{u}_i$.

2.4 $\Delta\ddot{u}_i = \dfrac{1}{\beta(\Delta t)^2}\Delta u_i - \dfrac{1}{\beta\Delta t}\dot{u}_i - \dfrac{1}{2\beta}\ddot{u}_i$.

2.5 $u_{i+1} = u_i + \Delta u_i,\ \dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i,\ \ddot{u}_{i+1} = \ddot{u}_i + \Delta\ddot{u}_i$.

3.0 *Repetition for the next time step.* Replace i by $i + 1$ and implement steps 2.1 to 2.5 for the next time step.

Method is stable if $\dfrac{\Delta t}{T_n} \leq \dfrac{1}{\pi\sqrt{2}}\dfrac{1}{\sqrt{\gamma - 2\beta}}$

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Wilson's Method

Start with Newmark's Method
for linear accln case (conditionally stable)

Modify by assuming accln is linear over an extended time step $\delta t = \theta \cdot \Delta t \quad (\theta > 1)$

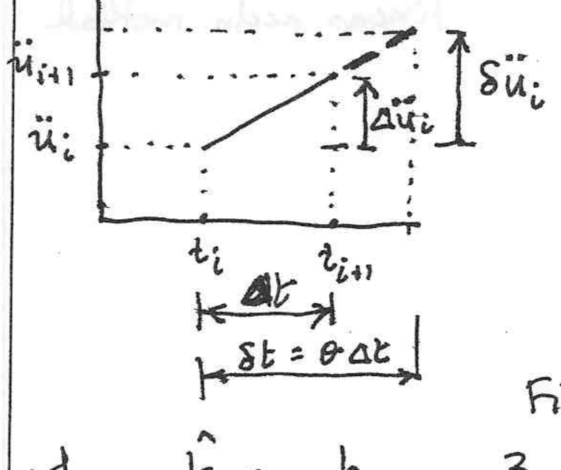
Using $\gamma = 1/2$, $\beta = 1/6$ and replacing Δt by δt in original formulation, we have:

$$\delta \dot{u}_i = (\delta t)\,\ddot{u}_i + \left(\frac{\delta t}{2}\right)\delta \ddot{u}_i$$

$$\delta u_i = (\delta t)\,\dot{u}_i + \frac{(\delta t)^2}{2}\,\ddot{u}_i + \frac{(\delta t)^2}{6}\,\delta \ddot{u}_i$$

$$m\,\delta \ddot{u}_i + c\,\delta \dot{u}_i + k\,\delta u_i = \delta p_i$$

where δu_i = increment in u from
t_i to $t_{i+\theta}$ $\quad (t_{i+\theta} = t_i + \theta\Delta t)$
... etc. (similar defns for $\delta \dot{u}_i$ & $\delta \ddot{u}_i$)



← Definition of $\delta \ddot{u}_i$

Also, define

$$\delta p_i = \theta \cdot \Delta p_i$$

Finally, $\hat{k}\,\delta u_i = \delta \hat{p}_i$

where $\hat{k} = k + \frac{3}{\theta \Delta t}\,c + \frac{6}{(\theta \cdot \Delta t)^2} \cdot m$

& $\delta \hat{p}_i = \theta \cdot \Delta p_i + \left(\frac{6}{\theta \cdot \Delta t}\,m + 3c\right)\dot{u}_i + \left(3m + \frac{\theta \cdot \Delta t}{2} \cdot c\right)\ddot{u}_i$

Stability and Computational Error

Newmark's Method is stable if

$$\frac{\Delta t}{T_n} \leq \frac{1}{\pi\sqrt{2}} \cdot \frac{1}{\sqrt{\gamma - 2\beta}}$$

Average Acceleration

$\gamma = 1/2,\ \beta = 1/4 \quad \Rightarrow \quad \frac{\Delta t}{T_n} < \infty$

(i.e., method is unconditionally stable)

Linear Acceleration

$\gamma = 1/2,\ \beta = 1/6 \quad \Rightarrow \quad \frac{\Delta t}{T_n} \leq 0.551 \quad \left(\text{or } \frac{\sqrt{3}}{\pi} \right)$

Accuracy requirements typically demand smaller time steps than stability requirements. (esp. for SDOF systems)

Recall for Central Difference

Solution stable if $\frac{\Delta t}{T_n} < \frac{1}{\pi} = 0.318$

For MDOF systems, there will be characteristic periods associated with different modes of vibration. Some of these periods will be short. Then, stability requirements might become more important than accuracy requirements

Computational Error

Consider free vibration problem: (Example from book)

$m\ddot{u} + ku = 0$ with $u(0) = 1$, $\dot{u}(0) = 0$

Theoretical Solution: $u(t) = \cos\omega_n t$

Numerical Methods display two types of errors

1. Amplitude Decay (AD)
2. Period Elongation (PE)

AD (Amplitude Decay)

Not a problem for Central Difference and Newmark's methods

PE (Period Elongation)

- Large Error for Central Difference Method

  Period shortening > 20% close to stability limit ($\frac{\Delta t}{T_n} \le 0.318$)

- Newmark's Method

  Linear Acceleration Period Elongation < Average Acceleration Period Elongation

  as long as Linear Acceleration Stability Requirements are met.

  $$\left(\text{i.e., } \frac{\Delta t}{T_n} < \frac{\sqrt{3}}{\pi}\right)$$

Because there is no AD error and because of smallest PE error, Nemark's Linear Acceleration method is usually most accurate of those discussed.

Generally speaking (with any method), for good accuracy $\frac{\Delta t}{T_n} = 0.1$ is reasonable

Excitation Function might often ~~to~~ require smaller time steps. Example: for earthquake records, $\Delta t = .02$ sec.

Usual Procedure: Try Δt; then $\frac{\Delta t}{2}$; ..... smaller Δt unit convergence

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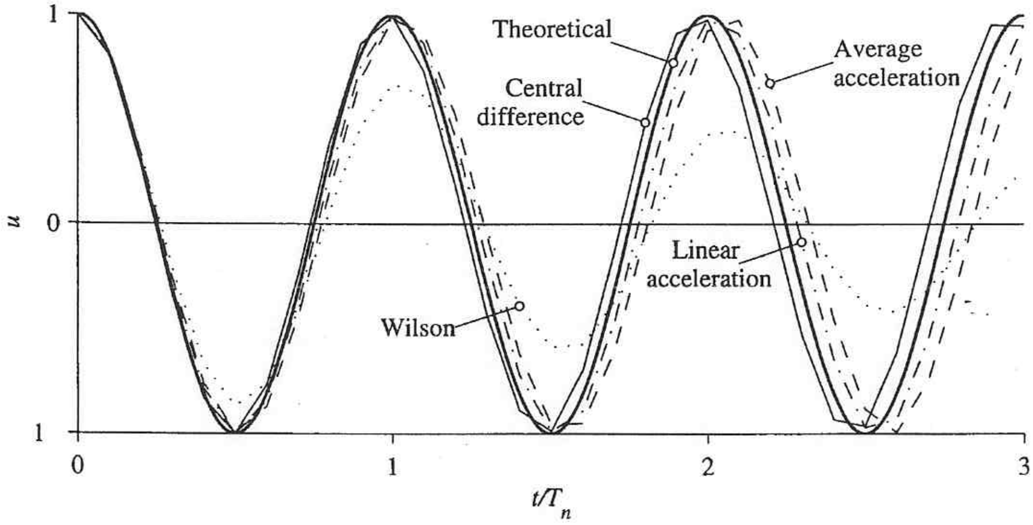

Figure 5.5.1 Free vibration solution by four numerical methods ($\Delta t/T_n = 0.1$) and the theoretical solution.

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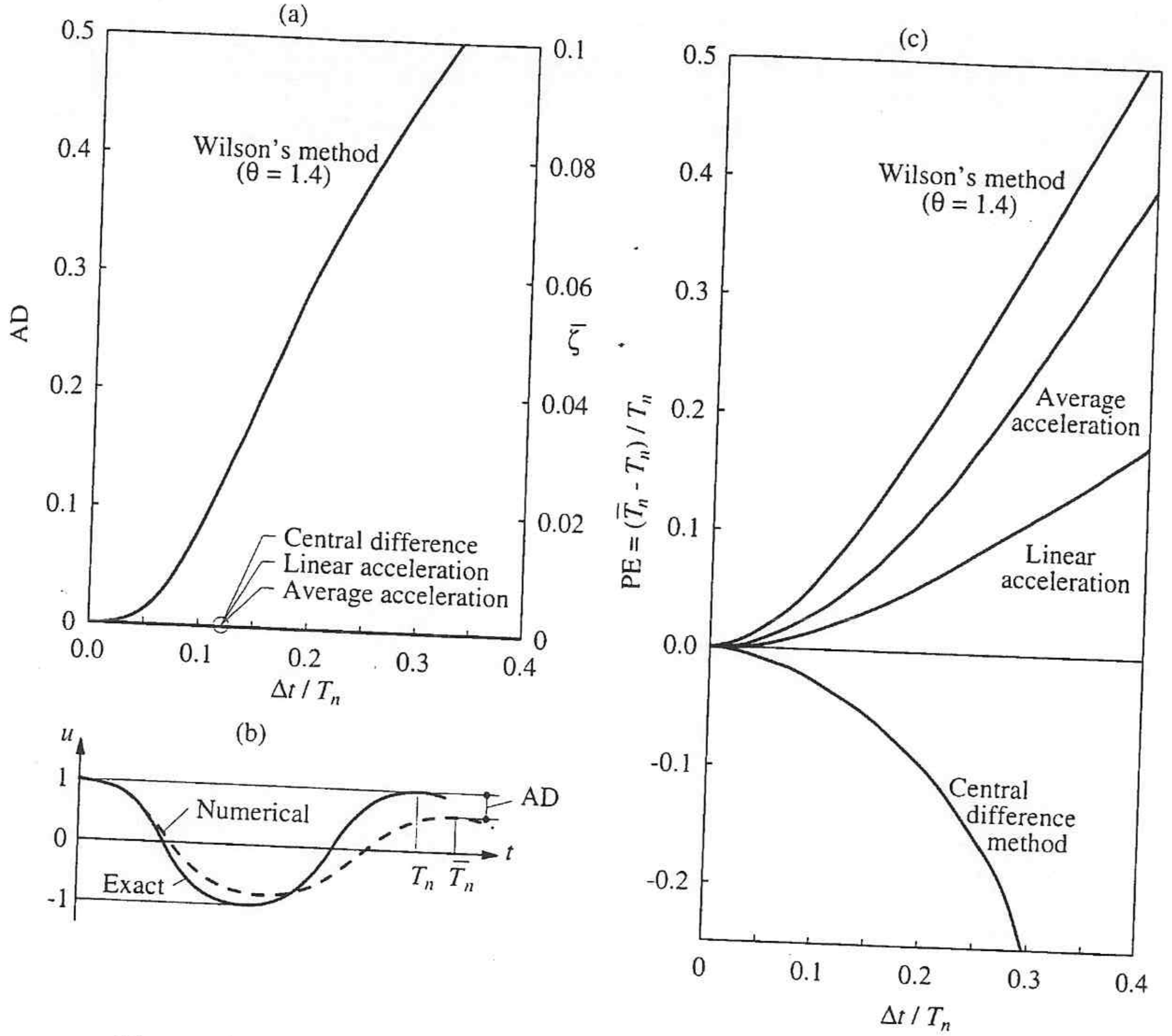

Figure 5.5.2 (a) Amplitude decay versus $\Delta t/T_n$; (b) definition of AD and PE; (c) period elongation versus $\Delta t/T_n$.

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Alternative Newmark Formulation (Non-incremental)

$$\begin{bmatrix} 1 & 0 & -\beta(\Delta t)^2 \\ 0 & 1 & -\gamma\Delta t \\ k & c & m \end{bmatrix} \begin{Bmatrix} u_{i+1} \\ \dot{u}_{i+1} \\ \ddot{u}_{i+1} \end{Bmatrix} = \begin{bmatrix} 1 & \Delta t & (\frac{1}{2}-\beta)(\Delta t)^2 \\ 0 & 1 & (1-\gamma)\Delta t \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_i \\ \dot{u}_i \\ \ddot{u}_i \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ p_{i+1} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} u_{i+1} \\ \dot{u}_{i+1} \\ \ddot{u}_{i+1} \end{Bmatrix} = \underset{3\times 3}{\underline{F}_N} \begin{Bmatrix} u_i \\ \dot{u}_i \\ \ddot{u}_i \end{Bmatrix} + \underset{3\times 1}{\underline{H}_N}\, p_{i+1}$$

$$\underline{F}_N = \underline{F}_N(m, c, k, \Delta t, \gamma, \beta)$$

$$= \frac{1}{A} \begin{bmatrix} A - \omega_n^2\beta(\Delta t)^2 & A\Delta t - 2\xi\omega_n\beta(\Delta t)^2 - \omega_n^2\gamma(\Delta t)^3 & \frac{1}{2}A(\Delta t)^2 - \beta(A+B)(\Delta t)^2 \\ -\omega_n^2\gamma\Delta t & A - 2\xi\omega_n\gamma\Delta t - \omega_n^2\gamma(\Delta t)^2 & A\Delta t - \gamma(A+B)\Delta t \\ -\omega_n^2 & -2\xi\omega_n - \omega_n^2\Delta t & -B \end{bmatrix}$$

where $A = A(\xi, \omega_n, \gamma, \beta) = 1 + 2\xi\omega_n\gamma\Delta t + \omega_n^2\beta(\Delta t)^2$

& $B = B(\xi, \omega_n, \gamma, \beta) = 2\xi\omega_n(1-\gamma)\Delta t + \omega_n^2(\frac{1}{2}-\beta)(\Delta t)^2$

$$\underline{H}_N = \underline{H}_N(m, c, k, \Delta t, \gamma, \beta) = \frac{1}{mA} \begin{Bmatrix} \beta(\Delta t)^2 \\ \gamma\Delta t \\ 1 \end{Bmatrix}$$

Supplementary Notes on Numerical Evaluation of Dynamic Response

A. Accuracy of Finite Difference Schemes

B. Notes on Numerical Analysis for Nonlinear Response

A. Accuracy of Finite Difference Schemes

- Central Difference — Three-point formulas

$$ {}_{3,CD}\dot{u}_i = \frac{1}{2\Delta t}\left[-u_{i-1} + u_{i+1}\right] $$

$$ {}_{3,CD}\ddot{u}_i = \frac{1}{(\Delta t)^2}\left[u_{i-1} - 2u_i + u_{i+1}\right] $$

Accuracy of these determined using Taylor series expansion

$$ u_{i-1} = u_i - \Delta t\cdot\dot{u}_i + \frac{(\Delta t)^2}{2}\ddot{u}_i - \frac{(\Delta t)^3}{6}\dddot{u}_i + \frac{(\Delta t)^4}{14}\ddddot{u}_i + \dots $$

$$ u_{i+1} = u_i + \Delta t\,\dot{u}_i + \frac{(\Delta t)^2}{2}\ddot{u}_i + \frac{(\Delta t)^3}{6}\dddot{u}_i + \frac{(\Delta t)^4}{24}\ddddot{u}_i + \dots $$

$$ \Rightarrow {}_{3,CD}\dot{u}_i = 0 + \dot{u}_i + 0 + \underbrace{\frac{(\Delta t)^2}{6}\dddot{u}_i}_{\substack{\uparrow \\ \text{error is of order } (\Delta t)^2}} + 0 + \dots $$

$$ {}_{3,CD}\dot{u}_i = \dot{u}_i + \underbrace{\frac{(\Delta t)^2}{6}\dddot{u}_i}_{\substack{\uparrow \\ \text{Biggest error is of order } (\Delta t)^2}} + \text{higher order terms} $$

$$ \Rightarrow {}_{3,CD}\dot{u}_i = \dot{u}_i + \underset{\uparrow\ \text{"oh" not zero}}{O}\left((\Delta t)^2\right) $$

OR 3-point central difference formula has error of the order $(\Delta t)^2$

Note: Error is even smaller if $\dddot{u}_i = 0$ since then the error will be of order $(\Delta t)^4$ ....

2nd derivative:

$$ \Rightarrow {}_{3,CD}\ddot{u}_i = 0 + 0 + \ddot{u}_i + 0 + \underbrace{\frac{(\Delta t)^2}{12}\ddddot{u}_i}_{\text{error of order } (\Delta t)^2} $$

$${}_{3,CD}\ddot{u}_i = \ddot{u}_i + \frac{(\Delta t)^2}{12}\ddddot{u}_i + \text{higher order terms}$$

$${}_{3,CD}\ddot{u}_i = \ddot{u}_i + O((\Delta t)^2)$$

Both 3-point central-difference formulas have errors of $O(\Delta t)^2)$

Pattern Formulas:

$${}_{3,CD}\dot{u} = \frac{1}{2\Delta t}\left\{ \textcircled{-1} — \textcircled{0} — \textcircled{1} \right\} + O((\Delta t)^2)$$

$${}_{3,CD}\ddot{u} = \frac{1}{(\Delta t)^2}\left\{ \textcircled{1} — \textcircled{-2} — \textcircled{1} \right\} + O((\Delta t)^2)$$

★ To improve accuracy, increase the no. of points in the pattern formulas.

5-point formulas

$${}_{5,CD}\dot{u}_i = \frac{1}{12\Delta t}\left\{ u_{i-2} - 8u_{i-1} + 8u_{i+1} - u_{i+2} \right\}$$

$${}_{5,CD}\ddot{u}_i = \frac{1}{12(\Delta t)^2}\left\{ -u_{i-2} + 16u_{i-1} - 30u_i + 16u_{i+1} + u_{i+2} \right\}$$

Pattern formulas:

$${}_{5,CD}\dot{u} = \frac{1}{12\Delta t}\left\{ \textcircled{1} — \textcircled{-8} — \textcircled{0} — \textcircled{8} — \textcircled{-1} \right\} + O((\Delta t)^4)$$

$${}_{5,CD}\ddot{u} = \frac{1}{12(\Delta t)^2}\left\{ \textcircled{-1} — \textcircled{16} — \textcircled{-30} — \textcircled{16} — \textcircled{-1} \right\} + O((\Delta t)^4)$$

Q: How would you verify that these 5-pt formulas have errors of the order, $(\Delta t)^4$?

A: Write Taylor series expansions to show that

$${}_{5,CD}\dot{u}_i = \dot{u}_i \underbrace{- \frac{1}{30}(\Delta t)^4 u_i^{V}}_{O[(\Delta t)^4]} \; ; \; {}_{5,CD}\ddot{u}_i = \ddot{u}_i \underbrace{- \frac{1}{90}(\Delta t)^4 u_i^{VI}}_{O((\Delta t)^4)}$$

B. Notes on Numerical Analysis for Nonlinear Response

Linear

$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + ku_{i+1} = p_{i+1}$

$m\ddot{u}_i + c\dot{u}_i + ku_i = p_i$

$\Rightarrow m\Delta\ddot{u}_i + c\Delta\dot{u}_i + k\Delta u_i = \Delta p_i$

Nonlinear

$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + f_{s\,i+1} = p_{i+1}$

$m\ddot{u}_i + c\dot{u}_i + f_{s_i} = p_i$

$m\Delta\ddot{u}_i + c\Delta\dot{u}_i + (\Delta f_s)_i = \Delta p_i$ — ①

where $\Delta x_i = x_{i+1} - x_i \quad \{x = u, \dot{u}, \ddot{u}, f_s, p\}$

with Newmark's Method given γ & β

can solve for Δu_i in linear case easily

using $\hat{k}\,\Delta u_i = \Delta\hat{p}_i$

$$\hat{k} = k + \frac{\gamma}{\beta\Delta t}c + \frac{1}{\beta(\Delta t)^2}m$$

$$\Delta\hat{p}_i = \Delta p_i + \left(\frac{1}{\beta\Delta t}m + \frac{\gamma}{\beta}c\right)\dot{u}_i + \left[\frac{1}{2\beta}m + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)c\right]\ddot{u}_i$$

Easy then to time-step through the duration of interest.

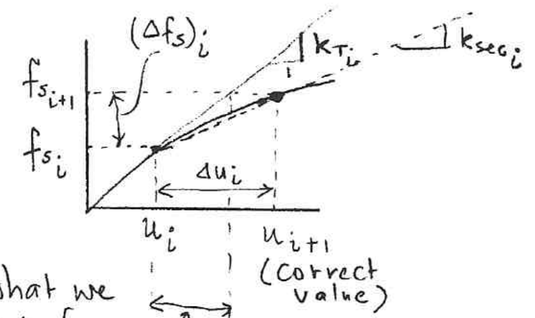
In nonlinear case, complications arise due to f_s

What we know at $t = t_i$ is the tangent stiffness k_{T_i}.

We really need the secant stiffness k_{sec_i} which will yield correct value of u_{i+1}

But replace $(\Delta f_s)_i$ by $k_{T_i}\Delta u_i$ in Eq. (1)

Graphically:



what we will get for Δu_i if we don't improve iteratively.

So, we need to iterate if we plan on using k_{T_i}

our setup now looks like this

$$\hat{k}_{T_i} \, \Delta u_i = \Delta \hat{p}_i \quad \left(\text{v/s} \quad \overset{\text{no subscript } i\,!}{\hat{k}} \, \Delta u_i = \Delta \hat{p}_i \text{ in the linear case} \right)$$

where $\hat{k}_{T_i} = k_{T_i} + \dfrac{\gamma}{\beta \Delta t} c + \dfrac{1}{\beta (\Delta t)^2} \cdot m$

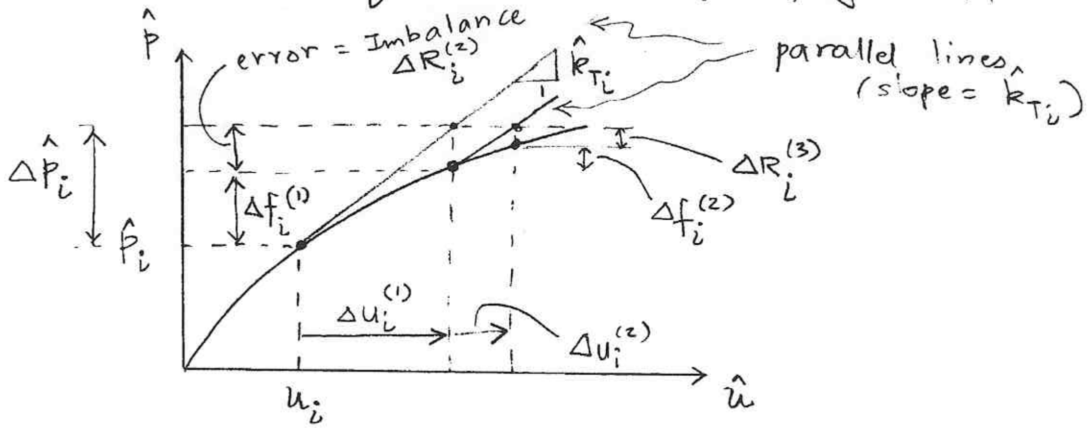
Recognizing we will have to iterate

Let's call our <u>First</u> calculation, $\Delta u_i^{(1)}$

$$\Delta u_i^{(1)} = \frac{\Delta \hat{p}_i}{k_{T_i}}$$

Why do we need to iterate?

Check whether equilibrium is satisfied if you stopped.



Note $\Delta f_i^{(1)} \neq \Delta \hat{p}_i$

where $\Delta f_i^{(1)} = m \Delta \ddot{u}_i^{(1)} + c \Delta \dot{u}_i^{(1)} + f_s(u_i + \Delta u_i^{(1)}) - f_s(u_i)$

Error = Imbalance = Residual force in this iteration

$= \Delta \hat{p}_i - \Delta f_i^{(1)}$

Call this Residual force for time step i, $\Delta R_i^{(2)} = \Delta \hat{P}_i - \Delta f_i^{(1)}$

(2) refers to the 2nd iteration which will try to improve our solution by solving $\hat{k}_{T_i} \Delta u_i^{(2)} = \Delta R_i^{(2)}$

Hence, $\Delta u_i^{(2)} = \dfrac{\Delta R_i^{(2)}}{\hat{k}_{T_i}}$

again determine

$$\Delta f_i^{(2)} = m \Delta \ddot{u}_i^{(2)} + c \Delta \dot{u}_i^{(2)} + f_s(u_i + \Delta u_i^{(1)} + \Delta u_i^{(2)}) - f_s(u_i + \Delta u_i^{(1)})$$

This time note that $\Delta f_i^{(2)} \neq \Delta R_i^{(2)}$ hence another iteration may be needed.

Compute new residual force for 3rd iteration, $\Delta R_i^{(3)} = \Delta R_i^{(2)} - \Delta f_i^{(2)}$

continue for as many steps as it takes to converge

For any tolerance, ε

Converge reached at iteration j if

$$\frac{\Delta u_i^{(j)}}{\sum_{k=1}^{j} \Delta u_i^{(k)}} < \varepsilon$$

$\Rightarrow$ no significant change results in Δu_i where $\Delta u_i = \sum_{k=1}^{j} \Delta u_i^{(k)}$

$u_{i+1} = u_i + \Delta u_i$ .... continue on to next time step.

This process is called the

Modified Newton - Raphson Iterative Process

since we always use $\hat{k}_{T_i}$

(i.e., we do not calculate corrected tangent stiffness at each iteration).

Original Newton - Raphson
uses updated tangent stiffness
and will converge faster BUT
requires more computations in each iteration.

Easy to implement this procedure for nonlinear analysis of dynamic systems.

Earthquake Response of Linear Systems

Our focus: Response of linear Elastic systems (SDOF idealizations)

$$\rightarrow \left[\begin{array}{l} \text{Deformations} \\ \text{Internal element forces, stresses} \end{array} \right.$$

as functions of time

and of system parameters (e.g., ξ, T_n)

Questions we'll address:

- What is a Response Spectrum?
- How does one use a Response Spectrum to determine the Peak Response?
- What is a Design Spectrum?
- How is a Design Spectrum different from a Response Spectrum?
- How are these spectra constructed?

Earthquake Excitation

Recall Equation of Motion: $m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g$

OR $\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = -\ddot{u}_g$

Given $\ddot{u}_g(t)$, ground acceleration as a function of time, how does one obtain $u(t)$?

IF m, c, k known OR if ξ, ω_n known

SDOF system response can be determined <u>numerically</u>

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Strong-Motion Accelerographs

- Record ground acceleration when triggered by the arrival of earthquake waves
- Usually three components of motion are recorded
  - 2 orthogonal Horizontal components
  - 1 vertical (up-down) component
- Generally, a transducer (m-c-k) element is used

  Analog: $f_n = 25\text{ Hz}$, $\xi = 60\%$

  can capture motions up to ~15 Hz

  Digital: $f_n = 50\text{ Hz}$, $\xi = 70\%$

  can capture motions up to ~30 Hz

  Other digital accelerographs are also available with different types of transducer elements. Many advances in technology to allow for wide frequency range, accuracy, size, ease of use, etc.

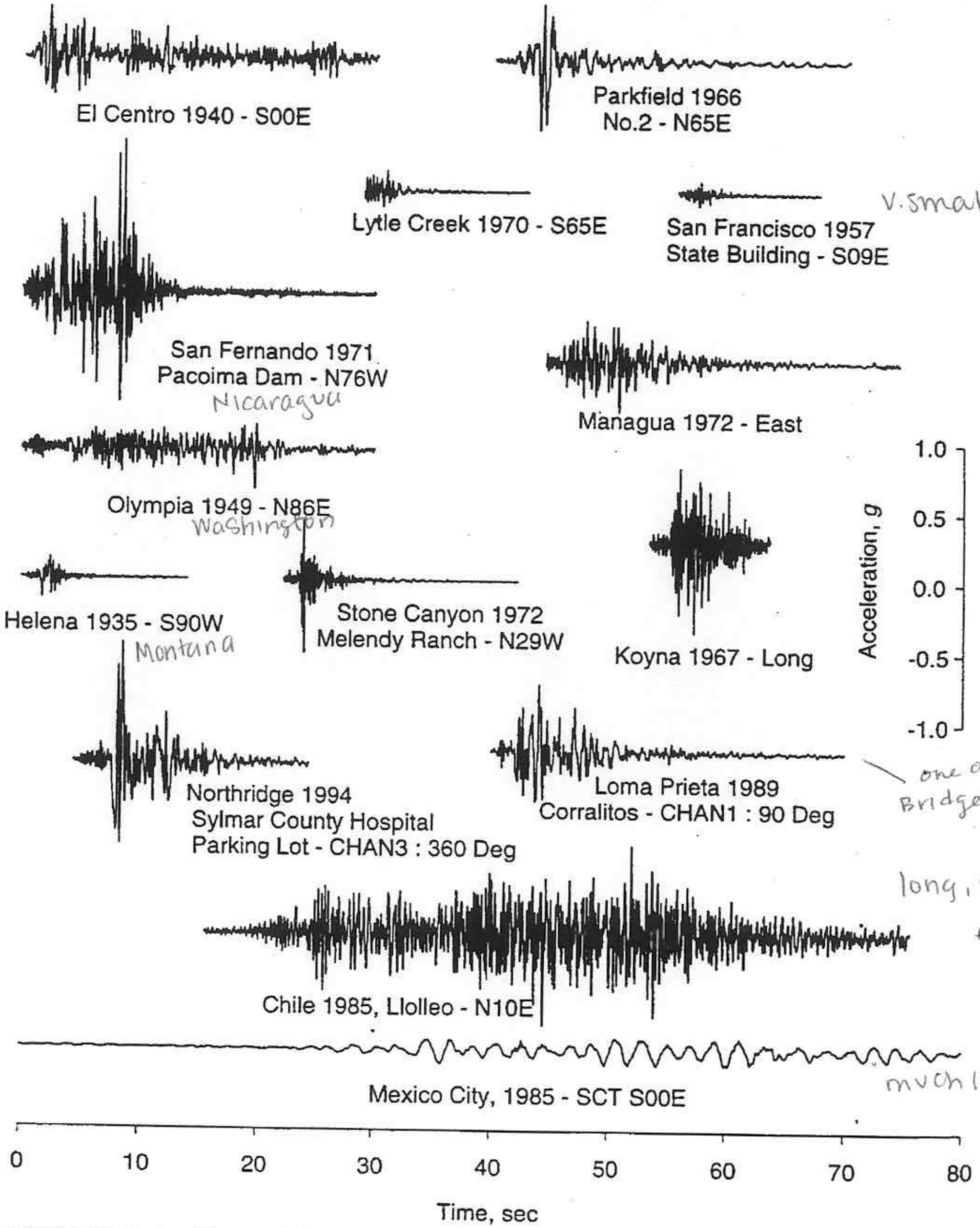
Peak Values of ground acceleration $\ddot{u}_g(t)$

- "Usually" decrease with distance from epicenter/point of rupture
- Depend on local soil conditions

Earthquake records display

Large variability in
- amplitude
- duration (of strong shaking)
- general appearance (frequency content, presence of pulses, spikiness, etc.)

Ground motions



California

V. small

Nicaragua

Washington

Montana

one of the Bay Bridge spans fell

long, rolling earthquake

much longer period

Figure 6.1.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

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EARTHQUAKE EXCITATION

Single DOF Response

$\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = -\ddot{u}_g(t)$

response depends on ω_n (k,m) and ξ (c)

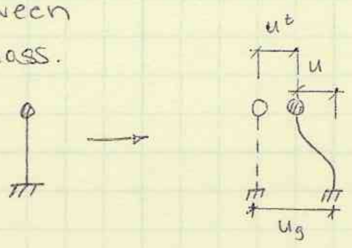
if two systems have the same ω_n and ξ, response will be the same (even if k, c, m are different)

for accurate approximations of u(t), use $\Delta t = 0.02$s or smaller

calculate

- $u(t), \dot{u}(t), \ddot{u}(t)$
- $u^t(t), \dot{u}^t(t), \ddot{u}^t(t)$ — totals. Recall: $\ddot{u}^t(t) = \ddot{u}(t) + \ddot{u}_g(t)$

u refers to the relative deflection between ground and mass.



Response values

total displacement of mass, $u^t(t)$

- needed to determine separation between buildings, to consider pounding

total acceleration of mass, $\ddot{u}^t(t)$

- important when considering transfer to sensitive instruments in building

From $\ddot{u}_g(t)$

$\hookrightarrow$ get $u(t) \equiv u(t, T_n, \xi)$

changing building characteristics

T_n : as it decreases, deformation response decreases, less time to max

ξ : as it increases, deformation decreases, less time to max

but remember: transmissibility has limits to ξ helping

generalizations are not necessarily accurate. oops. (see pg 209)

Response Spectra

$\ddot{u}_g(t) \longrightarrow \underset{T_n,\,\xi}{\text{SDOF}} \longrightarrow u(t)$

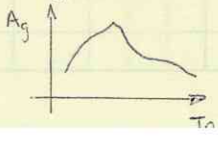
Equivalent static force, $f_S(t) = ku(t) = m\omega_n^2 u(t)$

$\hookrightarrow$ the external force that would produce the same deformation u(t), considering only k (m, c not important)

Define: $A(t) = \omega_n^2 u(t)$ — pseudo-acceleration response

$f(t) = mA(t)$

A convenient means to summarize the peak response of <u>all</u> possible linear SDOF systems to a particular component from earthquake records.



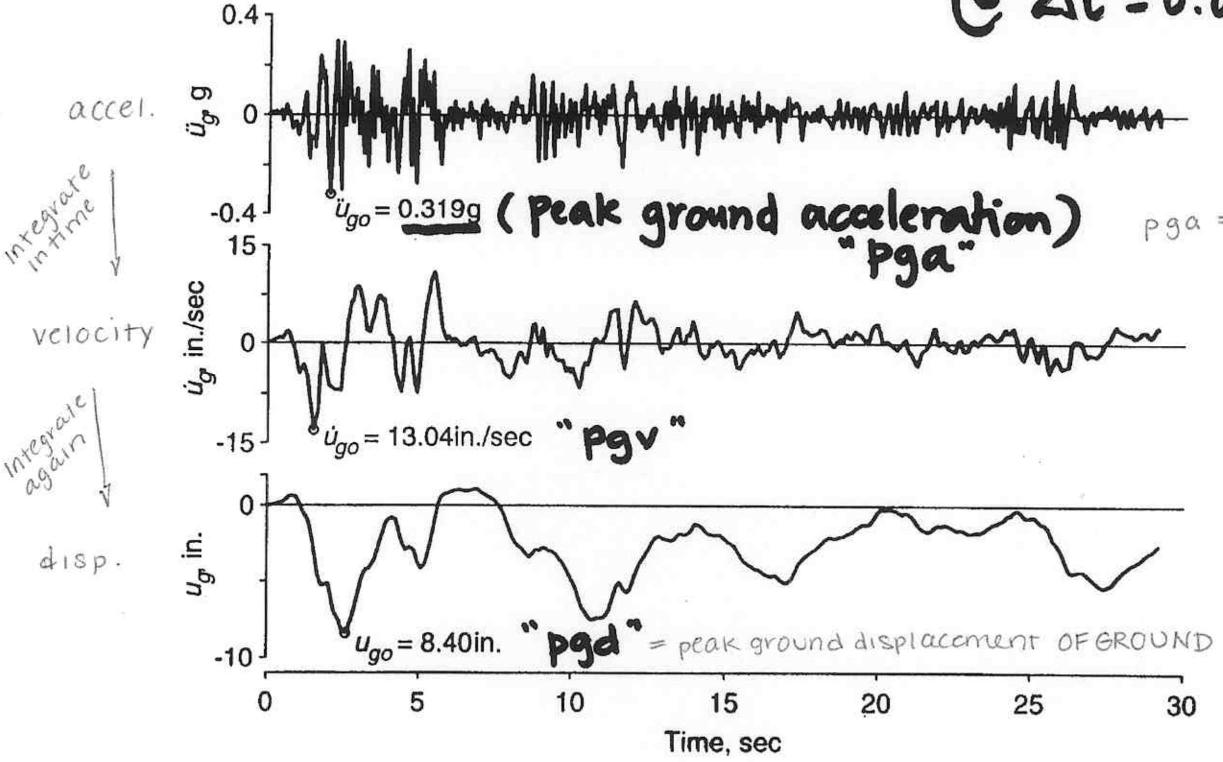
Remember:

$p_i = -m \cdot \ddot{u}_{gi}$

El Centro ground motion (N-S component)

(Imperial Valley EQ) May 18, 1940

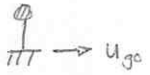
1559 data points @ $\Delta t = 0.02$ sec



$pga = \ddot{u}_{go} = \max|\ddot{u}_g(t)|$

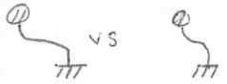
(negative sign not needed)

Figure 6.1.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.



strain in the column relates to transmissibilit

— how much does load move?



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14-4

Consider El Centro, California site

Imperial Valley Earthquake (May 18, 1940)

North-South component

(Site: Imperial Valley Irrigation District Substation)

Peak ground acceleration = pga = 0.319g $= \ddot{u}_{g_o}$

Peak ground velocity = pgv = 13.04 in/s $= \dot{u}_{g_o}$

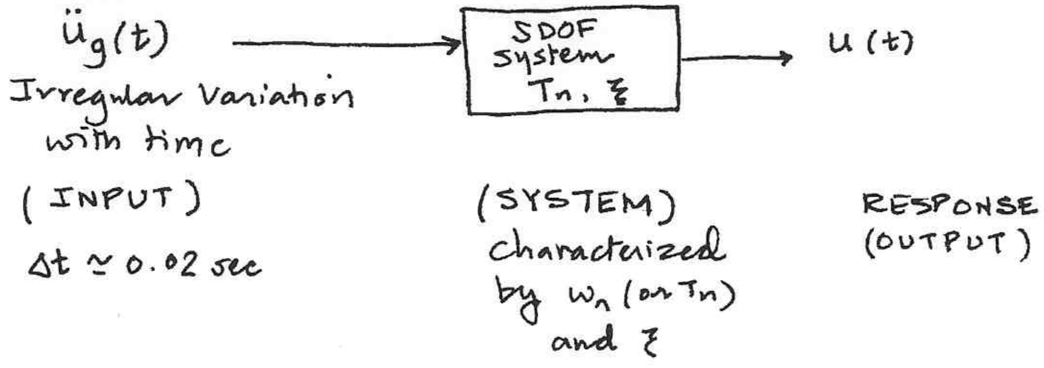
Peak ground displacement = pgd = 8.40 in. $= u_{g_o}$

(subscript "o" → peak)

Record has 1559 data points @ $\Delta t = 0.02$ sec intervals.

Typical records of ground motion have $\Delta t = 0.01$ to 0.02 seconds (to capture irregular variations) and ~ 1500 to 3000 ordinates

Computing SDOF Response



Solution to $\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = -\ddot{u}_g(t)$ needs numerical methods because of irregular $\ddot{u}_g(t)$ records. Use $\Delta t = 0.02$ sec or smaller

Numerical Solution can provide $u(t)$, $\dot{u}(t)$, $\ddot{u}(t)$ (RELATIVE) as well as $u^t(t)$, $\dot{u}^t(t)$, $\ddot{u}^t(t)$ (TOTAL)

Recall: $\ddot{u}^t(t) = \ddot{u}(t) + \ddot{u}_g(t)$, etc.

Response Quantities of Engineering Interest

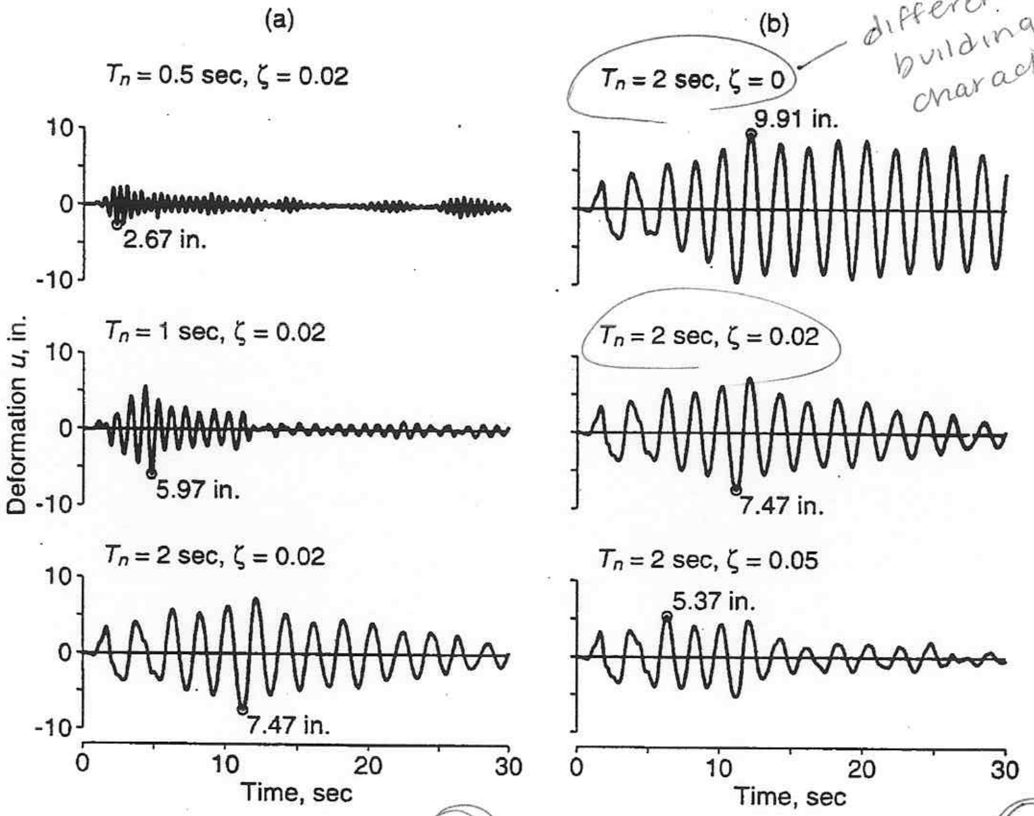
Displacement of the mass relative to the ground	$u(t)$
Internal forces/stresses (linearly related to displacement)	e.g., BM, SF in beams, columns
Total Displacement of Mass useful → to determine separation distance b/w buildings to prevent pounding	$u^t(t)$
Total Acceleration of Mass useful → for cases where structure supports sensitive equipment (ground motion "transmitted" to equipment)	$\ddot{u}^t(t)$

Response History

Given $\ddot{u}_g(t)$ ⟶ Determine $u(t) \equiv u(t; T_n, \xi)$

i.e., $u(t)$ for a specified T_n and ξ.

Deformation response of SDF systems to El Centro ground motion



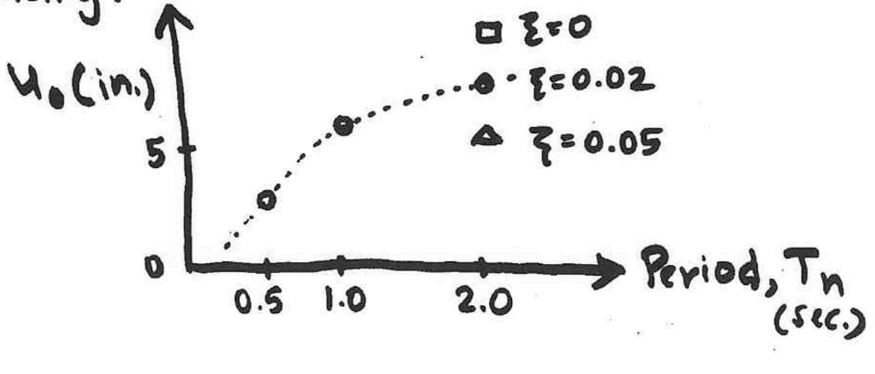
different building characteristics

Effect of Period, T_n
$\xi = 0.02$ fixed

Effect of damping, ξ
$T_n = 2$ sec, fixed

Figure 6.4.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Summary:



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Equivalent Static Force and Pseudo-acceleration Response

$$\ddot{u}_g(t) \longrightarrow \underset{\substack{T_n,\ \xi \\ \text{or } m, c, k}}{\boxed{\text{SDOF}}} \longrightarrow u(t)$$

Equivalent static force, $f_s(t) = k\,u(t)$ [See Note below]

$$= m\omega_n^2\,u(t)$$

Note: $f_s(t)$ is called the "equivalent" static force because it is the external force that would produce the same deformation $u(t)$ (at any time, t) in the stiffness component of the structure (i.e., the structure without mass or damping)

Define $A(t) = \omega_n^2\,u(t)$ as the Pseudo-acceleration response.

$$f_s(t) = m\,A(t)$$

Note: $A(t) \neq \ddot{u}(t)$ and $A(t) \neq \ddot{u}^t(t)$

$$\begin{tikzcd}
\ddot{u}_g(t) \arrow[r] & \underset{T_n,\ \xi}{\boxed{\text{SYSTEM}}} \arrow[r] & u(t) \arrow[r] & \boxed{*\ \omega_n^2} \arrow[r] & A(t) \arrow[d] \\
 & & & & \boxed{*\ m} \arrow[d] \\
 & & & & f_s(t)
\end{tikzcd}$$

$\Rightarrow$ $u(t)$ multiplied by $\omega_n^2 \left[= \left(\frac{2\pi}{T_n}\right)^2\right]$ gives $A(t)$; pseudo-acceleration response

Multiplier <u>decreases</u> with increasing period.

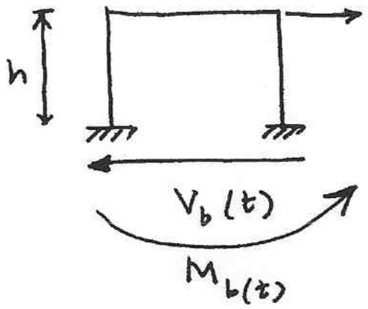
$A(t) = \omega_n^2\,u(t)$ can be plotted for El Centro motion directly (once $u(t)$ is known)

$\xi = 2\%$	T_n (sec)	0.5	1.0	2.0
	$u_o = \max u(t)$ (in.)	2.67	5.97	7.47
	$A_o = \max A(t)$	1.09g	0.61g	0.19g

Internal Forces

Using static analysis and equivalent static force (f_s), shears, moments, axial forces in beams, columns, etc. of structures or stresses at any locations can be computed.

Instantaneous values (i.e., at any specified time) can be obtained for any response quantity.



$$f_s(t) = m A(t) = m \cdot \left(\frac{2\pi}{T_n}\right)^2 u(t)$$

For 1-story frame,

- obtain $f_s(t)$
- Perform static analysis

here, Base Shear $V_b(t) = f_s(t)$

Base Overturning Moment $M_b(t) = h \cdot f_s(t)$

OR $\quad V_b(t) = m A(t) \quad$ Base Shear

and $\quad M_b(t) = h \cdot V_b(t) \quad$ Base Overturning Moment

Pseudo-acceleration response of SDF systems to El Centro ground motion

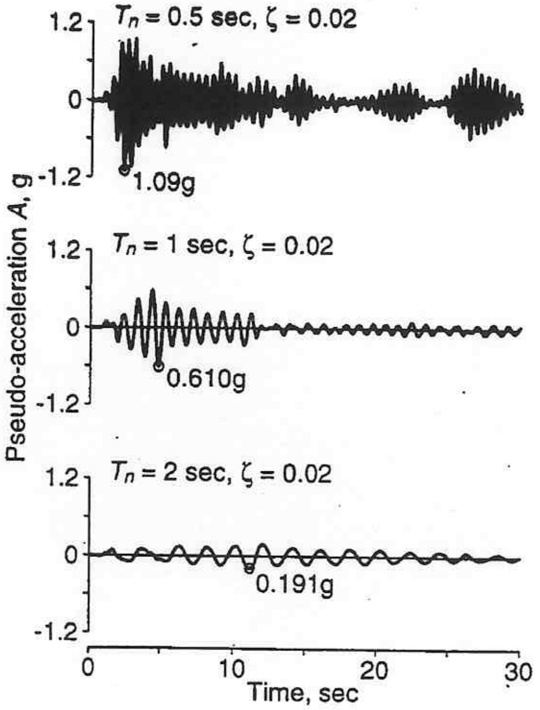
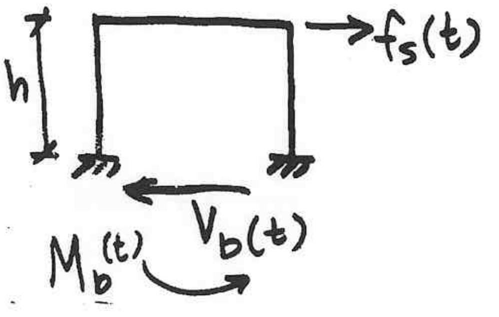


Figure 6.4.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$$A(t) = \omega_n^2 u(t)$$



$$V_b(t) = m A(t)$$

$$M_b(t) = h \cdot V_b(t)$$

$$= m \cdot h \cdot A(t)$$

Response Spectrum

A response spectrum is a plot of the peak value of a response quantity as a function of the natural period (T_n) or frequency of a SDOF with a fixed damping ratio, ξ.
Usually, several plots each for a different damping ratio are plotted.

Example:

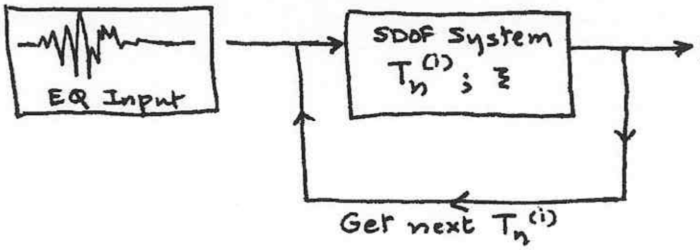
	Plot of	Peak Response
Displacement Response Spectrum	$u_o(T_n, \xi)$ v/s T_n	$u_o(T_n, \xi) \equiv \max_t \lvert u(t, T_n, \xi) \rvert$
Relative Velocity Response Spectrum	$\dot{u}_o(T_n, \xi)$ v/s T_n	$\dot{u}_o(T_n, \xi) \equiv \max_t \lvert \dot{u}(t, T_n, \xi) \rvert$
Relative Acceleration Response Spectrum	$\ddot{u}_o(T_n, \xi)$ v/s T_n	$\ddot{u}_o(T_n, \xi) \equiv \max_t \lvert \ddot{u}(t, T_n, \xi) \rvert$
Acceleration Response Spectrum	$\ddot{u}_o^t(T_n, \xi)$ v/s T_n	$\ddot{u}_o^t(T_n, \xi) \equiv \max_t \lvert \ddot{u}^t(t, T_n, \xi) \rvert$
Pseudo-acceleration Response spectrum	$A_o(T_n, \xi)$ v/s T_n	$A_o(T_n, \xi) \equiv \max_t \lvert A(t, T_n, \xi) \rvert$ $= \left(\frac{2\pi}{T_n}\right)^2 u_o(T_n, \xi)$

A Response Spectrum is a convenient means to summarize the peak response of all possible linear SDOF systems to a particular component of earthquake ground motion recorded at a site.

It is a practical means of characterizing ground motions and their effects on structures.

Types of Response Spectra

1. Interest : $\overset{\text{Peak}}{\text{Deformation}}$ (D)



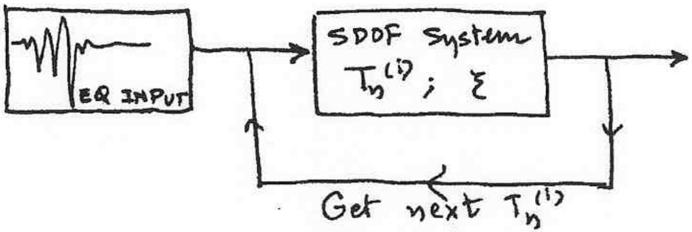
Save $u_o(T_n^{(i)}, \xi)$
$$\equiv \max_t \left| u(t, T_n^{(i)}, \xi) \right|$$

Plot $u_o(T_n, \xi)$ v/s T_n (Deformation Response Spectrum)

Repeat for other damping ratios, if necessary.

$$\left[u_o(T_n, \xi) \equiv D(T_n, \xi) \right]$$

2. Interest : $\overset{\text{Peak}}{\text{Strain Energy}}$ (E_{S_o})



Save $V(T_n^{(i)}, \xi)$
$$\equiv \left(\frac{2\pi}{T_n^{(i)}}\right) \cdot \max_t \left| u(t, T_n^{(i)}, \xi) \right|$$

Peak Strain Energy $= E_{S_o} = \frac{1}{2} k u_o^2$ (u_o = Peak Deformation)

$$= \frac{1}{2} m \cdot \omega_n^2 u_o^2$$

$$= \frac{1}{2} m \cdot V^2$$

where $V \equiv$ $\overset{\text{Peak}}{\text{pseudo-velocity}}$ response (define)

Note : V ($\overset{\text{peak}}{\text{pseudo-velocity}}$) is directly related to Peak Strain Energy. Also $V = \omega_n \cdot D$

Plot of $V(T_n, \xi)$ v/s T_n (Pseudo-Velocity Response Spectrum)

$$\left[V(T_n, \xi) \equiv \frac{2\pi}{T_n} \cdot D(T_n, \xi) \right]$$

Deformation response spectrum

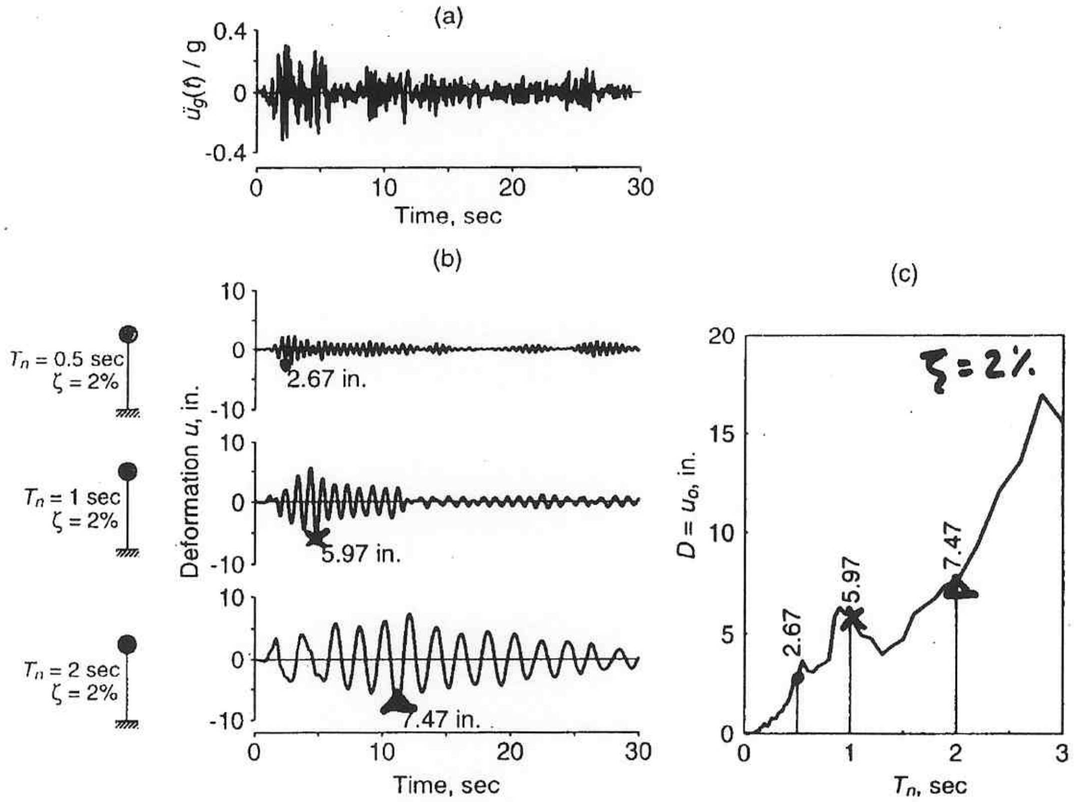
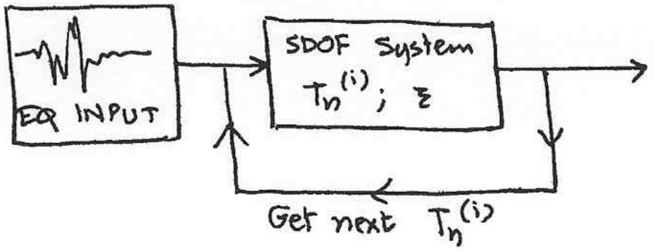


Figure 6.6.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

3. Interest : Peak Equivalent Static Force (f_{s_0})



Save $A(T_n^{(i)}, \xi)$

$$\equiv \left(\frac{2\pi}{T_n^{(i)}}\right)^2 \cdot \max_t \left|u(t, T_n^{(i)}, \xi)\right|$$

Peak Equivalent Static Force $= f_{s_0}$

$$= k \cdot u_0$$

$$= m \cdot \omega_n^2 u_0$$

$$= m \cdot A$$

where $A \equiv$ Peak pseudo-acceleration response (define)

Note : A (Peak pseudo-acceleration) is directly related to Peak Equivalent Static Force.

Also, $A = \omega_n^2 D$

Plot of $A(T_n, \xi)$ v/s T_n (Pseudo-acceleration Response Spectrum)

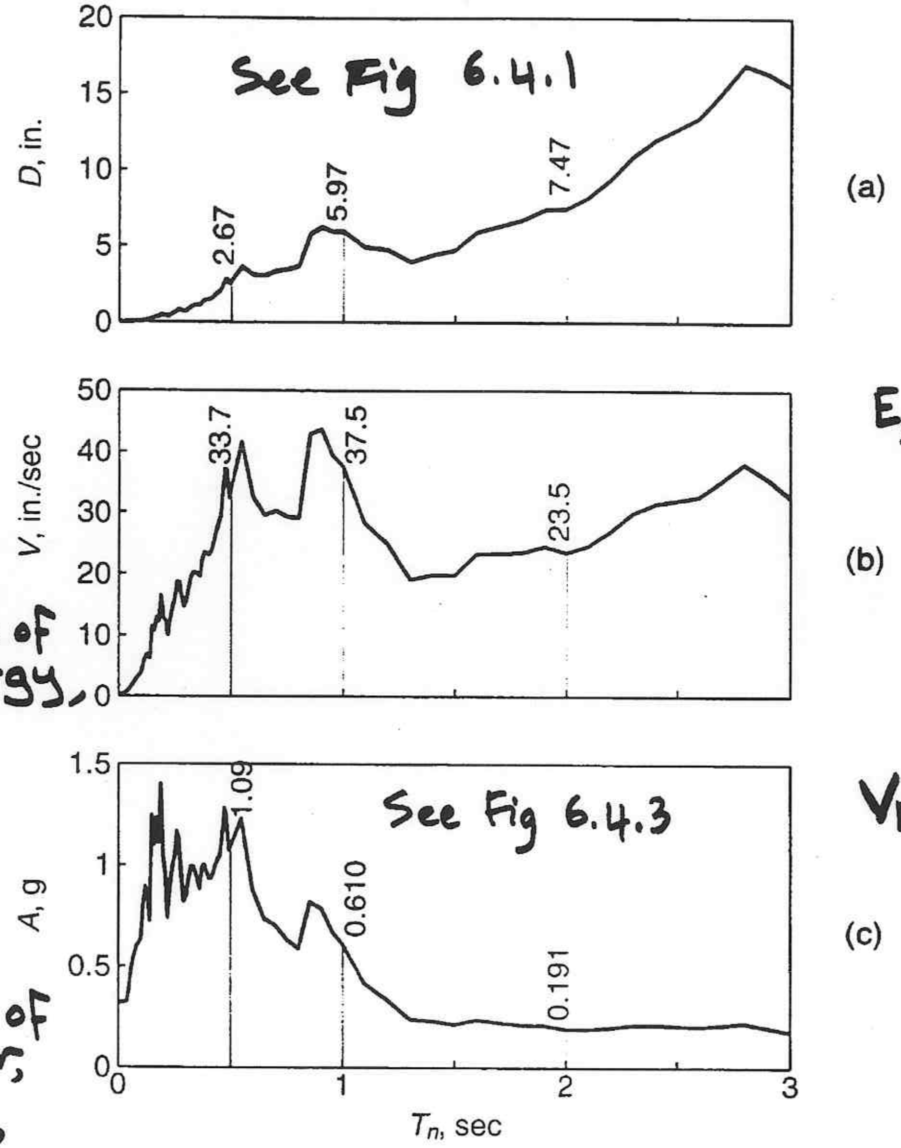
$$\left[\; A(T_n, \xi) \equiv \left(\frac{2\pi}{T_n}\right)^2 \cdot D(T_n, \xi) \; \right]$$

Example with El Centro N-S component and 2% damping

T_n (sec.)	D	$V = \frac{2\pi}{T_n} \cdot D$	$A = \left(\frac{2\pi}{T_n}\right)^2 \cdot D$
0.5	2.67 in.	33.7 in/sec	1.09 g
1.0	5.97 in.	37.5 in/sec	0.610 g
2.0	7.47 in.	23.5 in/sec	0.191 g

If other periods are included, plots of D v/s T_n, V v/s T_n, and A v/s T_n will yield Deformation, Pseudo-velocity, and pseudo-acceleration response spectra for the El Centro motion and for 2% damping.

Deformation, pseudo-velocity, and pseudo-acceleration response spectra ($\zeta = 2\%$)



See Fig 6.4.1

$u_0 = D$

$E_{s_0} = \frac{1}{2} k u_0^2$

$= \frac{1}{2} k D^2$

$= \frac{1}{2} m V^2$

Related to peak value of strain energy, E_{s_0}

See Fig 6.4.3

$V_{b_0} = f_{s_0}$

$= k u_0$

$= k D$

$= m A$

Related to peak value of Base shear, V_{b_0}

Figure 6.6.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$D \equiv u_0$ Peak displacement

$V = \omega_n D$ Pseudo-velocity

$A = \omega_n^2 D$ Pseudo-acceleration

Four-Way Log Paper for Spectra of D, V, A

D : Peak Deformation

V : Peak Pseudo-Velocity

A : Peak Pseudo-Acceleration

$$\frac{A}{\omega_n} = V = \omega_n D \qquad -(*)$$

⇒ All 3 spectra (for D, V, and A) provide the same information apart from a factor of ω_n (or $\frac{2\pi}{T_n}$)

⇒ Can use 4-way log paper to show all 3 spectra together

$$V = \frac{A}{\omega_n} = \frac{A T_n}{2\pi}$$

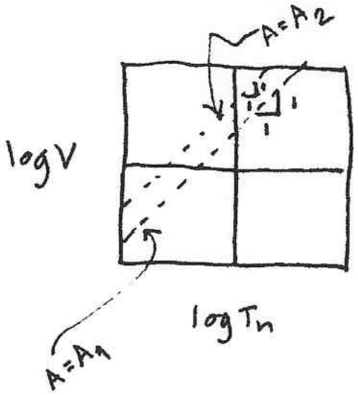
$$\Rightarrow \log V = \log T_n + \log\left(\frac{A}{2\pi}\right)$$

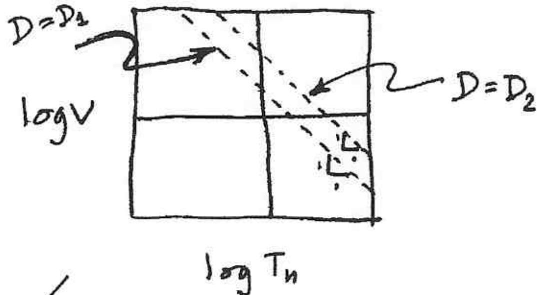
$\log V$ vs. $\log T_n$ is a straight line with slope = +1 if A = const.

$$V = \omega_n D = \frac{2\pi D}{T_n}$$

$$\Rightarrow \log V = -\log T_n + \log(2\pi D)$$

$\log V$ vs. $\log T_n$ is a straight line with slope = −1 if D = const.





combine on 4-way log paper

For any T_n,

Vertical Scale ↑ used to read off V

Inclined Scale ↖ used to read off A

Inclined Scale ↗ used to read off D

In practice, sufficient to read any one of V, D, A & use eqn (*) to get other two.

Combined *D-V-A* response spectrum (ζ = 2%)

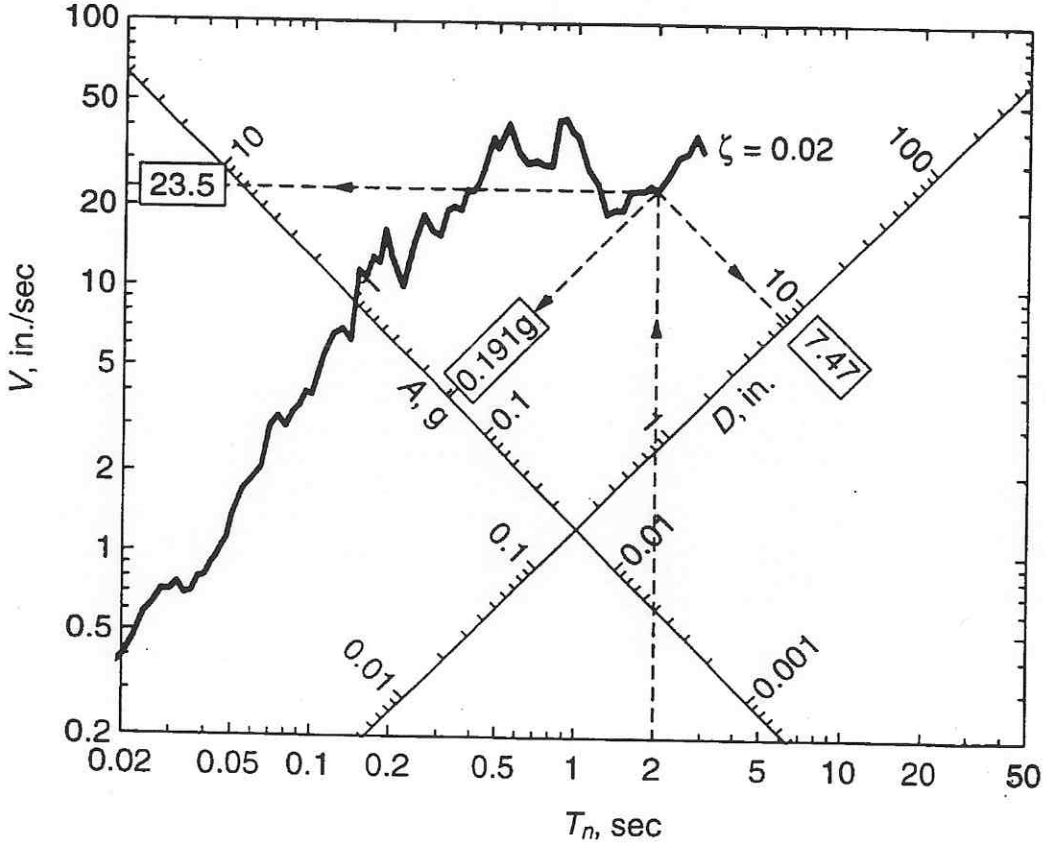


Figure 6.6.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$$V = \frac{2\pi}{T_n} \cdot D \Rightarrow \log V = -\log T_n + \log(2\pi D)$$

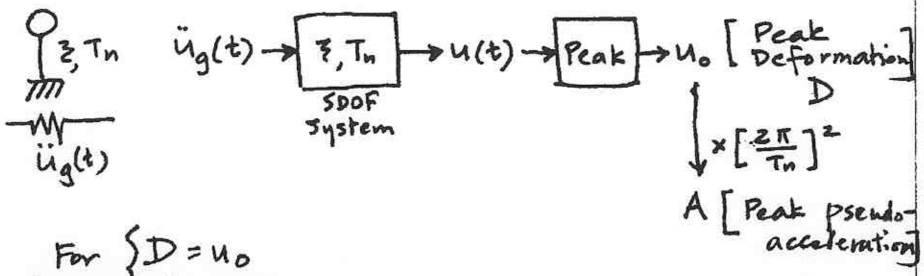
$$V = \frac{T_n}{2\pi} \cdot A \Rightarrow \log V = \log T_n + \log\left(\frac{A}{2\pi}\right)$$

How to Construct a Response Spectrum

① → Take the ground acceleration record, $\ddot{u}_g(t)$ typically available in digitized form (e.g., discretized at $\Delta t = 0.02$ sec time intervals)

② → Select a damping ratio, ξ

Select a natural period, T_n (for the given ξ)



For Response Spectra Ordinates
$$\begin{cases} D = u_o \\ V = \left(\dfrac{2\pi}{T_n}\right) u_o \\ A = \left(\dfrac{2\pi}{T_n}\right)^2 u_o \end{cases}$$

Take next T_n (for the given ξ)

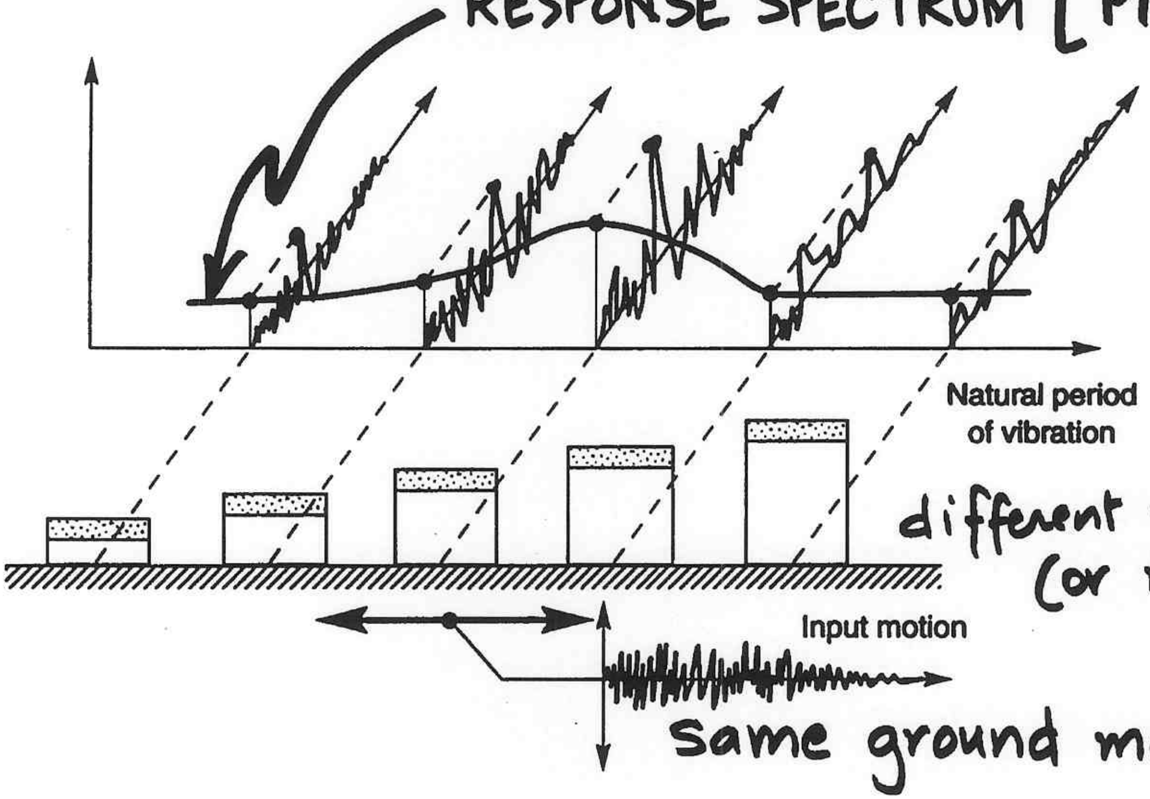
Take next damping ratio, ξ

③ Plot D, V, A versus T_n for each damping ratio

Plots can be made separately or together

Plots can be on log paper or on plain paper.

RESPONSE SPECTRUM [Plot of Peak Response vs. Period]



different structures (or natural periods)

same ground motion

Generally, Response Spectra for most earthquake records have been computed and published or are available in the public domain.

e.g. California Division of Mines and Geology

Web site :- http://docinet3.consrv.ca.gov/csmip/servlet/listsearch

Response Spectra typically cover:

Period range : 0.02 to 50 sec

Damping ratio range : 0 to 20%

(e.g., with El Centro N-S component from 1940 earthquake)

Effort required

112 periods b/w 0.02 & 50 sec

5 damping ratios 0, 2, 5, 10, & 20%

1559 data points @ 0.02 sec intervals for El Centro

⇒ $\ddot{u}_g(t)$ run thru' 560 (=112×5) different structures and peak response is recorded on spectra.

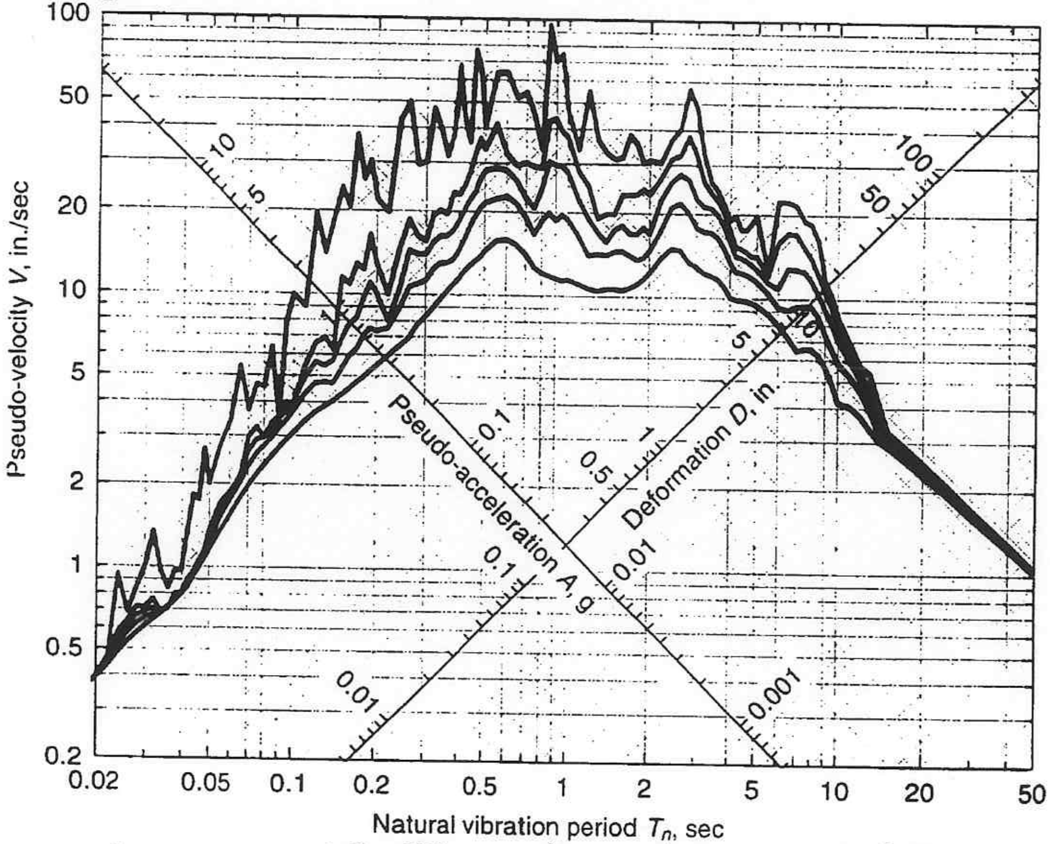
Response Spectra appearance depends on:

- Magnitude
- Distance from rupture/fault to station
- Local soil conditions
- Regional geology
- Type of rupture
- Propagation path

Effort Required to Create Response Spectra

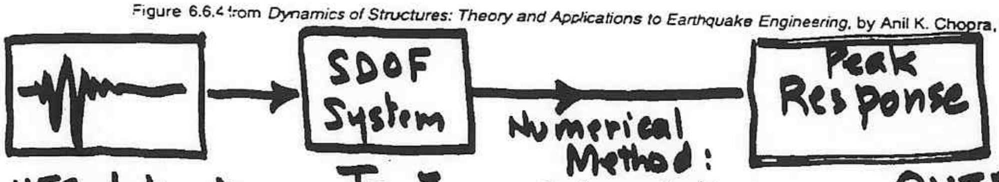
Response spectrum for El Centro ground motion

5 damping ratios $\zeta = 0, 2, 5, 10$, and 20%.



← **112** T_n values b/w 0.05 & 50 sec →

Figure 6.6.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.



1159 data pts @ $\Delta t = 0.02$ sec

INPUT

T_n, ζ

Numerical Method: Interpolation of Excitation

OUTPUT

112 × 5 = 560 times $\ddot{u}_g(t)$ run thru different structures

Pseudo-acceleration response spectrum

- El Centro ground motion
- ζ = 0, 2, 5, 10, and 20%

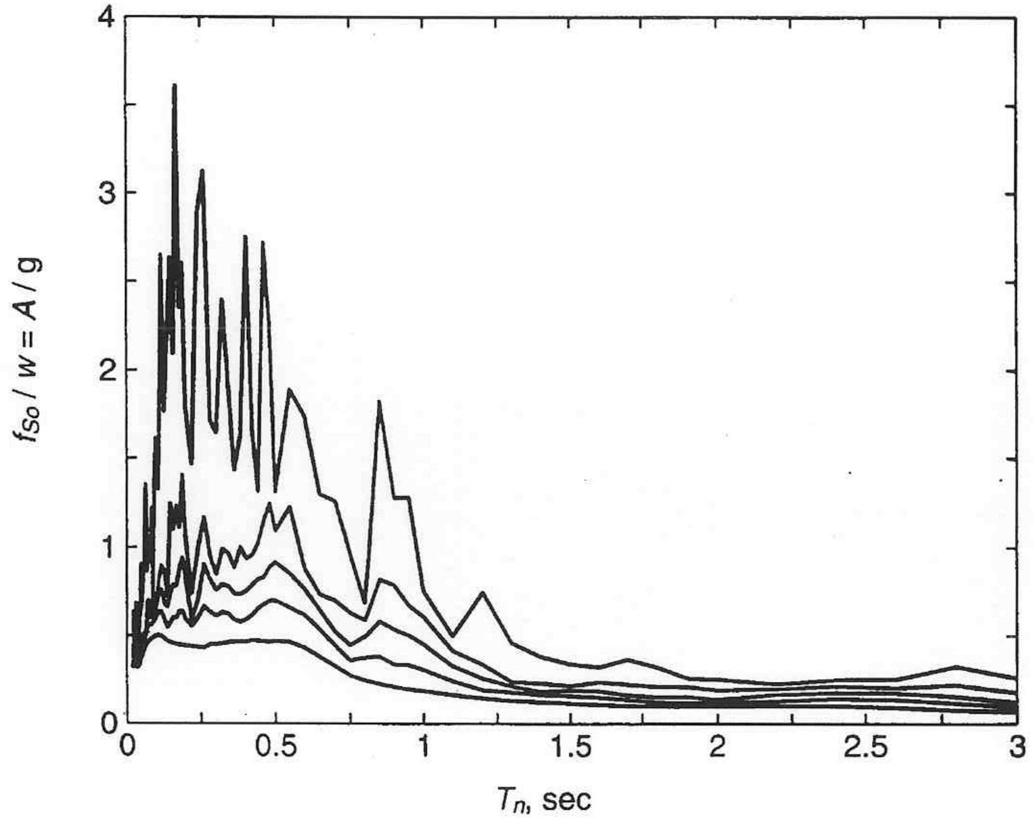
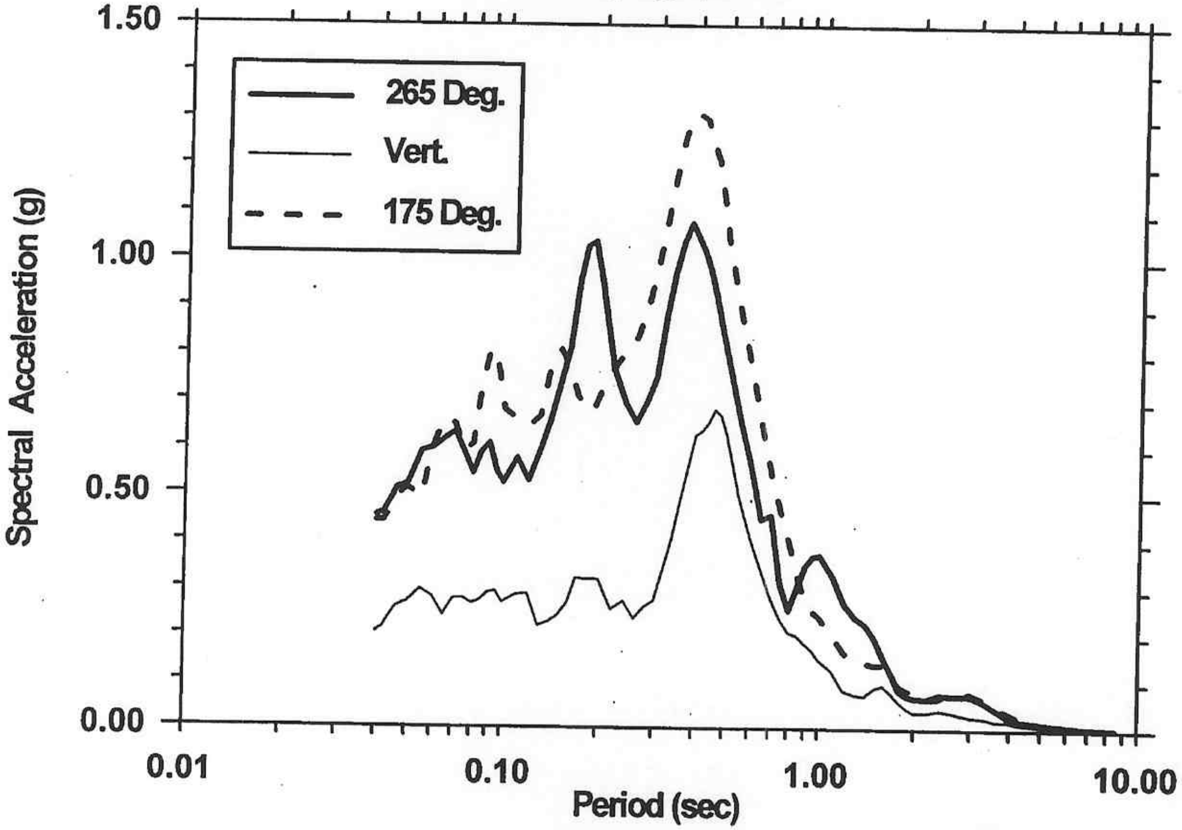
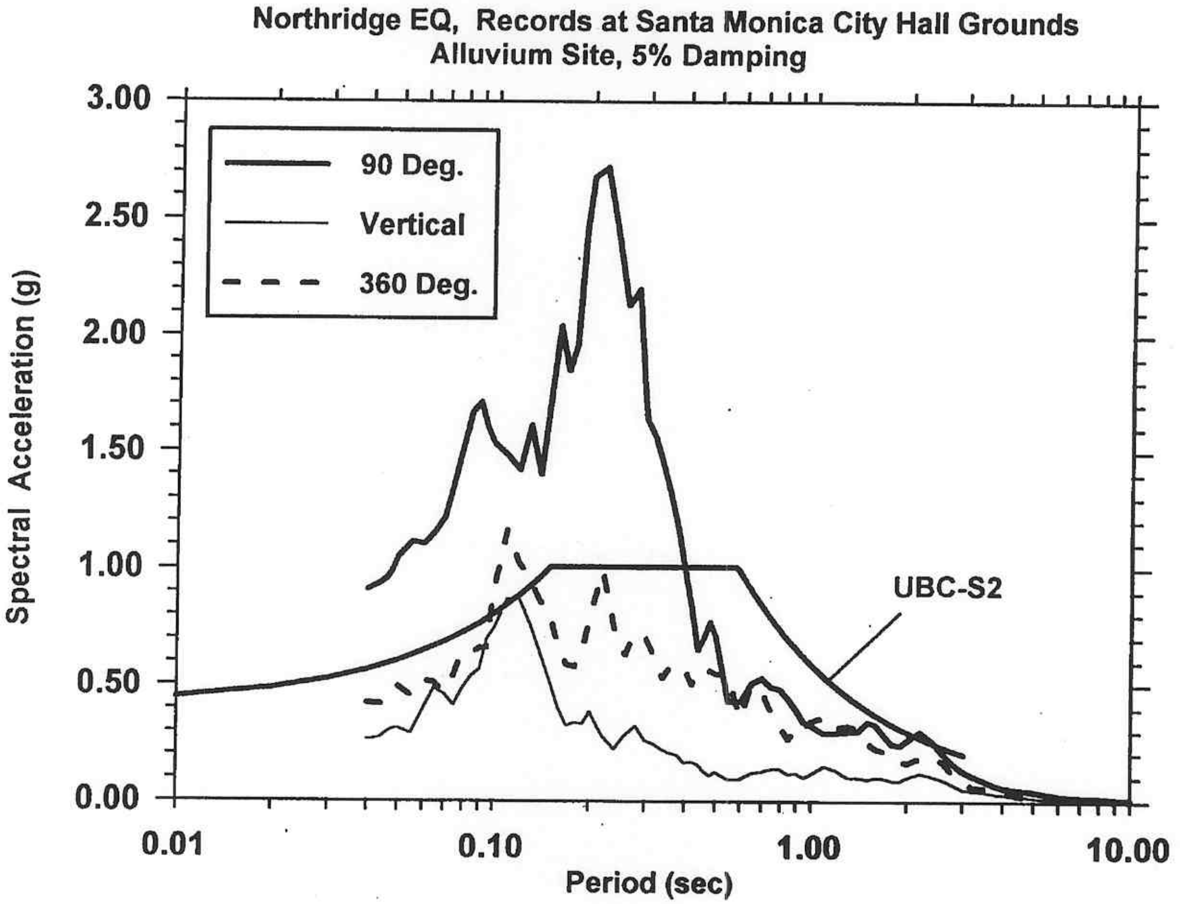


Figure 6.6.5 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Northridge EQ, Records at Pacoima Dam Downstream
5% Damping Spectra



based on 1500 pts @ 0.02 sec.



Northridge EQ, Records at Santa Monica City Hall Grounds
Alluvium Site, 5% Damping

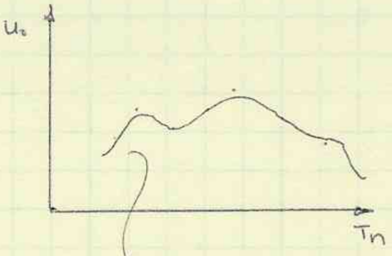
UBC - S2:

A soil profile with predominantly medium-dense to dense or medium-stiff to stiff soil conditions, where the soil depth exceeds 200 ft

STAEDTLER® No. 937 811E Engineer's Computation Pad

EARTHQUAKE!

Response Spectrum



lots of points from different systems, gathered

linear elastic — peak response of a SDOF system (T_n, ζ given) ~~ta~~ from a given excitation (ground accel. record, $\ddot{u}_g(t)$)

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = -\ddot{u}_g(t)$$

D = displacement response spectrum
V = pseudo-velocity RS
A = pseudo-acceleration RS

} scaled plots from original u_o plot, D

$$V = \left(\frac{2\pi}{T_n}\right) D$$

$$A = \left(\frac{2\pi}{T_n}\right)^2 D$$

V_{bo} (maximum ~~acc~~ velocity response)

$$V_{bo} = mA$$

peak demand on the base of the structure during an earthquake

vs. real response spectrum

$\dot{u}_o$ = velocity R.S.

Ground response

$\ddot{u}_{go}$ = peak ground acceleration

$$\underset{\text{response}}{\ddot{u}^t(t) = \ddot{u}(t)} + \underset{\text{ground}}{\ddot{u}_g(t)}$$

Homework comments

1. plot $\ddot{u}_g(t)$, find maximum
   3 curves — different for each ζ
   use interpolation and Newmark
2. shift data (3000 pts) to single column

How to Use a Response Spectrum to Determine Peak Response

Easy!

Work is already done in creating the Response Spectrum

Take T_n, ξ of the system/structure under consideration

Read off D, V, A (any one of these will do)

from plot on linear or 4-way log curves.

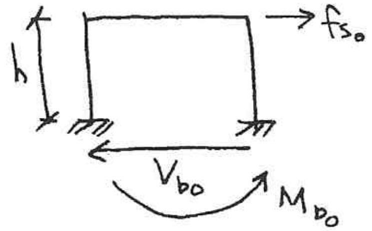
Peak Deformation $u_o = D = (T_n/2\pi)^2 \cdot A$

Peak Equivalent static Force $f_{s_o} = kD = mA$

(apply f_{s_o} in static analysis in order to find Peak Base Shear, V_{b_o}

Peak Overturning Moment, M_{b_o}, etc.)

e.g. One-story frame



$f_{s_o} = V_{b_o}$

$\Rightarrow V_{b_o} = kD = m \cdot A$

$M_{b_o} = h \cdot kD = h \cdot m \cdot A$

Example using Response Spectrum

A 12-ft-long vertical cantilever, a 4-in.-nominal-diameter standard steel pipe, supports a 5200-lb weight attached at the tip as shown in Fig. E6.2. The properties of the pipe are: outside diameter, $d_o = 4.500$ in., inside diameter $d_i = 4.026$ in., thickness $t = 0.237$ in., and second moment of cross-sectional area, $I = 7.23$ in^4, elastic modulus $E = 29,000$ ksi, and weight = 10.79 lb/foot length. Determine the peak deformation and bending stress in the cantilever due to the El Centro ground motion. Assume that $\boxed{\zeta = 2\%.}$

Solution The lateral stiffness of this SDF system is

$$k = \frac{3EI}{L^3} = \frac{3(29 \times 10^3)7.23}{(12 \times 12)^3} = 0.211 \text{ kip/in.}$$

The total weight of the pipe is 10.79 × 12 = 129.5 lb, which may be neglected relative to the lumped weight of 5200 lb. Thus

$$m = \frac{w}{g} = \frac{5.20}{386} = 0.01347 \text{ kip-sec}^2\text{/in.}$$

The natural vibration frequency and period of the system are

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.211}{0.01347}} = 3.958 \text{ rad/sec} \qquad \boxed{T_n = 1.59 \text{ sec}}$$

From the response spectrum curve for $\zeta = 2\%$ (Fig. E6.2b), for $T_n = 1.59$ sec, $\underline{D = 5.0 \text{ in.}}$ and $\underline{A = 0.20\ g}$. The peak deformation is

$$u_o = D = 5.0 \text{ in.}$$

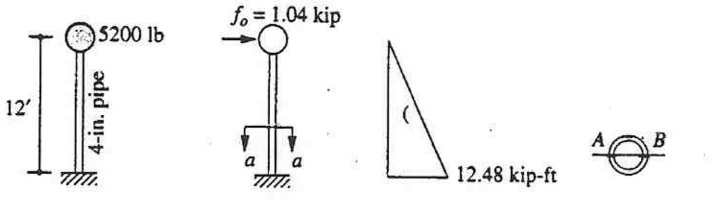
Note:
This could have been computed (instead of reading spectrum) using D

$$A = \left(\frac{2\pi}{T_n}\right)^2 \cdot D$$

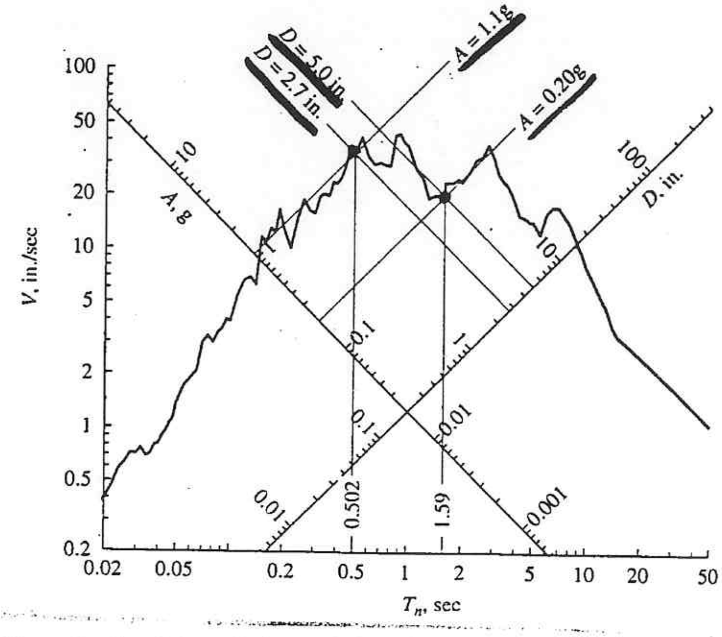
$$= \left(\frac{2\pi}{1.59}\right)^2 \cdot 5$$

$$= 78.1 \text{ in/s}^2$$

$$= 0.20g$$



(a) (c) (d) Moment (e) Section a-a



The peak value of the equivalent static force is

$$f_{So} = \frac{A}{g} w = 0.20 \times 5.2 = 1.04 \text{ kips}$$

The bending moment diagram is shown in Fig. E6.2d with the maximum moment at the base = 12.48 kip-ft. Points A and B shown in Fig. E6.2e are the locations of maximum bending stress:

$$\sigma_{\max} = \frac{Mc}{I} = \frac{(12.48 \times 12)(4.5/2)}{7.23} = 46.5 \text{ ksi}$$

As shown, $\sigma = +46.5$ ksi at A and $\sigma = -46.5$ ksi at B, where + denotes tension. The algebraic signs of these stresses are irrelevant because the direction of the peak force is not known, as the pseudo-acceleration spectrum is, by definition, positive.

Example 6.3

The stress computed in Example 6.2 exceeded the allowable stress and the designer decided to increase the size of the pipe to an 8-in.-nominal standard steel pipe. Its properties are $d_o = 8.625$ in., $d_i = 7.981$ in., $t = 0.322$ in., and $I = 72.5\ \text{in}^4$. Comment on the advantages and disadvantages of using the bigger pipe.

Solution

$$k = \frac{3(29 \times 10^3)72.5}{(12 \times 12)^3} = 2.112 \text{ kips/in.}$$

$$\omega_n = \sqrt{\frac{2.112}{0.01347}} = 12.52 \text{ rad/sec} \qquad \boxed{T_n = 0.502 \text{ sec}}$$

From the response spectrum (Fig. E6.2b): $\underline{D = 2.7 \text{ in.}}$ and $\underline{A = 1.1g}$. Therefore,

$$\hookrightarrow \text{check: } A = \left(\frac{2\pi}{0.5}\right)^2 \cdot (2.7) \frac{1}{(32.2)(12)} = 1.1g$$

$$u_o = D = 2.7 \text{ in.}$$

$$mA = f_{So} = 1.1 \times 5.2 = 5.72 \text{ kips}$$

$$f_{S_o} \cdot h = M_{\text{base}} = 5.72 \times 12 = 68.64 \text{ kip-ft}$$

$$\sigma_{\max} = \frac{(68.64 \times 12)(8.625/2)}{72.5} = 49.0 \text{ ksi}$$

Using the 8-in.-diameter pipe decreases the deformation from 5.0 in. to 2.7 in. However, contrary to the designer's objective, the bending stress increases slightly.

Dynamics v/s Statics

This example points out an important difference between the response of structures to earthquake excitation and to a fixed value of static force. In the latter case, the stress would decrease, obviously, by increasing the member size. In the case of earthquake excitation, the increase in pipe diameter shortens the natural vibration period from 1.59 sec to 0.50 sec, which for this response spectrum has the effect of increasing the equivalent static force f_{So}. Whether the bending stress decreases or increases by increasing the pipe diameter depends on the increase in section modulus, I/c, and the increase or decrease in f_{So}, depending on the response spectrum.

$$\sigma_{\max} = M_{\text{base}} \cdot \frac{c}{I} = m \cdot A \cdot h \cdot \frac{c}{I} = m \cdot h \cdot \left(\frac{A}{I/c}\right)$$

NEW [redacted] | [redacted] OLD

$$I/c = \frac{72.5}{8.625/2} = 16.81 \qquad = \frac{7.23}{4.5/2} = 3.21$$

Factor of 5.24 increase (old to new)

$A = 1.1g$ | $A = 0.20g$

Factor of 5.5 increase (old to new)

$$\sigma_{\max} \propto \left(\frac{A}{I/c}\right) \Rightarrow \sigma_{\max,\text{new}} > \sigma_{\max,\text{old}}$$

Example 6.4

A small one-story reinforced concrete building is idealized for purposes of structural analysis as a massless frame supporting a total dead load of 10 kips at the beam level (Fig. E6.4a). The frame is 24 ft wide and 12 ft high. Each column and the beam has a 10-in.-square cross section. Assume that the Young's modulus of concrete is 3×10^3 ksi and the damping ratio for the building is estimated as 5%. Determine the peak response of this frame to the El Centro ground motion. In particular, determine the peak lateral deformation at the beam level and plot the diagram of bending moments at the instant of peak response.

Solution The lateral stiffness of such a frame was calculated in Chapter 1: $k = 96EI/7h^3$, where EI is the flexural rigidity of the beam and columns and h is the height of the frame. For this particular frame,

$$k = \frac{96(3 \times 10^3)(10^4/12)}{7(12 \times 12)^3} = 11.48 \text{ kips/in.}$$

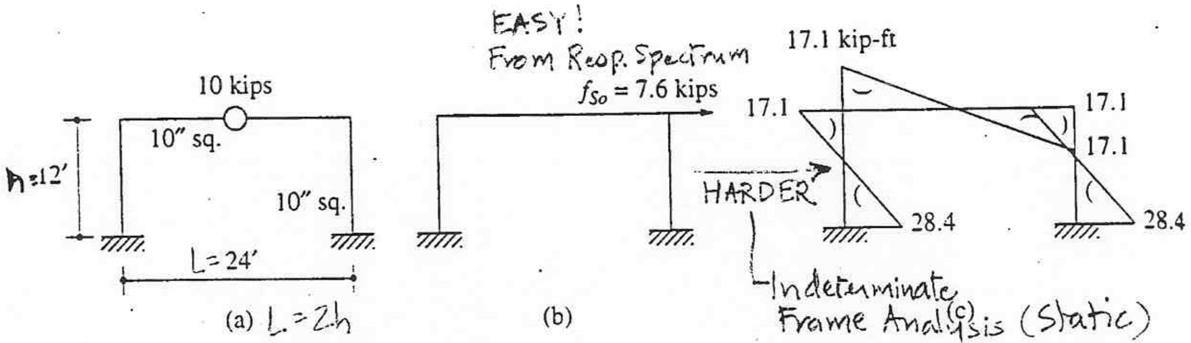


Figure E6.4 (a) Frame; (b) equivalent static force; (c) bending moment diagram.

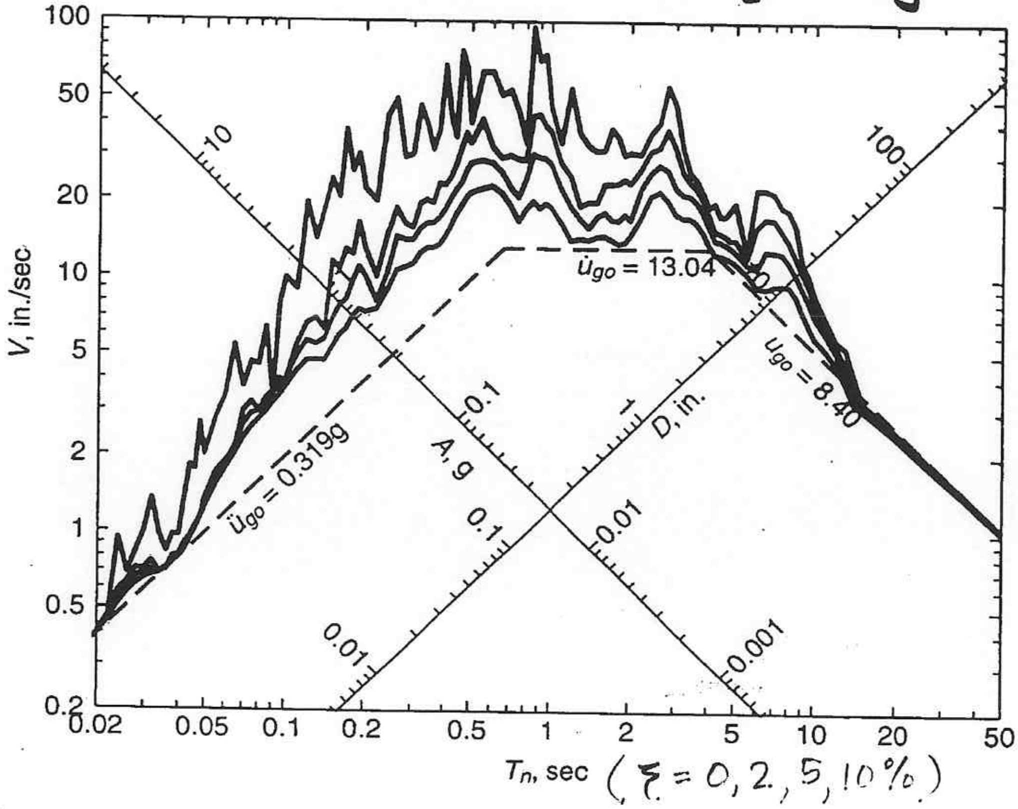
The natural vibration period is

$$T_n = \frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{\frac{10/386}{11.48}} = 0.30 \text{ sec}$$

For $\boxed{T_n = 0.3}$ and $\boxed{\zeta = 0.05}$ we read from the response spectrum of Fig. 6.6.4: $D = 0.67$ in. and $A = 0.76$g. Peak deformation: $u_o = D = 0.67$ in. Equivalent static force: $f_{So} = (A/g)w = 0.76 \times 10 = 7.6$ kips. Static analysis of the frame for this lateral force, shown in Fig. E6.4b, gives the bending moments that are plotted in Fig. E6.4c.

Response spectrum [Fig 6.6.4]

u_{go}, $\dot{u}_{go}$, and $\ddot{u}_{go}$ [see Fig 6.1.4]



(ξ = 0, 2, 5, 10%)

Figure 6.8.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Response Spectra show amplification of motion (relative to ground) as a function of period.

Response spectrum plotted on normalized scales

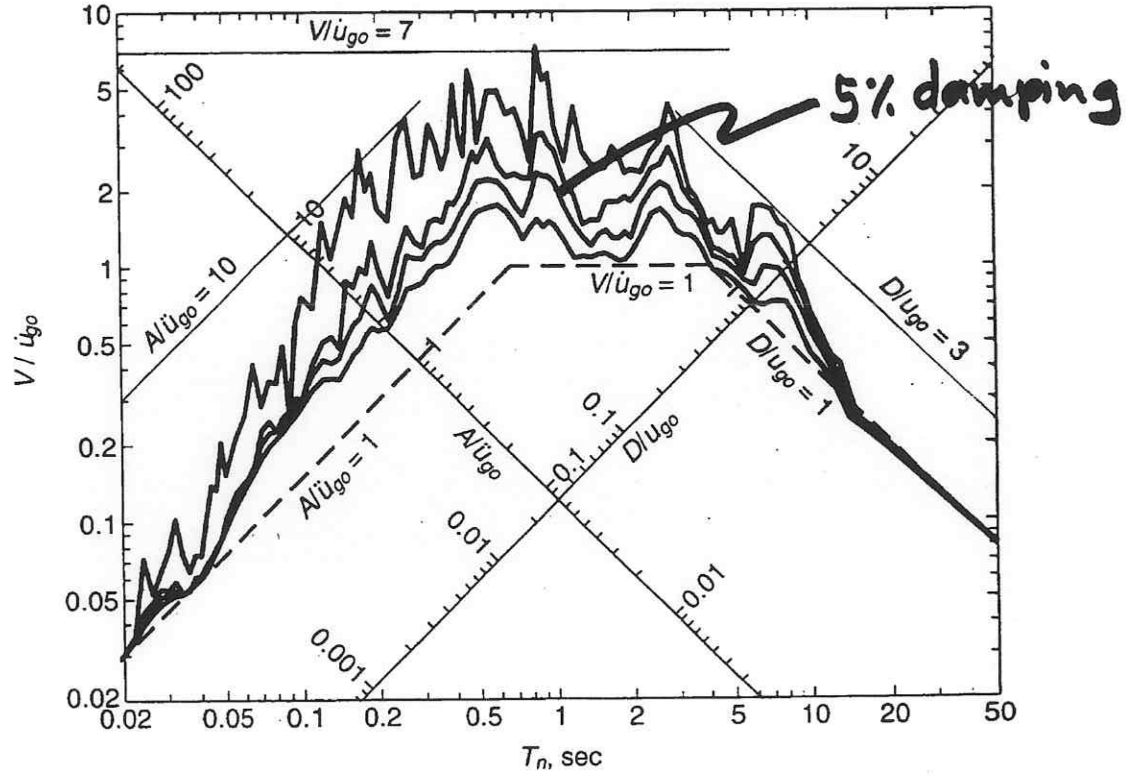


Figure 6.8.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra. Prentice-Hall. 1995.

Response spectrum and spectral regions

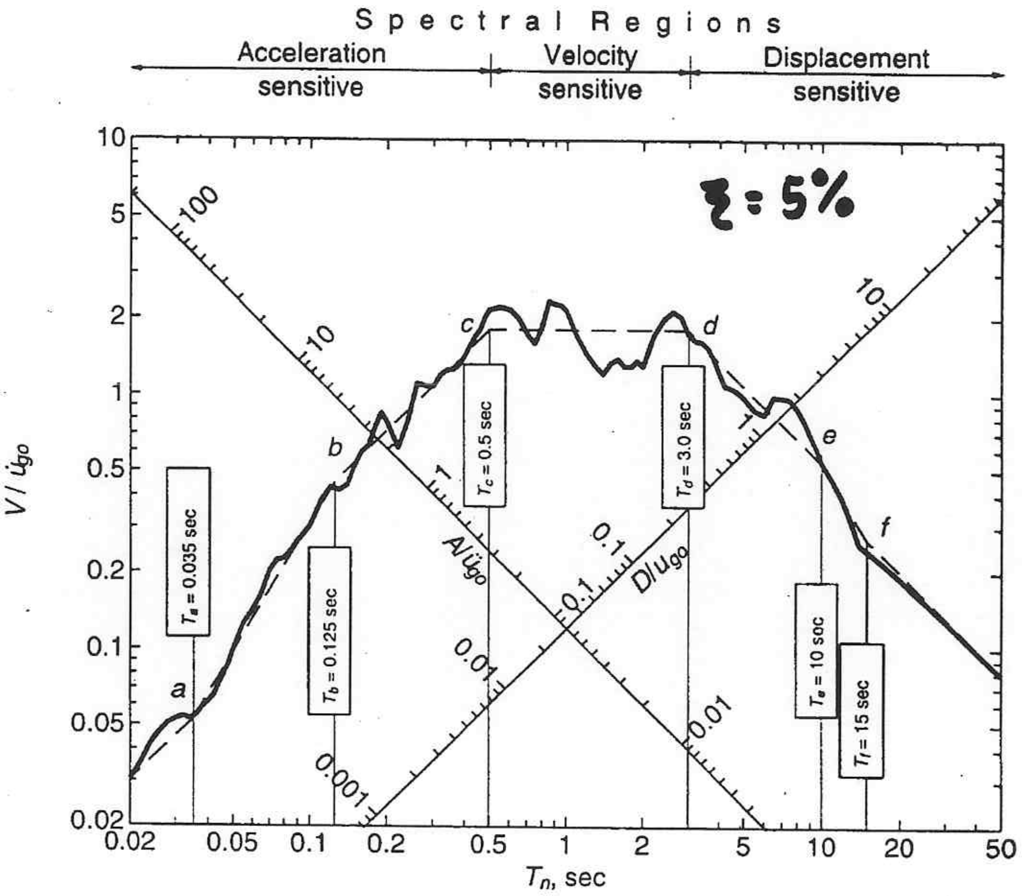


Figure 6.8.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

What insights can we gain from Studying Response Spectra?

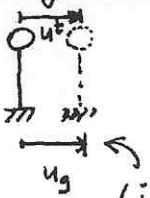
Plots of normalized spectra (D/u_{go} , $V/\dot{u}_{go}$, $A/\ddot{u}_{go}$) show that amplification (relative to ground) depends on the natural period under consideration. Also, different period ranges influence D, V, A in terms of extent of amplification.

e.g., 5%-damped "Normalized" Response Spectrum for the El Centro motion suggests different behavior in the following period ranges:

(1) $T_n < T_a = 0.035$ sec. Very short Period systems

$A \simeq \ddot{u}_{go}$; $D \simeq 0$ (Small deformations)

System is rigid (essentially) and mass moves rigidly with the ground.



For light damping, $\ddot{u}^t(t) \simeq -\overbrace{A(t)}^{\omega_n^2 u(t)}$

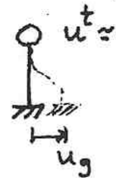
or $\ddot{u}^t_o \simeq \mathbf{A} \Rightarrow \mathbf{A} \simeq \ddot{u}_{go}$

$(\ddot{u}_g(t) \simeq \ddot{u}^t(t))$

(2) $T_n > T_f = 15$ sec. Very long Period systems

$D \simeq u_{go}$; $A \simeq 0$ (Small forces)

System is so flexible that the mass remains stationary while the ground below it moves.



$u^t \simeq 0 \rightarrow \ddot{u}^t(t) \simeq 0$ For light damping, $\ddot{u}^t(t) \simeq -A(t)$

$\ddot{u}^t(t) \simeq 0 \Rightarrow A(t) \simeq 0$ (& $A \simeq 0$)

$u^t(t) \simeq 0 \Rightarrow u(t) \simeq -u_g(t)$ OR $D \simeq u_{go}$

(3) $T_a < T_n < T_c$ ($T_a = 0.035$ sec, $T_c = 0.50$ sec) Short Period Systems

$A > \ddot{u}_{go}$ (amplification of acceleration)

For $0.035 \text{ sec} = T_a < T_n < T_b = 0.125 \text{ sec}$.

$A/\ddot{u}_{go}$ depends on T_n and ξ ; $A/\ddot{u}_{go} > 1$

For $0.125 \text{ sec} = T_b < T_n < T_c = 0.50 \text{ sec}$.

$A/\ddot{u}_{go}$ depends only on ξ ; $A/\ddot{u}_{go} > 1$
(constant w.r.t. T_n)

(4) $T_d < T_n < T_f$ ($T_d = 3$ sec, $T_f = 15$ sec) Long Period systems

$D > u_{go}$ (amplification of displacement)

For $10 \text{ sec} = T_e < T_n < T_f = 15 \text{ sec}$

D/u_{go} depends on T_n and ξ ; $D/u_{go} > 1$

For $3 \text{ sec} = T_d < T_n < T_e = 10 \text{ sec}$.

D/u_{go} depends only on ξ ; $D/u_{go} > 1$
(constant w.r.t. T_n)

(5) $T_c < T_n < T_d$ ($T_c = 0.50$ sec, $T_d = 3$ sec) Intermediate Period Systems/Structures.

$V/\dot{u}_{go} > 1$

$V/\dot{u}_{go}$ depends only on ξ
(constant w.r.t. T_n)

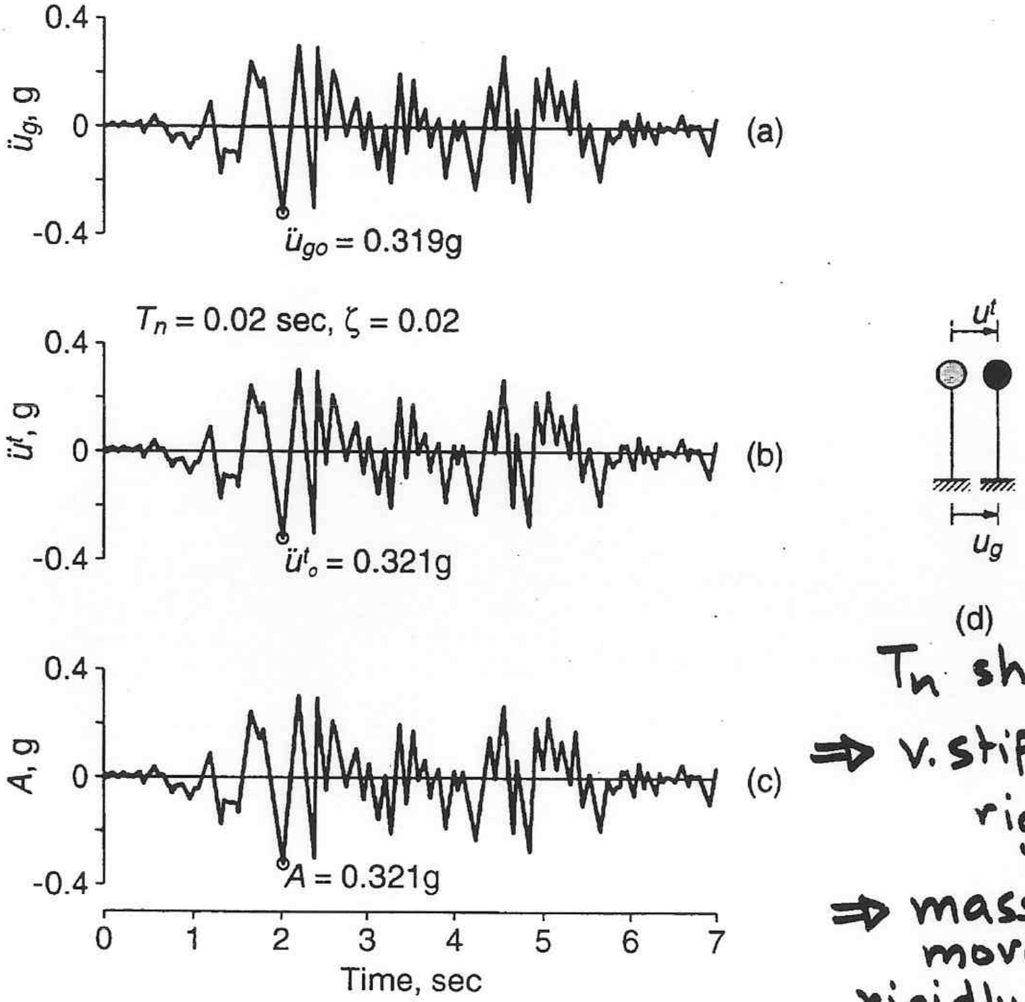
$T_n < T_c \longrightarrow$ Acceleration Sensitive Region
Structure Response most directly related to ground acceleration

$T_c < T_n < T_d \longrightarrow$ Velocity Sensitive Region
Structure Response most directly related to ground velocity

$T_n > T_d \longrightarrow$ Displacement Sensitive Region
Structure Response most directly related to ground displacement

Very Short Period System

Response $\ddot{u}^t(t)$ and $A(t)$ -- $T_n = 0.02$ sec



T_n short

⇒ v. stiff, rigid.

⇒ mass moves rigidly with ground

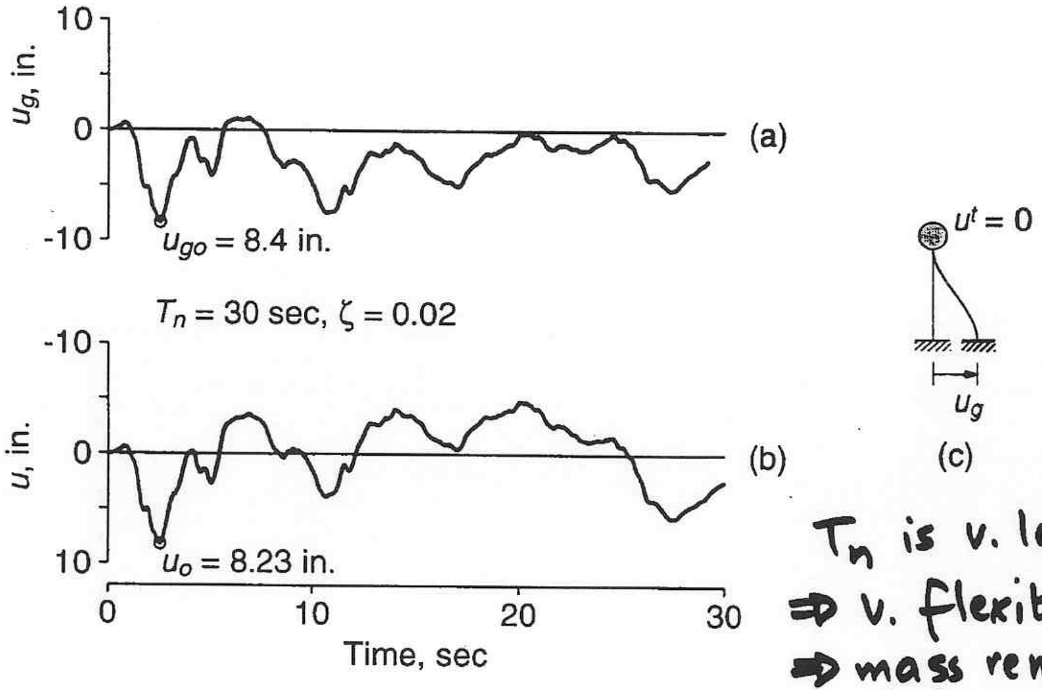
Figure 6.8.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$\ddot{u}^t_o \simeq A$ for small T_n values & light damping

$\ddot{u}^t_o \simeq \ddot{u}_{go}$ for small T_n values

Very Long Period System

Response $u(t)$ -- $T_n = 30$ sec, $\zeta = 2\%$



T_n is v. long
$\Rightarrow$ v. flexible
$\Rightarrow$ mass remains stationary while ground below moves

Figure 6.8.5 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$u_o \simeq u_{go}$ for large T_n values

$A \simeq \ddot{u}_o^t$ for light damping
$\simeq 0$ for large T_n values

Effect of Damping on Response Spectra

- Damping, in general, reduces response for a given natural period

- The extent of the change in response due to the same change in damping is different for different period structures.

- For very short period structures $T_n \to 0$
  damping does not affect response

Also, for very long period structures $T_n \to \infty$
damping does not affect response

Effect of damping greatest in the Velocity-sensitive region
Here, effect of damping depends on characteristics of the ground motion.

- Reduction in response when ξ is changed from 0 to 2% is much greater than when ξ is changed from 10 to 12%

i.e., Effect of damping is greatest for smaller damping values.

greater influence/effect of damping change

smaller effect

Variation of pseudo-acceleration with damping

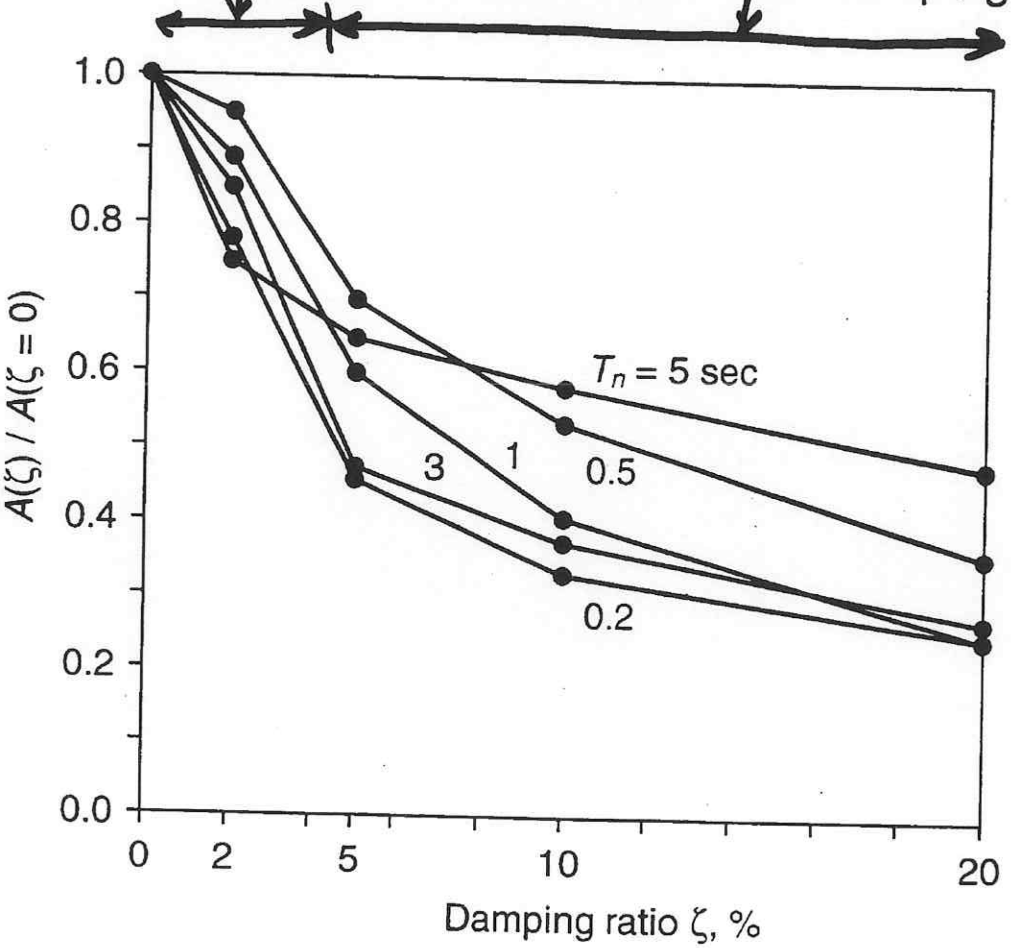
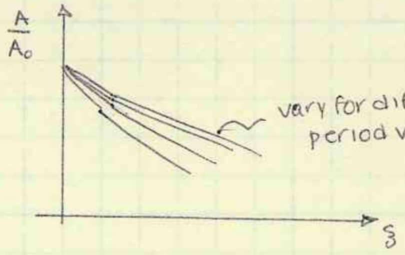


Figure 6.6.6 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

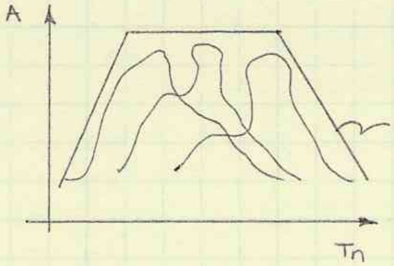
Damping reduces response

RESPONSE SPECTRA

Design spectra

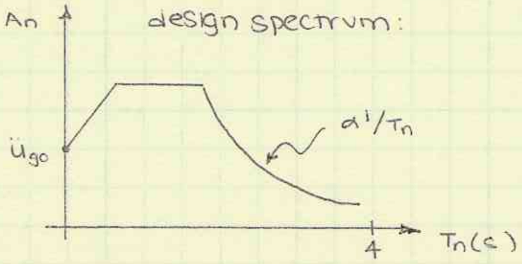


vary for different period values



find one smooth curve that encompasses individual responses

design spectrum:



USGS info $\longrightarrow$ FEMA 450 – document available
ASCE 7, IBC 2006

MCE: maximum considered earthquake

1886 – Charleston earthquake

$A = \omega_n^2 u$ → with light damping (<10%) and small periods, A approximates actual response very accurately.

Elastic Design Spectra

Why do we need them?

- Design of new structures.
- Safety evaluation of existing ones.
- A single ground motion record's spectrum is not appropriate since it displays a jaggedness that may be different from that due to other earthquakes.

What we need

- A smooth curve or series of straight lines (for a specified damping) that indicates levels of motion to be designed for as a function of natural period
- the design spectra must be representative of ground motions recorded at the site (or at similar sites) during past earthquakes.

How?

We try to match (for our site) in selecting earthquake records

- magnitude of earthquake
- distance of site from fault
- fault (rupture) mechanism
- propagation path / geology
- local soil conditions

17-2

Response spectra for three ground motions at El Centro site

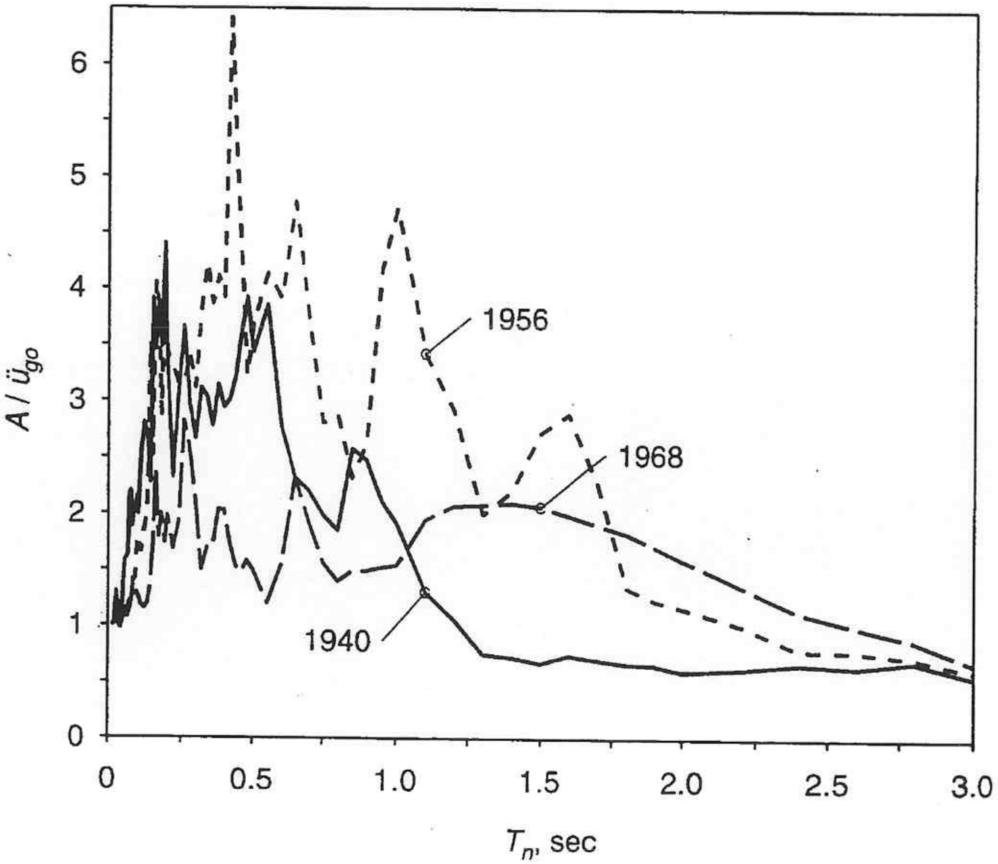


Figure 6.9.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

- Same site
- Different earthquakes

Process of development of a Design Spectrum that is representative of an ensemble of recorded motions

- Statistical analysis

e.g. Riddell & Newmark did this for 10 records in 1979.
Plots of mean / median (50th percentile) and mean $+1\sigma$ / (84th percentile) of these 10 spectra (at each period) yielded <u>smoother</u> spectra than each individual spectrum

Easy, then, to replace these smooth spectra by series of straight lines.

Result : Design Spectra

Procedure :

Take 4 recommended period values : T_a, T_b, T_e, T_f (equal to $1/33$, $1/8$, 10, and 33 sec., respectively)

Use Amplification Factors for either D, V, or A depending on period. Factors are function of ξ.

Newmark & Hall constructed design spectra (median and 84%-ile)

e.g., Median 5% damped spectra uses amplification factors $\alpha_A = 2.12$, $\alpha_V = 2.03$, $\alpha_D = 1.63$

Given $u_{go}, \dot{u}_{go}$, and $\ddot{u}_{go}$; $\alpha_A, \alpha_V, \alpha_D$; T_a, T_b, T_e, T_f

$T_b < T_n < T_c$ $\quad A = \alpha_A \ddot{u}_{go}$

$T_c < T_n < T_d$ $\quad V = \alpha_V \dot{u}_{go}$

$T_d < T_n < T_e$ $\quad D = \alpha_D u_{go}$

$T_n < T_a$ $\quad A = \ddot{u}_{go}$

$T_n > T_f$ $\quad D = u_{go}$

$T_a < T_n < T_b$ $\quad$ A increases from $\ddot{u}_{go}$ to $\alpha_A \ddot{u}_{go}$

$T_e < T_n < T_f$ $\quad$ D decreases from $\alpha_D u_{go}$ to u_{go}

α_A, α_V, α_D tabulated or available ~~as~~ functions of ξ

Use known $\ddot{u}_{go}$, $\dot{u}_{go}$, and u_{go}

~~and~~ OR $\ddot{u}_{go}$ with assumptions of how $\dot{u}_{go}$ and u_{go} relate to $\ddot{u}_{go}$

(e.g., firm ground

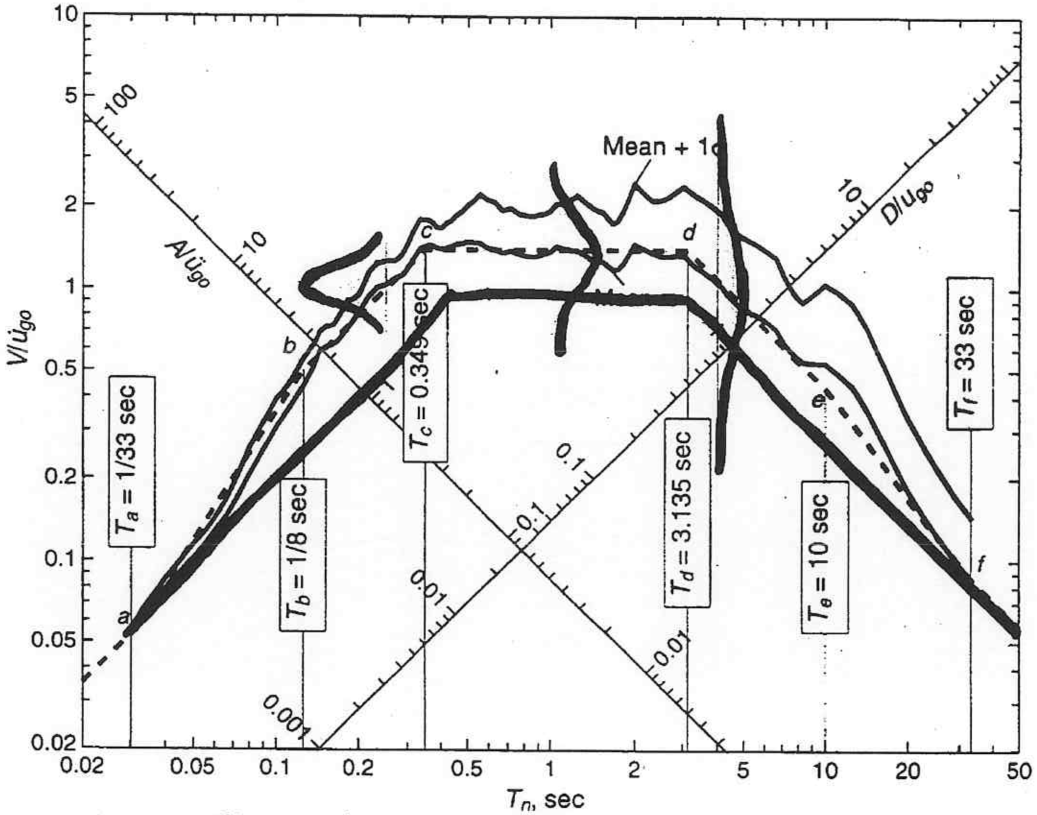
$$\frac{\dot{u}_{go}}{\ddot{u}_{go}} = \frac{48 \text{ in/sec}}{g}$$

$$\frac{\ddot{u}_{go} u_{go}}{\dot{u}_{go}^2} = 6$$

)

17-5

- Mean and mean + 1σ spectra
- Design spectrum
- Probability distributions for V



10 earthquake records
(Riddell & Newmark, 1979)

Figure 6.9.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

How to Construct a Design Spectrum

Summary. A procedure to construct a design spectrum is now summarized with reference to Fig. 6.9.3:

1. Plot the three dashed lines corresponding to the peak values of ground acceleration $\ddot{u}_{go}$, velocity $\dot{u}_{go}$, and displacement u_{go} for the design ground motion.
2. Obtain from Table 6.9.1 or 6.9.2 the values for α_A, α_V, and α_D for the ζ selected.
3. Multiply $\ddot{u}_{go}$ by the amplification factor α_A to obtain the straight line b–c representing a constant value of pseudo-acceleration A.
4. Multiply $\dot{u}_{go}$ by the amplification factor α_V to obtain the straight line c–d representing a constant value of pseudo-velocity V.
5. Multiply u_{go} by the amplification factor α_D to obtain the straight line d–e representing a constant value of deformation D.
6. Draw the line $A = \ddot{u}_{go}$ for periods shorter than T_a and the line $D = u_{go}$ for periods longer than T_f.
7. The transition lines a–b and e–f complete the spectrum.

Construction of elastic design spectrum

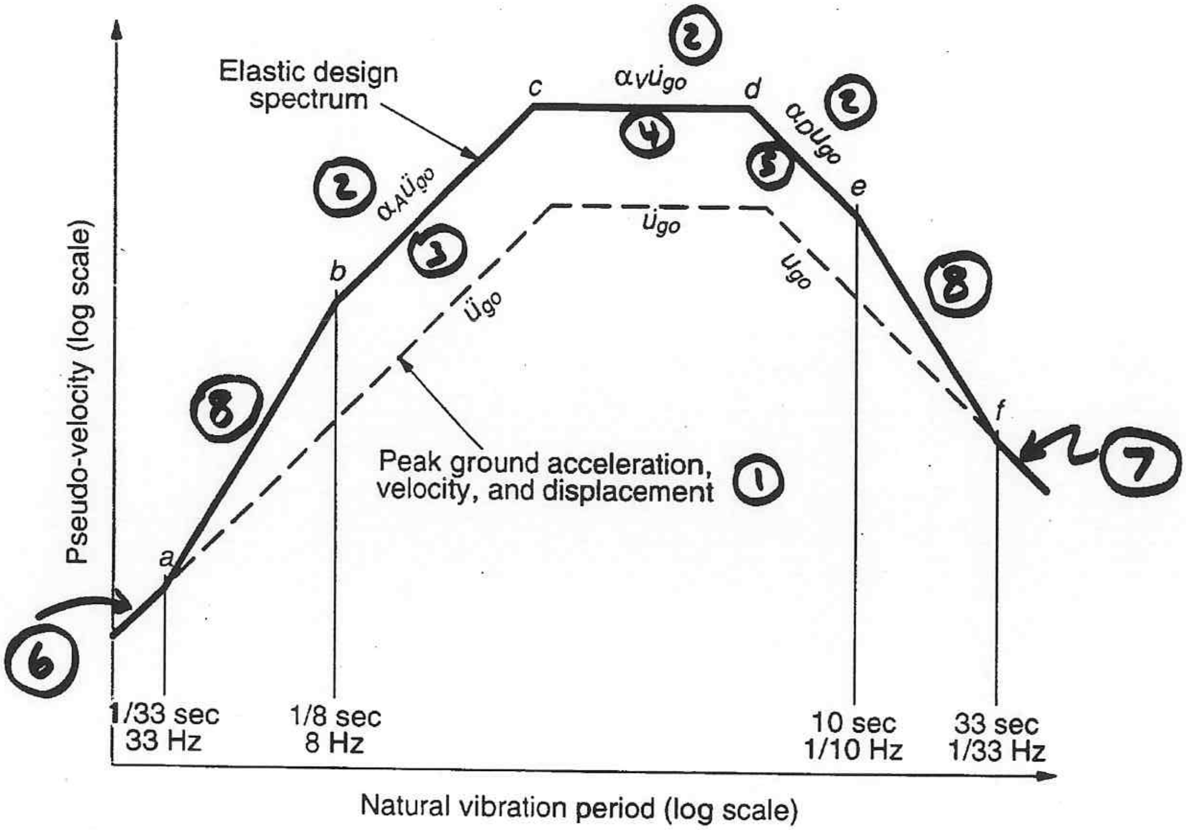


Figure 6.9.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Points c & d are at intersection of the constant - A, constant - V, and constant - D branches.

Recommended values for firm ground:

$$\frac{\dot{u}_{go}}{\ddot{u}_{go}} = \frac{48 \text{ in/sec}}{g} \quad ; \quad \frac{\ddot{u}_{go} u_{go}}{\dot{u}_{go}^2} = 6$$

Construction of elastic design spectrum (84%)

- $\ddot{u}_{go} = 1g$, $\dot{u}_{go} = 48$ in./sec, and $u_{go} = 36$ in. • $\zeta = 5\%$

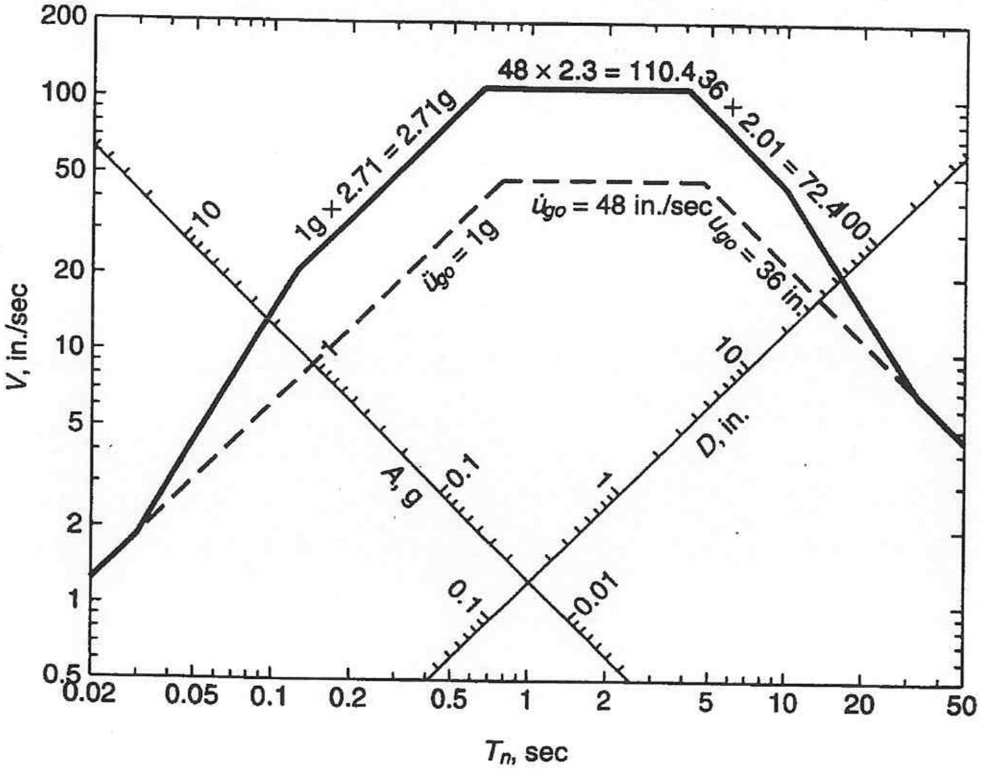


Figure 6.9.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Use Table 6.9.1 for $\alpha_A, \alpha_V, \alpha_D$ for $\xi = 5\%$

$\dot{u}_{go} = 48$ in/sec

$$u_{go} = \frac{6 \times 48^2}{1 \times 32 \times 12} = 36 \text{ in.}$$

Elastic pseudo-acceleration design spectrum

- $\ddot{u}_{go} = 1g$, $\dot{u}_{go} = 48$ in./sec, and $u_{go} = 36$ in. • $\zeta = 5\%$

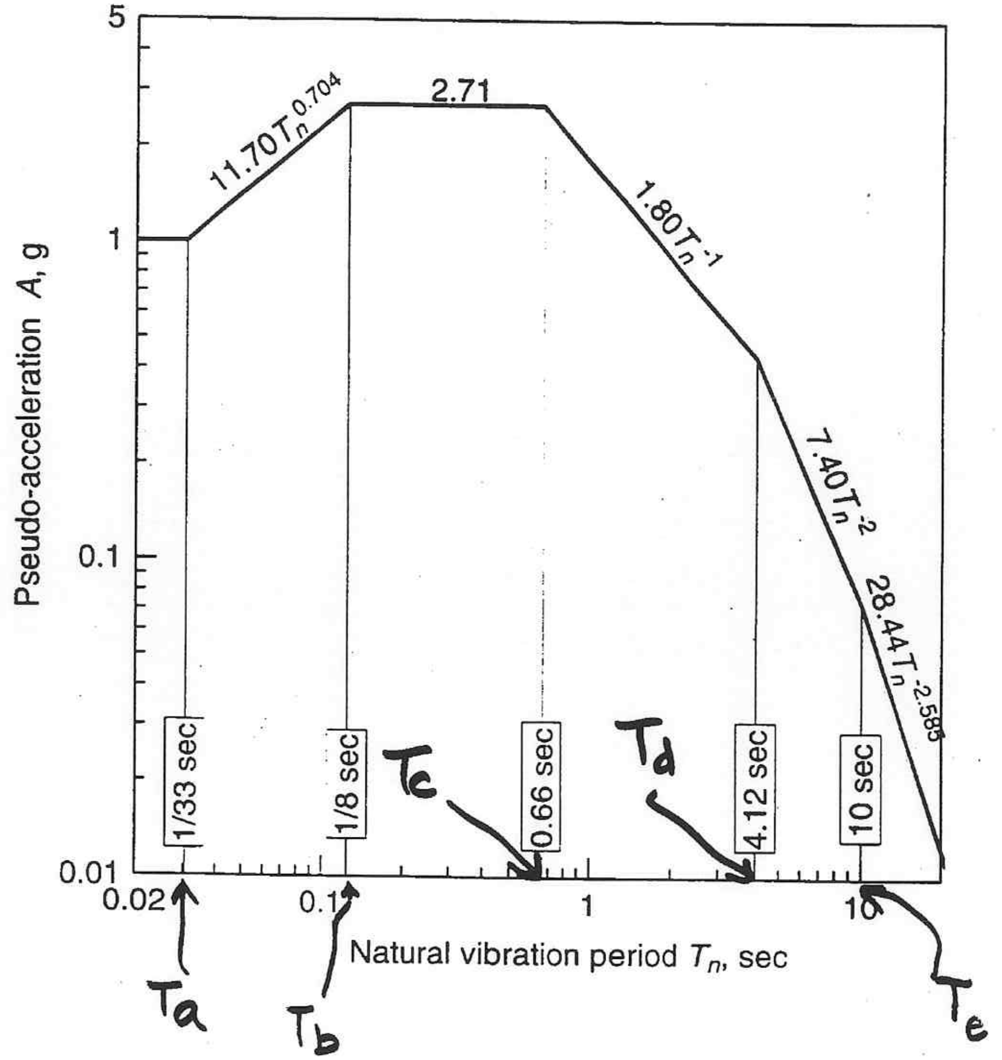
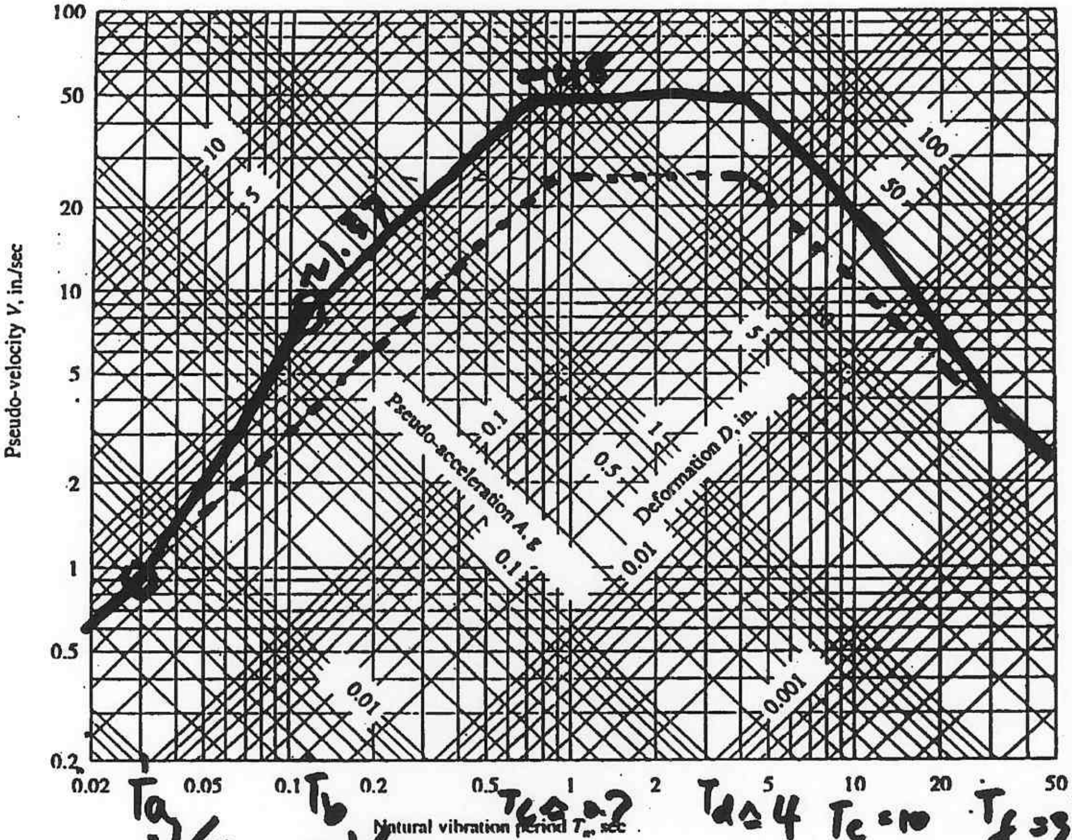


Figure 6.9.5 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

For $T_a < T_n < T_b$ $$\frac{\log 2.71 - \log 1}{\log A - \log 1} = \frac{\log(1/8) - \log(\frac{1}{33})}{\log T_n - \log(\frac{1}{33})}$$

Simplify: $A = 11.70T_n^{0.704}$

$\alpha_A = 2.74$
$\alpha_V = 2.03$
$\alpha_D = 1.63$



$T_a = 1/33$ $T_b = 1/8$ $T_c \triangleq 0.?$ $T_d \triangleq 4$ $T_e = 10$ $T_f = 33$

Figure A6.1 Graph paper with four-way logarithmic scales.

Exercise: Construct a <u>2%</u>-damped <u>median</u> design spectrum for a <u>rock</u> site where $\ddot{u}_{go} = 0.5g$

How is a Design Spectrum different from a Response Spectrum?

Response Spectrum ≡ Plot of the peak response of all possible SDOF systems to **one** specific ground motion.

- Usually a jagged (non-smooth) plot

Design Spectrum ≡ Plot of the level of seismic design force / deformation / acceleration as a function of natural period and damping ratio

- Usually a smooth plot
- Sometimes it may have the general appearance of a response spectrum but it may also be more like an envelope of two very dissimilar response spectra.

For design spectrum

$\ddot{u}_{go} = 0.319g$; $\dot{u}_{go} = 15.3$ in/s ; $u_{go} = 11.5$ in. $\left[\begin{matrix}\text{Firm}\\ \text{Soil}\end{matrix}\right]$

For response spectrum

$\ddot{u}_{go} = 0.319g$; $\dot{u}_{go} = 13.0$ in/s ; $u_{go} = 8.4$ in.

Standard design spectrum and El Centro response spectrum

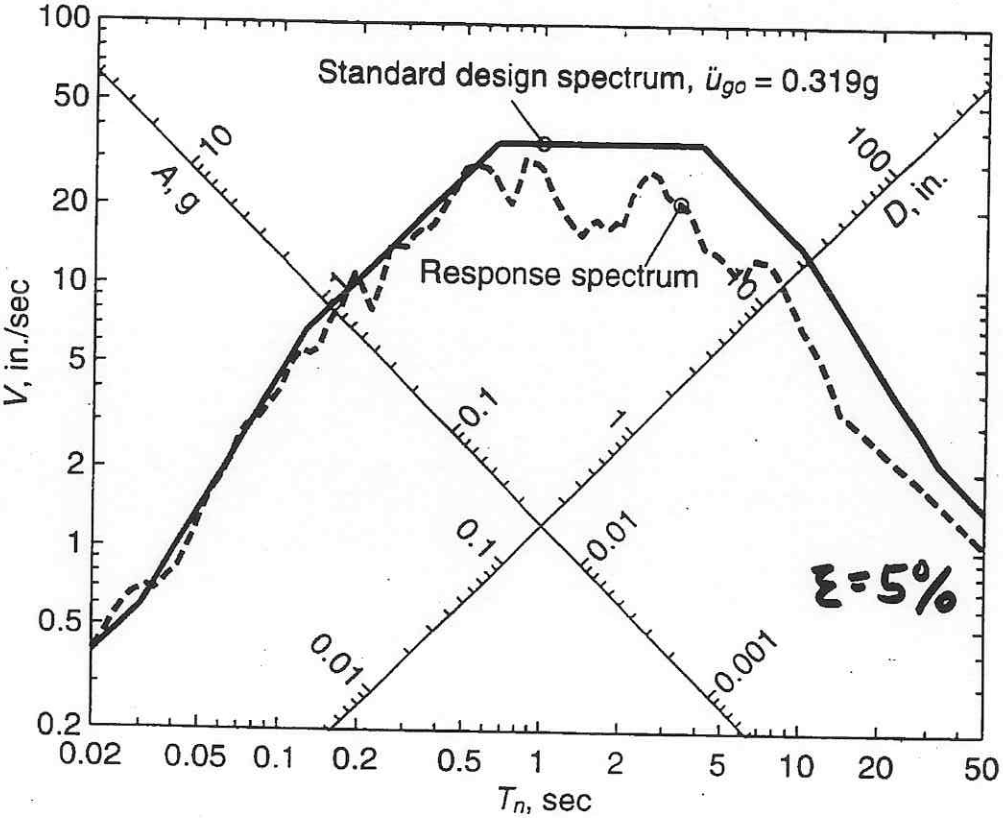


Figure 6.10.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Standard Design Spectrum above is 84%-ile spectrum (for ξ = 5%) with $\ddot{u}_{go} = 0.319g$ & FIRM SOIL assumptions

$\Rightarrow \dot{u}_{go}/\ddot{u}_{go} = 48 \frac{\text{in/sec}}{g}$; $\frac{\ddot{u}_{go} \cdot u_{go}}{\dot{u}_{go}^2} = 6$

and $\alpha_A = 2.71$, $\alpha_V = 2.30$, $\alpha_D = 2.01$

Design spectra and El Centro response spectrum
$\ddot{u}_{go} = 0.319g$, $\dot{u}_{go} = 13.04$ in./sec, $u_{go} = 8.40$ in.

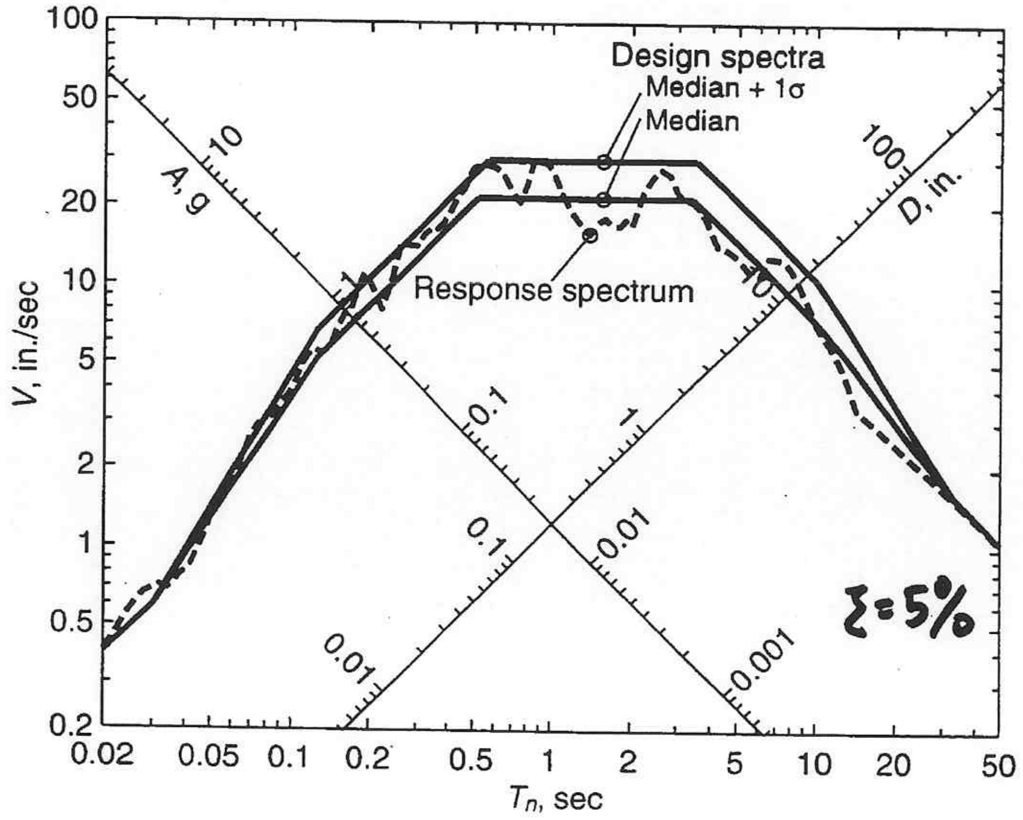


Figure 6.10.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Site design spectrum: Envelope of two design spectra

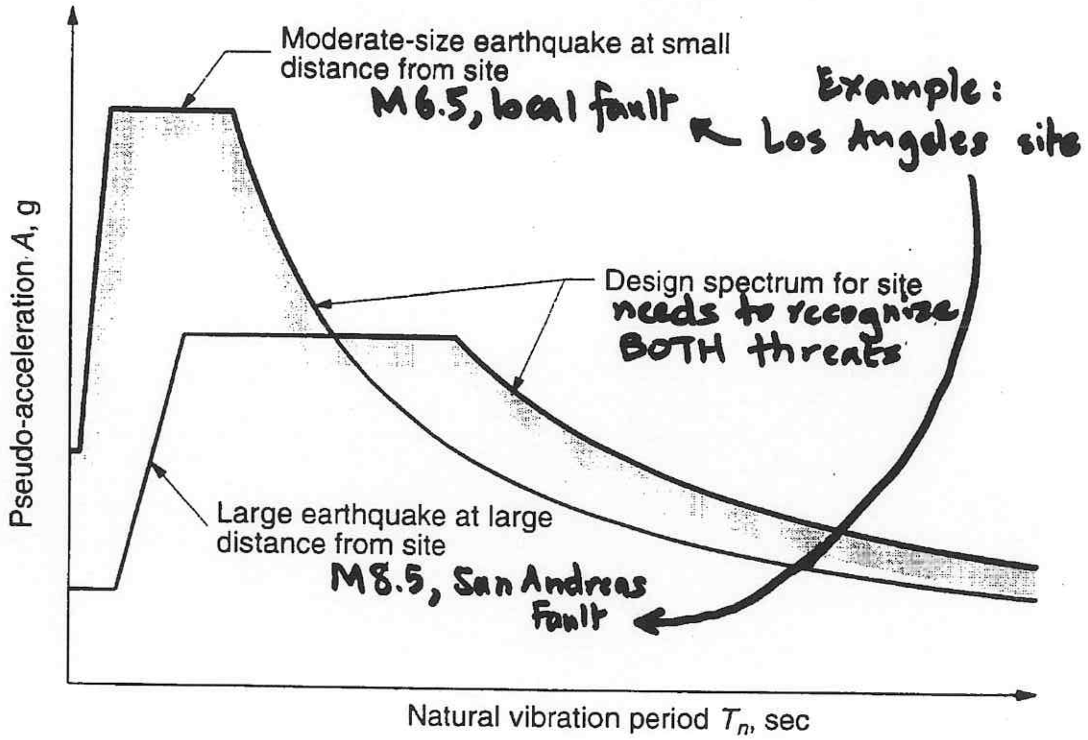


Figure 6.11.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Peak Relative Deformation = $u_0 \equiv D$

Peak Relative Velocity = $\dot{u}_0$

Peak Pseudo-Velocity = $\omega_n u_0 = \left(\frac{2\pi}{T_n}\right) \cdot u_0 \equiv \boxed{V}$

Relative-velocity and pseudo-velocity response spectra

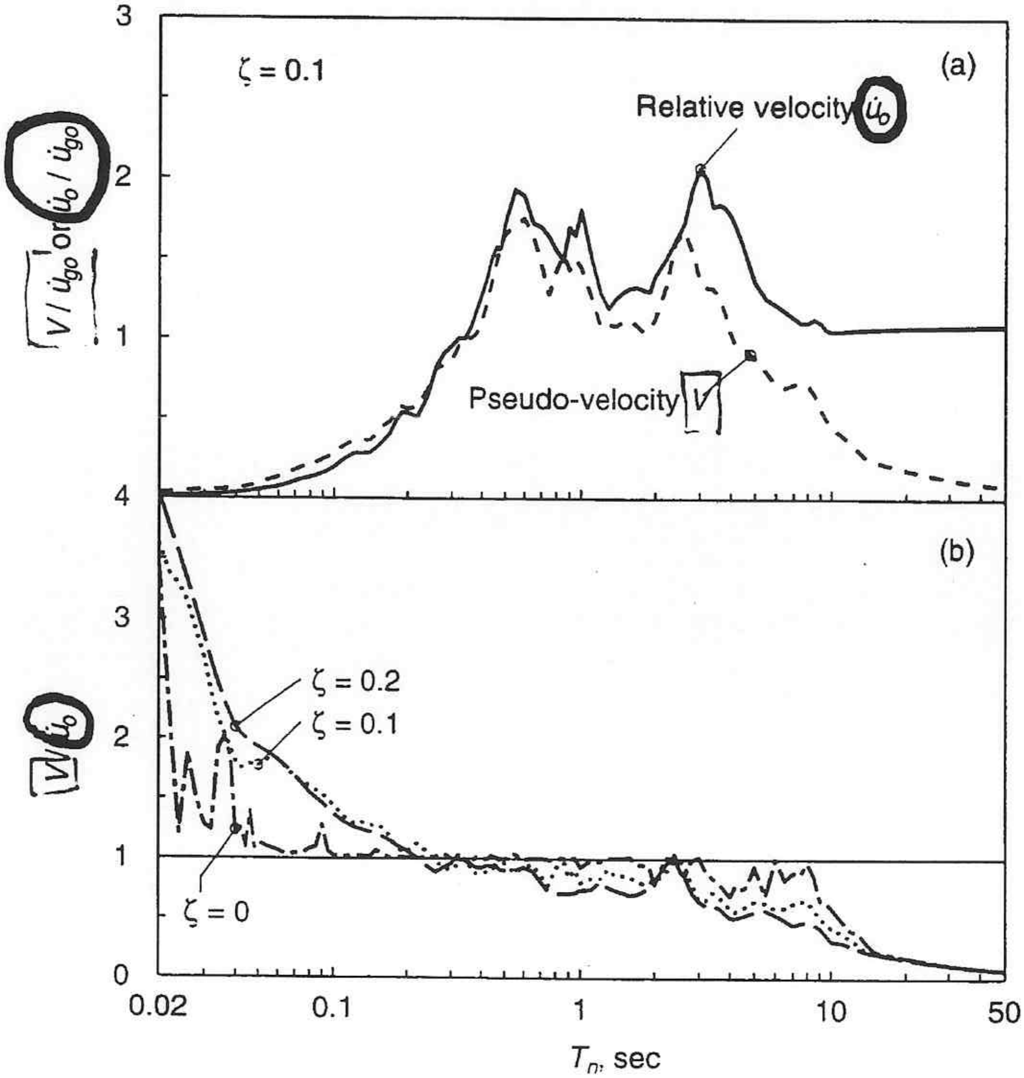


Figure 6.12.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall. 1995.

Conclusions:

$V \sim \dot{u}_0$ for intermediate periods ($\sim$0.2 to 3 sec.)

$V > \dot{u}_0$ for very short periods

$V < \dot{u}_0$ for very long periods

Differences are smaller for lighter damping

Recall Equation of Motion: $m\ddot{u}^t + c\dot{u} + ku = 0$

Peak Relative Deformation = $u_0 \equiv D$

Peak (Total) Acceleration = $\boxed{\ddot{u}_0^t}$

Peak Pseudo-Acceleration = $\omega_n^2 u_0 = \left(\frac{2\pi}{T_n}\right)^2 \cdot u_0 \equiv \boxed{A}$

Acceleration and pseudo-acceleration response spectra

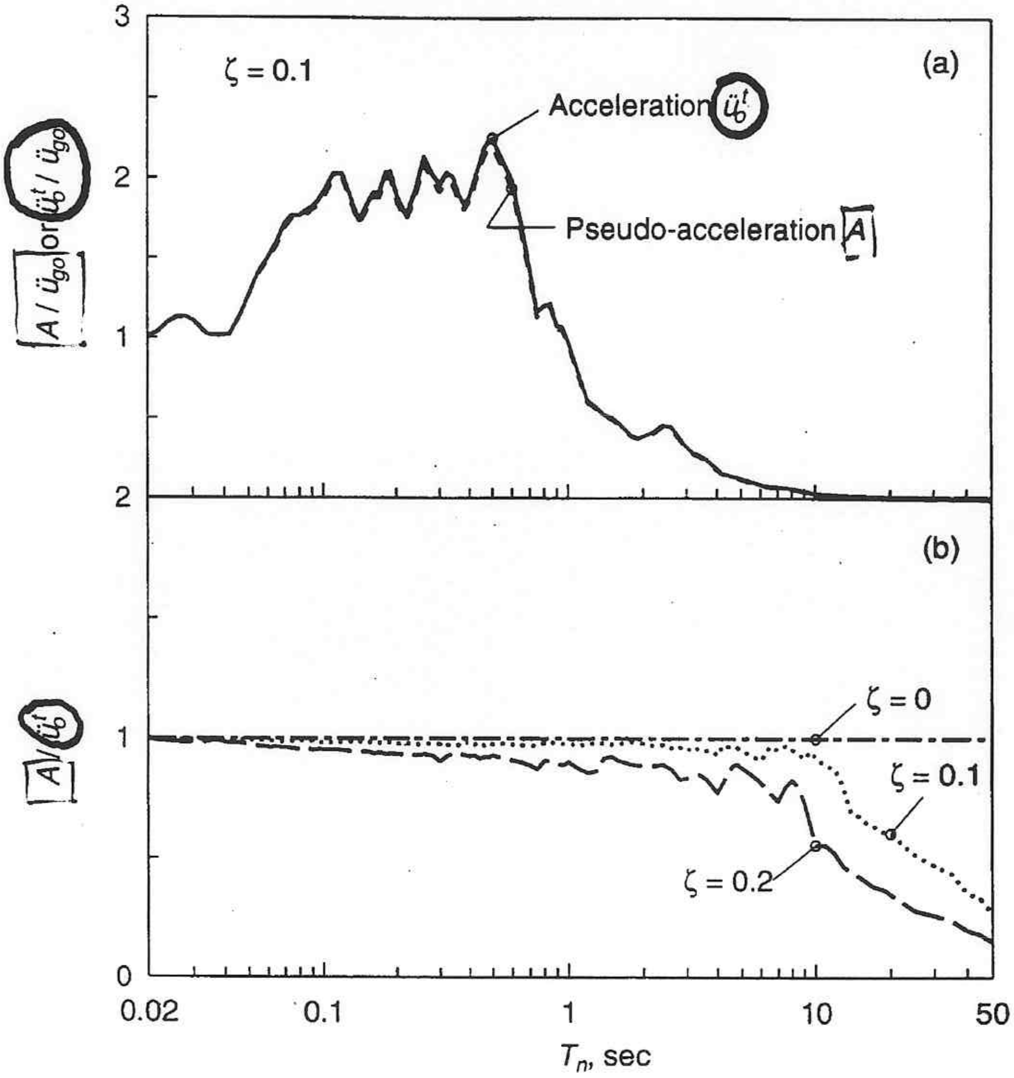


Figure 6.12.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Conclusions: $A \sim \ddot{u}_0^t$ for most cases
except very long periods and
heavy damping (>10%)

$A = \ddot{u}_0^t$ if $\zeta = 0$

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0.15g and where the mapped spectral response acceleration at 1 second period, S_1, determined in accordance with Section 1615.1, is less than or equal to 0.04g shall only be required to comply with Section 1616.4.

5. Structures located where the short period design spectral response acceleration, S_{DS}, determined in accordance with Section 1615.1 is less than or equal to 0.167g and the design spectral response acceleration at 1 second period, S_{D1}, determined in accordance with Section 1615.1 is less than or equal to 0.067g, shall only be required to comply with Section 1616.4.

1614.1.1 Additions to existing buildings. An addition that is structurally independent from an existing structure shall be designed and constructed as required for a new structure in accordance with the seismic requirements for new structures. An addition that is not structurally independent from an existing structure shall be designed and constructed such that the entire structure conforms to the seismic force resistance requirements for new structures unless the following conditions are satisfied:

1. The addition conforms with the requirements for new structures, and
2. The addition does not increase the seismic forces in any structural element of the existing structure by more than 5 percent, unless the element has the capacity to resist the increased forces determined in accordance with Sections 1613 through 1622.

1614.2 Change of occupancy. When a change of occupancy results in a structure being reclassified to a higher Seismic Use Group, the structure shall conform to the seismic requirements for a new structure.

Exception: Specific detailing provisions required for a new structure are not required to be met where it can be shown an equivalent level of performance and seismic safety contemplated for a new structure is obtained. Such analysis shall consider the regularity, overstrength, redundancy and ductility of the structure within the context of the specific detailing provided.

1614.3 Alterations. Existing structures being altered need not comply with Sections 1613 through 1622 provided that the following conditions are met:

1. The alterations do not create a structural irregularity as defined in Section 1616.5 or make an existing structural irregularity more severe.
2. The alteration does not increase the seismic forces in any structural element of the existing structure by more than 5 percent, unless the capacity of the element subject to the increased forces is still in compliance with Sections 1613 through 1622.
3. The alteration does not decrease the seismic resistance of any structural element of the existing structure to less than that required for a new structure.
4. The alterations do not result in the creation of an unsafe condition.

1614.4 Quality assurance. A Quality Assurance Plan shall be provided where required by Chapter 17.

1614.5 Seismic and wind. When the code-prescribed wind design produces greater effects, the wind design shall govern, but detailing requirements and limitations prescribed in this and referenced sections shall be followed.

SECTION 1615
EARTHQUAKE LOADS—SITE GROUND MOTION

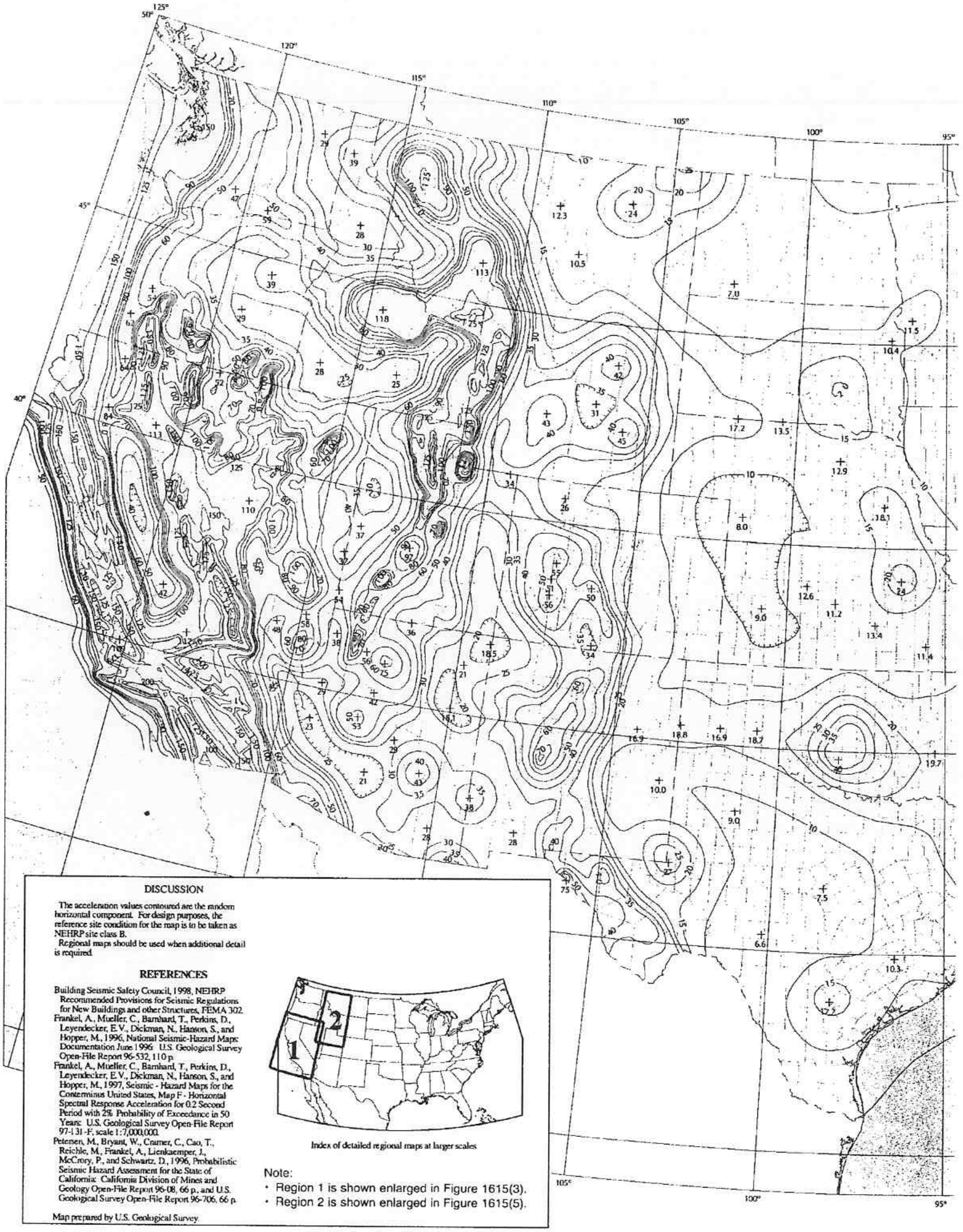
1615.1 General procedure for determining maximum considered earthquake and design spectral response accelerations. Ground motion accelerations, represented by response spectra and coefficients derived from these spectra, shall be determined in accordance with the general procedure of Section 1615.1 or the site-specific procedure of Section 1615.2. The site-specific procedure of Section 1615.2 shall be used for structures on sites classified as Site Class F, in accordance with Section 1615.1.1.

The mapped maximum considered earthquake spectral response acceleration at short periods, S_S, and at 1-second period, S_1, shall be determined from Figures 1615(1) through (10). Where a site is between contours, straight line interpolation or the value of the higher contour shall be used.

The Site Class shall be determined in accordance with Section 1615.1.1. The maximum considered earthquake spectral response accelerations at short period and 1-second period adjusted for site class effects, S_{MS} and S_{M1}, shall be determined in accordance with Section 1615.1.2. The design spectral response accelerations at short period, S_{DS}, and at 1-second period, S_{D1}, shall be determined in accordance with Section 1615.1.3. The general response spectrum shall be determined in accordance with Section 1615.1.4.

Exception: For structures located on sites with mapped spectral response acceleration at short period, S_S, less than or equal to 0.15g and mapped spectral response acceleration at 1-second period, S_1, less than or equal to 0.04g, the Site Class, maximum considered earthquake spectral response accelerations at short period and at 1-second period adjusted for site class effects (S_{MS} and S_{M1}), and the design spectral response accelerations at short period and at 1-second period (S_{DS} and S_{D1}) need not be determined. Such structures shall be categorized as Seismic Design Category A and need only comply with the requirements of Section 1616.4.

18-2



DISCUSSION

The acceleration values contoured are the random horizontal component. For design purposes, the reference site condition for the map is to be taken as NEHRP site class B.

Regional maps should be used when additional detail is required.

REFERENCES

Building Seismic Safety Council, 1998, NEHRP Recommended Provisions for Seismic Regulations for New Buildings and other Structures, FEMA 302.

Frankel, A., Mueller, C., Barnhard, T., Perkins, D., Leyendecker, E.V., Dickman, N., Hanson, S., and Hopper, M., 1996, National Seismic-Hazard Maps: Documentation June 1996: U.S. Geological Survey Open-File Report 96-532, 110 p.

Frankel, A., Mueller, C., Barnhard, T., Perkins, D., Leyendecker, E.V., Dickman, N., Hanson, S., and Hopper, M., 1997, Seismic - Hazard Maps for the Conterminus United States, Map F - Horizontal Spectral Response Acceleration for 0.2 Second Period with 2% Probability of Exceedance in 50 Years: U.S. Geological Survey Open-File Report 97-131-F, scale 1:7,000,000.

Petersen, M., Bryant, W., Cramer, C., Cao, T., Reichle, M., Frankel, A., Lienkaemper, J., McCrory, P., and Schwartz, D., 1996, Probabilistic Seismic Hazard Assessment for the State of California: California Division of Mines and Geology Open-File Report 96-08, 66 p., and U.S. Geological Survey Open-File Report 96-706, 66 p.

Map prepared by U.S. Geological Survey.

Index of detailed regional maps at larger scales

Note:
- Region 1 is shown enlarged in Figure 1615(3).
- Region 2 is shown enlarged in Figure 1615(5).

FIGURE 1615(1)
MAXIMUM CONSIDERED EARTHQUAKE GROUND MOTION FOR THE CONTERMINOUS UNITED STATES OF 0.2 SEC SPECTRAL RESPONSE ACCELERATION (5 PERCENT OF CRITICAL DAMPING), SITE CLASS B

18-3

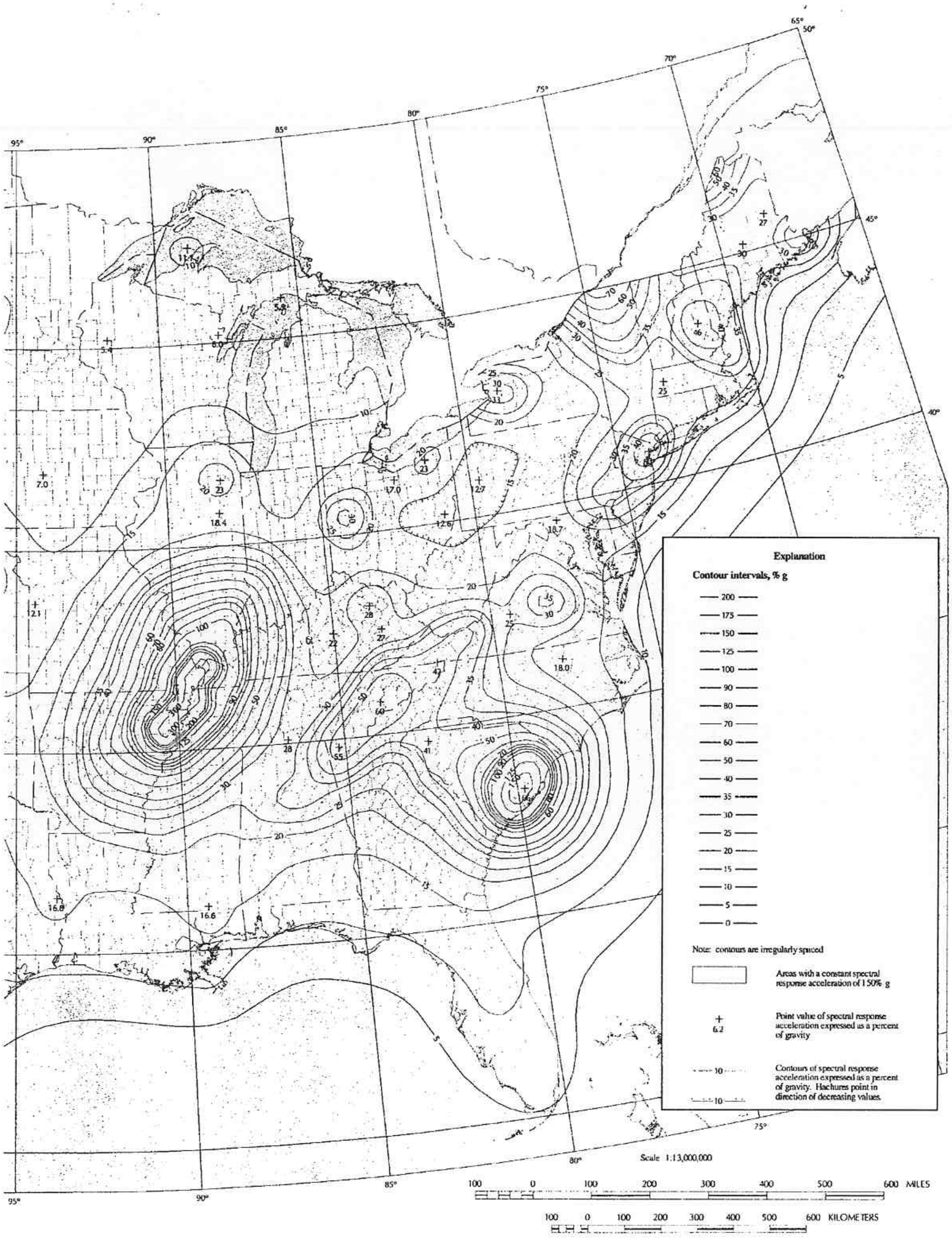
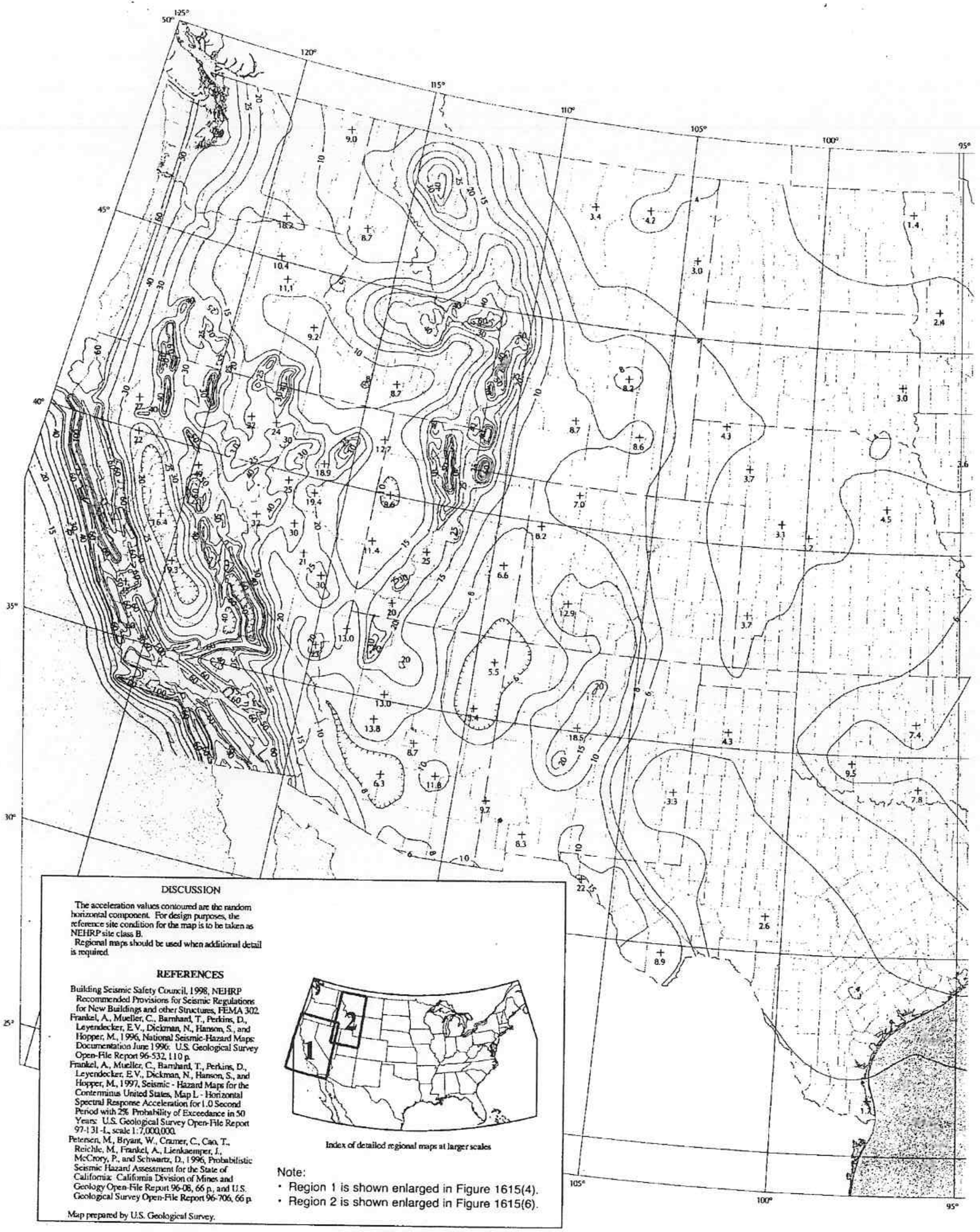


FIGURE 1615(1)—continued
MAXIMUM CONSIDERED EARTHQUAKE GROUND MOTION FOR THE CONTERMINOUS UNITED STATES OF 0.2 SEC SPECTRAL RESPONSE ACCELERATION (5 PERCENT OF CRITICAL DAMPING), SITE CLASS B

18-4



DISCUSSION

The acceleration values contoured are the random horizontal component. For design purposes, the reference site condition for the map is to be taken as NEHRP site class B.

Regional maps should be used when additional detail is required.

REFERENCES

Building Seismic Safety Council, 1998, NEHRP Recommended Provisions for Seismic Regulations for New Buildings and other Structures, FEMA 302.

Frankel, A., Mueller, C., Barnhard, T., Perkins, D., Leyendecker, E.V., Dickman, N., Hanson, S., and Hopper, M., 1996, National Seismic-Hazard Maps: Documentation June 1996: U.S. Geological Survey Open-File Report 96-532, 110 p.

Frankel, A., Mueller, C., Barnhard, T., Perkins, D., Leyendecker, E.V., Dickman, N., Hanson, S., and Hopper, M., 1997, Seismic - Hazard Maps for the Conterminus United States, Map L - Horizontal Spectral Response Acceleration for 1.0 Second Period with 2% Probability of Exceedance in 50 Years: U.S. Geological Survey Open-File Report 97-131-L, scale 1:7,000,000.

Petersen, M., Bryant, W., Cramer, C., Cao, T., Reichle, M., Frankel, A., Lienkaemper, J., McCrory, P., and Schwartz, D., 1996, Probabilistic Seismic Hazard Assessment for the State of California: California Division of Mines and Geology Open-File Report 96-08, 66 p., and U.S. Geological Survey Open-File Report 96-706, 66 p.

Map prepared by U.S. Geological Survey.

Index of detailed regional maps at larger scales

Note:
- Region 1 is shown enlarged in Figure 1615(4).
- Region 2 is shown enlarged in Figure 1615(6).

FIGURE 1615(2)
MAXIMUM CONSIDERED EARTHQUAKE GROUND MOTION FOR THE CONTERMINOUS UNITED STATES OF 1.0 SEC SPECTRAL RESPONSE ACCELERATION (5 PERCENT OF CRITICAL DAMPING), SITE CLASS B

18-5

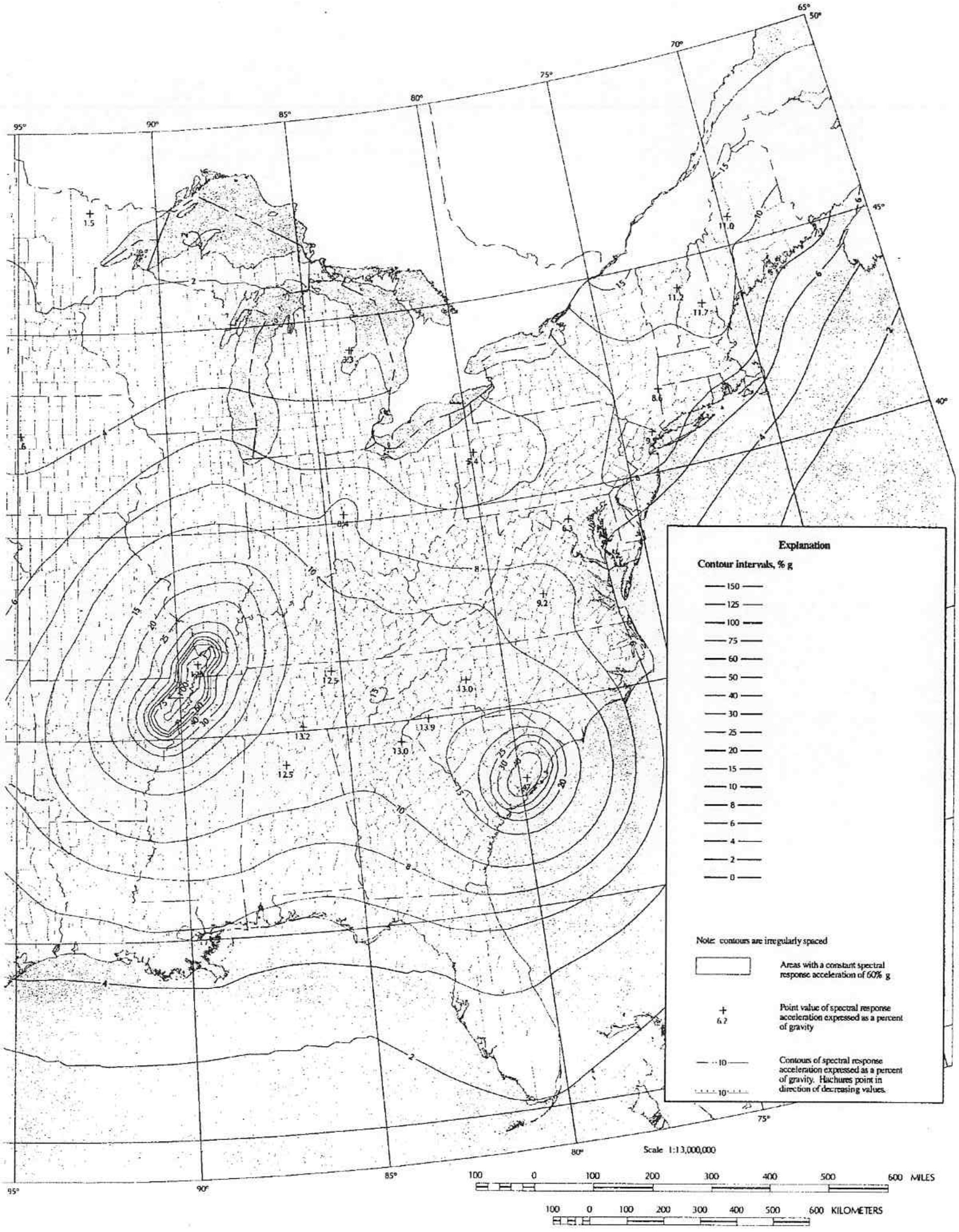


FIGURE 1615(2)—continued
MAXIMUM CONSIDERED EARTHQUAKE GROUND MOTION FOR THE CONTERMINOUS UNITED STATES OF 1.0 SEC SPECTRAL RESPONSE ACCELERATION (5 PERCENT OF CRITICAL DAMPING), SITE CLASS B

1615.1.1 Site class definitions. The site shall be classified as one of the site classes defined in Table 1615.1.1. Where the soil shear wave velocity, $\bar{v}_s$, is not known, site class shall be determined, as permitted in Table 1615.1.1, from standard penetration resistance, $\bar{N}$, or from soil undrained shear strength, $\bar{s}_u$, calculated per Section 1615.1.5. Where-site specific data are not available to a depth of 100 feet (30 480 mm), appropriate soil properties are permitted to be estimated by the registered design professional preparing the soils report based on known geologic conditions.

When the soil properties are not known in sufficient detail to determine the site class, Site Class D shall be used unless the building official determines that Site Class E or F soil is likely to be present at the site.

1615.1.2 Site coefficients and adjusted maximum considered earthquake spectral response acceleration parameters. The maximum considered earthquake spectral response acceleration for short periods, S_{MS}, and at 1-second period, S_{M1}, adjusted for site class effects, shall be determined by Equations 16-16 and 16-17, respectively:

$$S_{MS} = F_a S_s \qquad \textbf{(Equation 16-16)}$$

$$S_{M1} = F_v S_1 \qquad \textbf{(Equation 16-17)}$$

where:

F_a = Site coefficient defined in Table 1615.1.2(1).

F_v = Site coefficient defined in Table 1615.1.2(2).

S_S = The mapped spectral accelerations for short periods as determined in Section 1615.1.

S_1 = The mapped spectral accelerations for a 1-second period as determined in Section 1615.1.

1615.1.3 Design spectral response acceleration parameters. Five-percent damped design spectral response acceleration at short periods, S_{DS}, and at 1 second period, S_{D1}, shall be determined from Equations 16-18 and 16-19, respectively:

$$S_{DS} = \frac{2}{3} S_{MS} \qquad \textbf{(Equation 16-18)}$$

$$S_{D1} = \frac{2}{3} S_{M1} \qquad \textbf{(Equation 16-19)}$$

where:

S_{MS} = The maximum considered earthquake spectral response accelerations for short period as determined in Section 1615.1.2.

S_{M1} = The maximum considered earthquake spectral response accelerations for 1 second period as determined in Section 1615.1.2.

TABLE 1615.1.1
SITE CLASS DEFINITIONS

SITE CLASS	SOIL PROFILE NAME	AVERAGE PROPERTIES IN TOP 100 feet, AS PER SECTION 1615.1.5		
		Soil shear wave velocity, $\bar{v}_s$, (ft/s)	Standard penetration resistance, $\bar{N}$	Soil undrained shear strength, $\bar{s}_u$, (psf)
A	Hard rock	$\bar{v}_s > 5{,}000$	Not applicable	Not applicable
B	Rock	$2{,}500 < \bar{v}_s \le 5{,}000$	Not applicable	Not applicable
C	Very dense soil and soft rock	$1{,}200 < \bar{v}_s \le 2{,}500$	$\bar{N} > 50$	$\bar{s}_u \ge 2{,}000$
D	Stiff soil profile	$600 \le \bar{v}_s \le 1{,}200$	$15 \le \bar{N} \le 50$	$1{,}000 \le \bar{s}_u \le 2{,}000$
E	Soft soil profile	$\bar{v}_s < 600$	$\bar{N} < 15$	$\bar{s}_u < 1{,}000$
E	—	Any profile with more than 10 feet of soil having the following characteristics: 1. Plasticity index $PI > 20$; 2. Moisture content $w \ge 40\%$, and 3. Undrained shear strength $\bar{s}_u < 500$ psf		
F	—	Any profile containing soils having one or more of the following characteristics: 1. Soils vulnerable to potential failure or collapse under seismic loading such as liquefiable soils, quick and highly sensitive clays, collapsible weakly cemented soils. 2. Peats and/or highly organic clays ($H > 10$ feet of peat and/or highly organic clay where H = thickness of soil) 3. Very high plasticity clays ($H > 25$ feet with plasticity index $PI > 75$) 4. Very thick soft/medium stiff clays ($H > 120$ ft)		

For SI: 1 foot = 304.8 mm, 1 square foot = 0.0929 m², 1 pound per square foot = 0.0479 kPa.

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TABLE 1615.1.2(1)
VALUES OF SITE COEFFICIENT F_a AS A FUNCTION OF SITE CLASS AND MAPPED SPECTRAL RESPONSE ACCELERATION AT SHORT PERIODS (S_S)[a]

SITE CLASS	MAPPED SPECTRAL RESPONSE ACCELERATION AT SHORT PERIODS				
	$S_S \le 0.25$	$S_S = 0.50$	$S_S = 0.75$	$S_S = 1.00$	$S_S \ge 1.25$
A	0.8	0.8	0.8	0.8	0.8
B	1.0	1.0	1.0	1.0	1.0
C	1.2	1.2	1.1	1.0	1.0
D	1.6	1.4	1.2	1.1	1.0
E	2.5	1.7	1.2	0.9	Note b
F	Note b	Note b	Note b	Note b	Note b

a. Use straight line interpolation for intermediate values of mapped spectral acceleration at short period, S_S.

b. Site-specific geotechnical investigation and dynamic site response analyses shall be performed to determine appropriate values.

TABLE 1615.1.2(2)
VALUES OF SITE COEFFICIENT F_V AS A FUNCTION OF SITE CLASS AND MAPPED SPECTRAL RESPONSE ACCELERATION AT 1 SECOND PERIOD (S_1)[a]

SITE CLASS	MAPPED SPECTRAL RESPONSE ACCELERATION AT 1 SECOND PERIOD				
	$S_1 \le 0.1$	$S_1 = 0.2$	$S_1 = 0.3$	$S_1 = 0.4$	$S_1 \ge 0.5$
A	0.8	0.8	0.8	0.8	0.8
B	1.0	1.0	1.0	1.0	1.0
C	1.7	1.6	1.5	1.4	1.3
D	2.4	2.0	1.8	1.6	1.5
E	3.5	3.2	2.8	2.4	Note b
F	Note b	Note b	Note b	Note b	Note b

a. Use straight line interpolation for intermediate values of mapped spectral acceleration at 1-second period, S_1.

b. Site-specific geotechnical investigation and dynamic site response analyses shall be performed to determine appropriate values.

1615.1.4 General procedure response spectrum. The general design response spectrum curve shall be developed as indicated in Figure 1615.1.4 and as follows:

1. For periods less than or equal to T_0, the design spectral response acceleration, S_a, shall be given by Equation 16-20.
2. For periods greater than or equal to T_0 and less than or equal to the T_S, the design spectral response acceleration, S_a, shall be taken equal to S_{DS}.
3. For periods greater than T_S, the design spectral response acceleration, S_a, shall be given by Equation 16-21.

$$S_a = 0.6\frac{S_{DS}}{T_O}T + 0.4S_{DS} \qquad \textbf{(Equation 16-20)}$$

$$S_a = \frac{S_{D1}}{T} \qquad \textbf{(Equation 16-21)}$$

where:

S_{DS} = The design spectral response acceleration at short periods as determined in Section 1615.1.3.

S_{D1} = The design spectral response acceleration at 1 second period as determined in Section 1615.1.3.

T = Fundamental period (in seconds) of the structure (Section 1617.4.2).

$T_O = 0.2\ S_{D1}/S_{DS}$

$T_S = S_{D1}/S_{DS}$

1615.1.5 Site classification for seismic design. The notations presented below apply to the upper 100 feet (30 480 mm) of the site profile. Profiles containing distinctly different soil layers shall be subdivided into those layers designated by a number that ranges from 1 to *n* at the bottom where there are a total of *n* distinct layers in the upper 100 feet (30 480 mm). The symbol, *i*, then refers to any one of the layers between 1 and *n*.

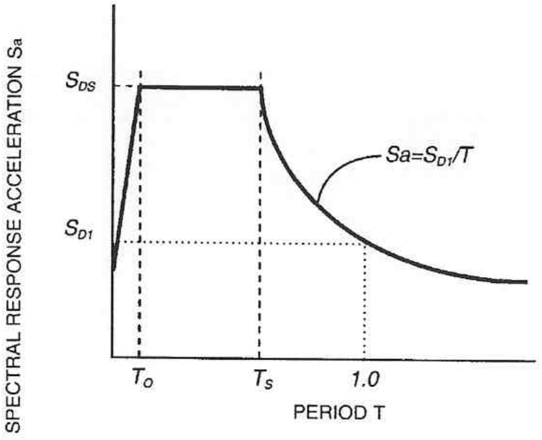

FIGURE 1615.1.4
DESIGN RESPONSE SPECTRUM

where:

v_{si} = The shear wave velocity in feet per second (m/s).

d_i = The thickness of any layer between 0 and 100 feet (30 480 mm).

$$\bar{v}_s = \frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} \frac{d_i}{v_{si}}} \qquad \textbf{(Equation 16-22)}$$

$$\sum_{i=1}^{n} d_i = 100 \text{ feet (30 480 mm)}$$

N_i is the Standard Penetration Resistance (ASTM D 1586-84) not to exceed 100 blows/feet (mm) as directly measured in the field without corrections.

$$\bar{N} = \frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} \frac{d_i}{N_i}} \qquad \textbf{(Equation 16-23)}$$

$$\bar{N}_{ch} = \frac{d_s}{\sum_{i=1}^{m} \frac{d_i}{N_i}} \qquad \textbf{(Equation 16-24)}$$

where:

$$\sum_{i=1}^{m} d_i = d_s$$

Use only d_i and N_i for cohesionless soils.

d_s = The total thickness of cohesionless soil layers in the top 100 feet (30 480 mm).

s_{ui} = The undrained shear strength in pounds per square foot (kPa), not to exceed 5,000 pounds per square foot (240 kPa), ASTM D 2166-91 or D 2850-87.

$$\bar{s}_u = \frac{d_c}{\sum_{i=1}^{k} \frac{d_i}{s_{ui}}} \qquad \textbf{(Equation 16-25)}$$

where:

$$\sum_{i=1}^{k} d_i = d_c$$

d_c = The total thickness ($100 - d_s$) (For SI: $30\,480 - d_s$) of cohesive soil layers in the top 100 feet (30 480 mm).

PI = The plasticity index, ASTM D 4318.

w = The moisture content in percent, ASTM D 2216.

The shear wave velocity for rock, Site Class B, shall be either measured on site or estimated by a geotechnical engineer or engineering geologist/seismologist for competent rock with moderate fracturing and weathering. Softer and more highly fractured and weathered rock shall either be measured on site for shear wave velocity or classified as Site Class C.

The hard rock, Site Class A, category shall be supported by shear wave velocity measurements either on site or on profiles of the same rock type in the same formation with an equal or greater degree of weathering and fracturing. Where hard rock conditions are known to be continuous to a depth of 100 feet (30 480 mm), surficial shear wave velocity measurements may be extrapolated to assess $\bar{v}_s$.

The rock categories, Site Classes A and B, shall not be used if there is more than 10 feet (3048 mm) of soil between the rock surface and the bottom of the spread footing or mat foundation.

1615.1.5.1 Steps for classifying a site.

1. Check for the four categories of Site Class F requiring site-specific evaluation. If the site corresponds to any of these categories, classify the site as Site Class F and conduct a site-specific evaluation.
2. Check for the existence of a total thickness of soft clay > 10 feet (3048 mm) where a soft clay layer is defined by: $\bar{s}_u < 500$ pounds per square foot (25 kPa), $w \geq 40$ percent, and $PI > 20$. If these criteria are satisfied, classify the site as Site Class E.
3. Categorize the site using one of the following three methods with, $\bar{v}_s$, $\bar{N}$, and $\bar{s}_u$, computed in all cases as specified.
   - 3.1. $\bar{v}_s$ for the top 100 feet (30 480 mm) $\bar{v}_s$ method).
   - 3.2. $\bar{N}$ for the top 100 feet (30 480 mm) ($\bar{N}$ method).

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TABLE 1615.1.5
SITE CLASSIFICATION[a]

SITE CLASS	$\bar{v}_s$	$\bar{N}$ or $\bar{N}_{ch}$	$\bar{s}_u$
E	< 600 ft/s	< 15	< 1,000 psf
D	600 to 1,200 ft/s	15 to 50	1,000 to 2,000 psf
C	1,200 to 2,500 ft/s	> 50	> 2,000

For SI: 1 foot per second = 304.8 mm per second, 1 pound per square foot = 0.0479 kN/m².

a. If the $\bar{s}_u$ method is used and the $\bar{N}_{ch}$ and $\bar{s}_u$ criteria differ, select the category with the softer soils (for example, use Site Class E instead of D).

3.3. $\bar{N}_{ch}$, for cohesionless soil layers (PI < 20) in the top 100 feet (30 480 mm) and average, $\bar{s}_u$, for cohesive soil layers (PI > 20) in the top 100 feet (30 480 mm) ($\bar{s}_u$ method).

1615.2 Site-specific procedure for determining ground motion accelerations. A site-specific study shall account for the regional seismicity and geology; the expected recurrence rates and maximum magnitudes of events on known faults and source zones; the location of the site with respect to these; near source effects if any; and the characteristics of subsurface site conditions.

1615.2.1 Probabilistic-maximum considered earthquake. Where site-specific procedures are used as required or permitted by Section 1615, the maximum considered earthquake ground motion shall be taken as that motion represented by an acceleration response spectrum having a 2-percent probability of exceedance within a 50-year period. The maximum considered earthquake spectral response acceleration at any period, S_{aM}, shall be taken from the 2-percent probability of exceedance within a 50-year period spectrum.

Exception: Where the spectral response ordinates at 0.2 second or 1 second for a 5-percent damped spectrum having a 2-percent probability of exceedance within a 50-year period exceeds the corresponding ordinates of the deterministic limit of Section 1615.2.2, the maximum considered earthquake ground motion spectrum shall be taken as the lesser of the probabilistic maximum considered earthquake ground motion or the deterministic maximum considered earthquake ground motion spectrum of Section 1615.2.3, but shall not be taken as less than the deterministic limit ground motion of Section 1615.2.2.

1615.2.2 Deterministic limit on maximum considered earthquake ground motion. The deterministic limit for the maximum considered earthquake ground motion shall be the response spectrum determined in accordance with Figure 1615.2.2, where site coefficients, F_a and F_v, are determined in accordance with Section 1615.1.2, with the value of the mapped short period spectral response acceleration, S_S, taken as 1.5g and the value of the mapped spectral response acceleration at 1 second, S_1, taken as 0.6g.

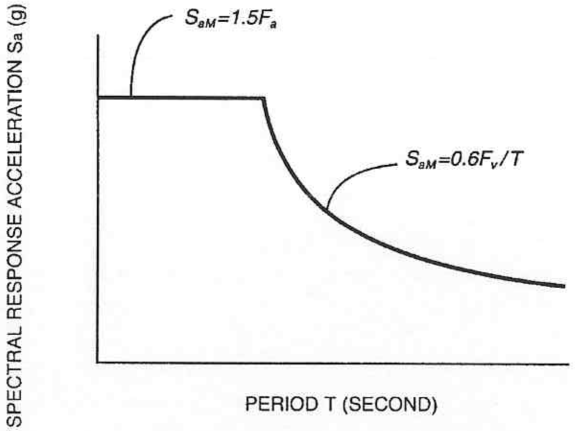

FIGURE 1615.2.2
DETERMINISTIC LIMIT ON MAXIMUM CONSIDERED EARTHQUAKE RESPONSE SPECTRUM

1615.2.3 Deterministic maximum considered earthquake ground motion. The deterministic maximum considered earthquake ground motion response spectrum shall be calculated as 150 percent of the median spectral response accelerations, S_{aM}, at all periods resulting from a characteristic earthquake on any known active fault within the region.

1615.2.4 Site-specific design ground motion. Where site-specific procedures are used to determine the maximum considered earthquake ground motion response spectrum, the design spectral response acceleration, S_a, at any period shall be determined from Equation 16-26:

$$S_a = \frac{2}{3} S_{aM} \qquad \textbf{(Equation 16-26)}$$

and shall be greater than or equal to 80 percent of the design spectral response acceleration, S_a, determined by the general response spectrum in Section 1615.1.4.

18-10

1615.2.5 Design spectral response coefficients. Where site-specific procedures are used as required or permitted by Section 1615, the design spectral response acceleration coefficient at short periods, S_{DS}, and the design spectral response acceleration at 1-second period, S_{D1}, shall be taken as the values of the design spectral response acceleration, S_a, obtained in accordance with Section 1615.2.4, at periods of 0.2 second and 1.0 second, respectively. The values so obtained shall not be taken as less than 80 percent of the values obtained from the general procedures of Section 1615.1.

SECTION 1616
EARTHQUAKE LOADS—CRITERIA SELECTION

1616.1 Structural design criteria. Each structure shall be assigned to a seismic design category in accordance with Section 1616.3. Seismic design categories are used in this code to determine permissible structural systems, limitations on height and irregularity, those components of the structure that must be designed for seismic resistance, and the types of lateral force analysis that must be performed.

Each structure shall be provided with complete lateral- and vertical-force-resisting systems capable of providing adequate strength, stiffness and energy dissipation capacity to withstand the design earthquake ground motions determined in accordance with Section 1615 within the prescribed deformation limits of Section 1617.3. The design ground motions shall be assumed to occur along any horizontal direction of a structure. A continuous load path, or paths, with adequate strength and stiffness to transfer forces induced by the design earthquake ground motions from the points of application to the final point of resistance, shall be provided.

Allowable Stress Design is permitted to be used to evaluate sliding, overturning and soil bearing at the soil-structure interface regardless of the design approach used in the design of the structure, provided load combinations of Section 1605.3 are utilized. When using Allowable Stress Design for proportioning foundations, the value of 0.2 $S_{DS}D$ in Equations 16-28, 16-29, 16-30 and 16-31 is permitted to be taken equal to zero. When the load combinations of Section 1605.3.2 are utilized, a one-third increase in soil allowable stresses is permitted for all load combinations that include W or E.

1616.2 Seismic use groups and occupancy importance factors. Each structure shall be assigned a seismic use group and a corresponding occupancy importance factor (I_E) as indicated in Table 1604.5.

1616.2.1 Seismic Use Group I. Seismic Use Group I structures are those not assigned to either Seismic Use Group II or III.

1616.2.2 Seismic Use Group II. Seismic Use Group II structures are those, the failure of which would result in a substantial public hazard due to occupancy or use as indicated by Table 1604.5, or as designated by the building official.

1616.2.3 Seismic Use Group III. Seismic Use Group III structures are those, the failure of which would result in having essential facilities that are required for postearthquake recovery and those containing substantial quantities of hazardous substances, as indicated in Table 1604.5, or as designated by the building official.

Where operational access to a Seismic Use Group III structure is required through an adjacent structure, the adjacent structure shall conform to the requirements for Seismic Use Group III structures. Where operational access is less than 10 feet (3048 mm) from an interior lot line or less than 10 feet (3048 mm) from another structure, access protection from potential falling debris shall be provided by the owner of the Seismic Use Group III structure.

1616.2.4 Multiple occupancies. Where a structure is occupied for two or more occupancies not included in the same seismic use group, the structure shall be assigned the classification of the highest seismic use group corresponding to the various occupancies.

Where structures have two or more portions that are structurally separated in accordance with Section 1620, each portion shall be separately classified. Where a structurally separated portion of a structure provides required access to, required egress from, or shares life safety components with another portion having a higher seismic use group, both portions shall be assigned the higher seismic use group.

1616.3 Determination of seismic design category. All structures shall be assigned to a seismic design category based on their seismic use group and the design spectral response acceleration coefficients, S_{DS} and S_{D1}, determined in accordance with Section 1615.1.3 or 1615.2.5. Each building and structure shall be assigned to the most severe seismic design category in accordance with Table 1616.3(1) or 1616.3(2), irrespective of the fundamental period of vibration of the structure, T.

TABLE 1616.3(1)
SEISMIC DESIGN CATEGORY BASED ON
SHORT PERIOD RESPONSE ACCELERATIONS

VALUE OF S_{DS}	SEISMIC USE GROUP		
	I	II	III
$S_{DS} < 0.167g$	A	A	A
$0.167g \le S_{DS} < 0.33g$	B	B	C
$0.33g \le S_{DS} < 0.50g$	C	C	D
$0.50g \le S_{DS}$	D[a]	D[a]	D[a]

a. Seismic Use Groups I and II structures located on sites with mapped maximum considered earthquake spectral response acceleration at 1-second period, S_1, equal to or greater than 0.75g, shall be assigned to Seismic Design Category E, and Seismic Use Group III structures located on such sites shall be assigned to Seismic Design Category F.

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MDOF SYSTEMS

Generalized SDOF problems

- many DOFs replaced by one
- assumed shape function

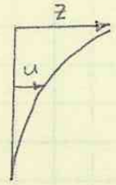
$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \hat{k}z = \tilde{p}(t)$$, where the ~s indicate generalization

Used for:

- rigid body assemblages
  distributed mass systems with discrete springs and dampers
- systems with distributed mass and stiffness
  infinite DOF probs
- lumped mass systems
  finite number of DOF

Distributed mass (handout 19)

height L
EI(x)



$$u^t(x,t) = u(x,t) + u_g(t)$$

$$u(x,t) = \psi(x)\,z(t)$$

$z(t)$ — generalized coordinate

$\psi(x)$ — shape function

choose $z(t)$, $\psi(t)$ to accurately model true behavior

$\psi(x)$ selection

- needs to match boundary conditions

$$\frac{x^2}{L^2} \longrightarrow \psi(0)=0,\ \psi'(0)=0\ ✓$$

find equation of motion by forcing internal work = external work

$$\underline{\tilde{m}\ddot{z} + \tilde{k}z = -\tilde{L}\,\ddot{u}_g(t)}$$

generalized values

$$\tilde{m} = \int_0^L m(x)\left[\psi(x)\right]^2 dx$$

$$\tilde{k} = \int_0^L EI(x)\left[\psi''(x)\right]^2 dx$$, $\tilde{c} = 2\zeta\omega_n\tilde{m}$, if ζ is specified

$$\tilde{L} = \int_0^L m(x)\,\psi(x)\,dx$$

$\omega_n^2 = \tilde{k}/\tilde{m}$, $\tilde{\Gamma} = \tilde{L}/\tilde{m}$ — scale factor on $\ddot{u}_g$

$$\ddot{z} + \omega_n^2 z = -\tilde{\Gamma}\,\ddot{u}_g(t)$$

Chapter 8 - Generalized SDOF Systems

Chapters 1-7 SDOF Systems with single point mass	→	Chapter 8 "Generalized" SDOF systems [Distributed mass & stiffness OR lumped-mass systems]	→	Chapters 9,10,.... Multi-degree-of-freedom systems (MDOF systems)
SDOF	→	TRANSITION	→	MDOF

Generalized SDOF systems

In such systems, there are actually many (sometimes, infinite) degrees of freedom which are replaced by one single <u>generalized coordinate</u> along with an assumed <u>shape function</u>

our equation of motion given in terms of the generalized coordinate, $z(t)$, will usually be of the form:

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{p}(t)$$

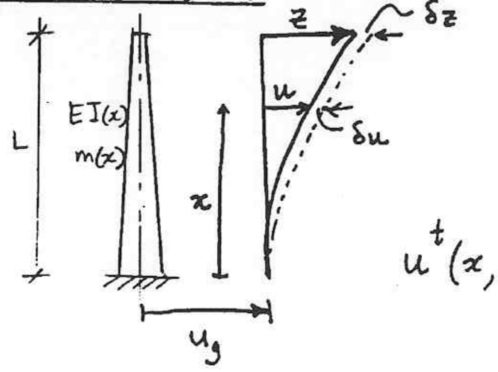
where $\tilde{m}$, $\tilde{c}$, $\hat{k}$, and $\tilde{p}(t)$ are "generalized" mass, damping, stiffness, and forces that depend on the assumed shape function.

This can be applied to:

- Rigid-Body Assemblages
  Distributed mass systems
  with discrete springs & dampers
- Systems with Distributed Mass & Stiffness
  Infinite degree of system systems
- Lumped-Mass Systems
  Finite No. of degrees of freedom (lumped at specific locations)
  e.g. Shear Building models

Distributed Mass & Elasticity Systems

Cantilever Tower



$m(x)$: mass per unit length [Note: Units]

$EI(x)$: flexural rigidity

$$u^t(x,t) = u(x,t) + u_g(t)$$

Choose a generalized coordinate $z(t)$
and a shape function $\psi(x)$ such that

$$u(x,t) = \psi(x)\, z(t)$$

$\psi(x)$ should satisfy the boundary conditions.

e.g., Cantilever should have $\psi(0) = 0$; $\psi'(0) = 0$

e.g., $\psi(x) = \frac{x^2}{L^2}$ would work.

D'Alembert's Principle: $f_I(x,t) = -m(x)\, \ddot{u}^t(x,t)$

Principle of Virtual Displacements :

→ Apply virtual displacement $\delta u(x)$

$\delta W_E = \delta W_I$ (External Virtual Work due to inertial forces acting thru' virtual displs $\delta u(x)$ = Internal Virtual Work)

Note: These must be consistent with support & continuity conditions

$$\delta W_E = \int_0^L f_I(x,t)\, \delta u(x)\, dx \leftarrow$$

. For Earthquakes

$$f_I(x,t) = -m(x)\ddot{u}^t(x,t)$$

$$\delta W_I = \int_0^L \mathcal{M}(x,t)\, \delta K(x)\, dx$$

For general loads, $p(x,t)$

$$\delta W_e = \int_0^L \left[-m(x)\ddot{u}(x,t) + p(x,t)\right] \delta u(x)\, dx$$

Bending Moment, $\mathcal{M}(x,t) = EI(x)u''(x,t)$

Curvature associated with virtual displacement, $\delta K(x) = \delta\left[u''(x)\right]$

where $u'' \equiv \dfrac{\partial^2 u}{\partial x^2}$

Using $\delta u(x) = \psi(x)\,\delta z$; $\delta\left[u''(x)\right] = \psi''(x)\,\delta z$

$$\boxed{\tilde{m}\ddot{z} + \tilde{k}z = -\tilde{L}\ddot{u}_g(t)} \text{ — } (\mathrm{I})$$

or

$$\boxed{\ddot{z} + \omega_n^2 z = -\tilde{\Gamma}\ddot{u}_g(t)} \text{ — } (\mathrm{II})$$

where $\tilde{m} = \displaystyle\int_0^L m(x)\left[\psi(x)\right]^2 dx$, Generalized Mass

$$\tilde{k} = \int_0^L EI(x)\left[\psi''(x)\right]^2 dx \text{ , Generalized Stiffness}$$

With specified ξ, we can define $\tilde{c} = 2\xi\omega_n\tilde{m}$ if we need to.

$$\tilde{L} = \int_0^L m(x)\,\psi(x)\, dx$$

$$\omega_n^2 = \tilde{k}/\tilde{m} \quad ; \quad \tilde{\Gamma} = \frac{\tilde{L}}{\tilde{m}}$$

(scale factor on $\ddot{u}_g$; see Eq. II)

In (I), $-\tilde{L}\ddot{u}_g(t)$ is a generalized excitation.

Use I or II to solve for $z(t)$.

$$u(x,t) = \psi(x)\, z(t).$$

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Equivalent static forces

Exact → $f_S(x,t) = \left[EI(x)\, u''(x,t)\right]''$ from beam theory.
at all locations

Easier → $f_S(x,t) = \omega_n^2\, m(x)\, \psi(x)\, z(t)$ ← no derivatives
But only approximate.
Still, when integrated over length, it is "exact"

better because $\psi(x)$ is an assumed shape function whose derivatives may not be accurate.

From Ⓘⓘ, $\tilde{\Gamma}$ may be thought of as a scale factor on ground acceleration $\ddot{u}_g(t)$.

Peak value of z $= Z_0 = \tilde{\Gamma} D = \dfrac{\tilde{\Gamma}}{\omega_n^2} A$
z_0

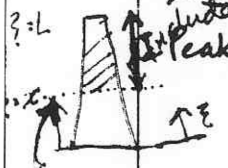
$u(x,t) = \psi(x)\, z(t)$
$\Rightarrow u_0(x) = \psi(x)\, z_0$

$\hookrightarrow u_0(x) = \tilde{\Gamma}\, D\, \psi(x)$ Peak Displacement at any x

$f_{s_0}(x) = \tilde{\Gamma}\, m(x)\, \psi(x)\, A$ Peak Equivalent Static force at x

Internal Forces

Shears to be included



Shear at height x desired

Peak Shear Force at height x, $\quad \mathcal{V}_0(x) = \displaystyle\int_x^L f_{s_0}(\xi)\, d\xi = \tilde{\Gamma} A \int_x^L m(\xi)\, \psi(\xi)\, d\xi$

Peak Bending Moment at height x, $\quad \mathcal{M}_0(x) = \displaystyle\int_x^L (\xi - x)\, f_{s_0}(\xi)\, d\xi$

$$= \tilde{\Gamma} A \int_x^L (\xi - x)\, m(\xi)\, \psi(\xi)\, d\xi$$

At base, $x = 0$ $\quad \mathcal{V}_{b_0} = \mathcal{V}_0(0) = \tilde{L}\, \tilde{\Gamma} A$

$\mathcal{M}_{b_0} = \mathcal{M}_0(0) = \tilde{L}^\theta\, \tilde{\Gamma} A \quad \left[\tilde{L}^\theta = \displaystyle\int_0^L x\, m(x)\, \psi(x)\, dx\right]$

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For any given force $p(x,t)$
including non-seismic forces, we have:

$$\boxed{\tilde{m}\ddot{z} + \tilde{k}z = \tilde{p}(t)}$$

where
$$\tilde{p}(t) = \int_0^L p(x,t)\,\psi(x)\,dx$$, Generalized Force

In special case where $p(x,t) = -m(x)\,\ddot{u}_g(t)$

as before, we have $\tilde{m}\ddot{z} + \tilde{k}z = -\tilde{L}\,\ddot{u}_g(t)$

where $\tilde{L} = \int_0^L m(x)\,\psi(x)\,dx$ (see Eq. I)

Utilizing Eq. (8.3.15) and dropping δz, Eq. (d) can be rewritten as

$$\int_0^L \left[\tilde{f}_S(x,t) - \omega_n^2 m(x)\psi(x)z(t)\right]\psi(x)\,dx = 0 \qquad \text{(e)}$$

Setting the quantity in brackets to zero gives

$$f_S(x,t) = \omega_n^2 m(x)\psi(x)z(t) \qquad \text{(f)}$$

where the tilde above f_S has now been dropped. This completes the derivation of Eq. (8.3.19).

Example 8.2

A uniform cantilever tower of length L has mass per unit length $= m$ and flexural rigidity EI (Fig. E8.2). Assuming the shape function $\underline{\psi(x) = 1 - \cos(\pi x/2L)}$, formulate the equation of motion for the system excited by ground motion, and determine its natural frequency.

Solution

1. *Determine the generalized properties.*

$$\tilde{m} = m\int_0^L \left(1 - \cos\frac{\pi x}{2L}\right)^2 dx = 0.227\ \text{mL} \qquad \text{(a)}$$

$$\tilde{k} = EI\int_0^L \left(\frac{\pi^2}{4L^2}\right)^2 \cos^2\frac{\pi x}{2L}\,dx = 3.04\frac{EI}{L^3} \qquad \text{(b)}$$

$$\tilde{L} = m\int_0^L \left(1 - \cos\frac{\pi x}{2L}\right) dx = 0.363\ \text{mL} \qquad \text{(c)}$$

The computed $\tilde{k}$ is close to the stiffness of the tower under a concentrated lateral force at the top.

2. *Determine the natural vibration frequency.*

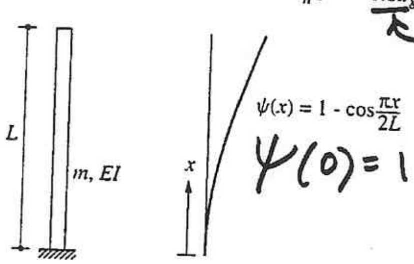
$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \frac{3.66}{L^2}\sqrt{\frac{EI}{m}} \qquad \text{(d)}$$

This approximate result is close to the exact natural frequency, $w_{\text{exact}} = (3.516/L^2)\sqrt{EI/m}$, determined in Chapter 16. The error is only 4%.

3. *Formulate the equation of motion.* Substituting $\tilde{L}$ and $\tilde{m}$ in Eq. (8.3.14) gives $\tilde{\Gamma} = 1.6$ and Eq. (8.3.13b) becomes

$$\ddot{z} + \omega_n^2 z = -\underline{1.6}\ddot{u}_g(t) \xleftarrow{\quad} \tilde{\Gamma} = \tilde{L}/\tilde{m} \qquad \text{(e)}$$

(scale factor on gr. motion)



$\psi(0) = 1 - 1 = 0$

$\psi'(x) = \frac{\pi}{2L}\sin\frac{\pi x}{2L}$

$\psi'(0) = 0$

$\psi(L) = 1$

Figure E8.2

To find response of tower:
use response spectrum
with known T_n.
Multiply D, V, or A obtained
by $\underline{\underline{1.6}}$ (only difference relative to ordinary SDOF systems)

Example 8.3

A reinforced-concrete chimney, 600 ft high, has a uniform hollow circular cross section with outside diameter 50 ft and wall thickness 2 ft 6 in. (Fig. E8.3a). For purposes of preliminary earthquake analysis, the chimney is assumed clamped at the base, the mass and flexural rigidity are computed from the gross area of the concrete (neglecting the reinforcing steel), and the damping is estimated as 5%. The unit weight of concrete is 150 lb/ft³ and its elastic modulus $E_c = 3600$ ksi.

$\psi(x) = 1 - \cos\left(\frac{\pi x}{2L}\right)$

Assuming the shape function as in Example 8.2, estimate the peak displacements, shear forces, and bending moments for the chimney due to ground motion characterized by the design spectrum of Fig. 6.9.5 scaled to a peak acceleration 0.25g.

Solution

1. *Determine the chimney properties.*

Length: $L = 600$ ft

Cross-sectional area: $A = \pi(25^2 - 22.5^2) = 373.1$ ft²

Mass/foot length: $m = \dfrac{150 \times 373.1}{32.2} = 1.738$ kip-sec²/ft²

Second moment of area: $I = \dfrac{\pi}{4}(25^4 - 22.5^4) = 105{,}507$ ft⁴

Flexural rigidity: $EI = 5.469 \times 10^{10}$ kip-ft²

2. *Determine the natural period.* From Example 8.2,

$$\omega_n = \frac{3.66}{L^2}\sqrt{\frac{EI}{m}} = 1.80 \text{ rad/sec}$$

$$T_n = \frac{2\pi}{\omega_n} = 3.49 \text{ sec}$$

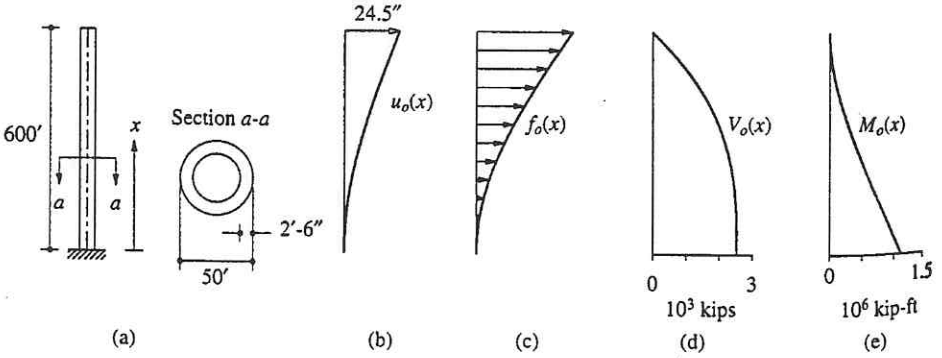


Figure E8.3

$$V_o(x) = \int_x^L f_o(\xi)\,d\xi$$

$$M_o(x) = \int_x^L (\xi - x) f_o(\xi)\,d\xi$$

19-8

3. *Determine the peak value of $z(t)$.* For $T_n = 3.49$ sec and $\zeta = 0.05$, the design spectrum gives $A/g = 0.25(1.80/3.49) = 0.129$. The corresponding deformation is $D = A/\omega_n^2 = 15.3$ in. Equation (8.3.20) gives the peak value of $z(t)$:

←

$A = 0.25 [1.80 T_n^{-1}]$

for $0.66 < T_n < 4.12$

$\tilde{\Gamma} = 1.6$ (scale factor)

$$z_o = 1.6D = 1.6 \times 15.3 = 24.5 \text{ in.}$$

4. *Determine the peak displacements $u_o(x)$ of the tower (Fig. E8.3b).*

$$u_o(x) = \psi(x)z_o = 24.5\left(1 - \cos\frac{\pi x}{2L}\right) \text{ in.}$$

5. *Determine the equivalent static forces.*

$$f_o(x) = \tilde{\Gamma} m(x)\psi(x)A = (1.6)(1.738)\left(1 - \cos\frac{\pi x}{2L}\right)0.129g$$

$$= 11.58\left(1 - \cos\frac{\pi x}{2L}\right) \text{ kips/ft}$$

These forces are shown in Fig. E8.3c.

6. *Compute the shears and bending moments.* Static analysis of the chimney subjected to external forces $f_o(x)$ gives the shear forces and bending moments. The results using Eq. (8.3.22) are presented in Fig. E8.3d and e. If we were interested only in the forces at the base of the chimney, they could be computed directly from Eq. (8.3.23). In particular, the base shear is

$$V_{bo} = \tilde{L}\tilde{\Gamma}A = (0.363mL)(1.6)0.129g$$

$$= 0.0749mLg = 2518 \text{ kips}$$

This is 7.49% of the total weight of the chimney.

Example 8.4

A simply supported bridge with a single span of L feet has a deck of uniform cross section with mass m per foot length and flexural rigidity EI. A single wheel load p_o travels across the bridge at a uniform velocity of v, as shown in Fig. E8.4. Neglecting damping and assuming the shape function as $\psi(x) = \sin(\pi x/L)$, determine an equation for the deflection at midspan as a function of time. The properties of a prestressed-concrete box-girder elevated-freeway connector are $L = 200$ ft, $m = 11$ kips/g per foot, $I = 700$ ft^4, and $E = 576{,}000$ kips/ft^2. If $v = 55$ mph, determine the impact factor defined as the ratio of maximum deflection at midspan and the static deflection.

Solution We assume that the mass of the wheel load is small compared to the bridge mass, and it can be neglected.

1. *Determine the generalized mass, generalized stiffness, and natural frequency.*

$$\psi(x) = \frac{\sin \pi x}{L} \qquad \psi''(x) = -\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}$$

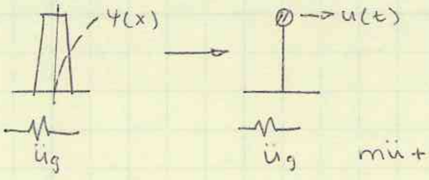
$$\tilde{m} = \int_0^L m \sin^2\frac{\pi x}{L}\,dx = \frac{mL}{2} \tag{a}$$

$$\tilde{k} = \int_0^L EI\left(\frac{\pi^2}{L^2}\right)^2 \sin^2\frac{\pi x}{L}\,dx = \frac{\pi^4 EI}{2L^3} \tag{b}$$

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{m}} \tag{c}$$

GENERALIZED SYSTEMS

SDOF models to represent MDOF structures — Handout 20, 19



$m\ddot{u} + ku = -m\ddot{u}_g$

$u(x,t) = \psi(x)\, z(t)$

Base forces (shear, moment)

$V_0(x) = \int_x^L \tilde{\Gamma} A m(\xi)\psi(\xi)\,d\xi$ ($\tilde{\Gamma}$ and A can be outside $\int$)

→ at $x=0$, $V_0 = \tilde{L}\tilde{\Gamma}A$

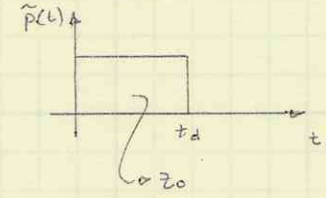
$M_0(x) = \tilde{\Gamma}A \int_x^L (\xi - x)\, m(\xi)\psi(\xi)\,d\xi$

→ at $x=0$, $M_0 = \tilde{L}^\theta \tilde{\Gamma} A$, where $\tilde{L}^\theta = \int_0^L x\, m(x)\psi(x)\,dx$

Key Assumption:
shape function cannot change over time!

Homework-ish comments

$u_0(x) = \psi(x) z_0$



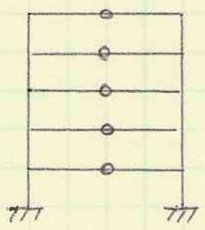
$f_{S0}(x) = \omega_n^2 m(x)\psi(x) z_0 \cdot \tilde{\Gamma}\frac{A}{\omega_n^2}$
↑
in book as f_0

$m_t(m,t) := \int_0^L m(x)\left[\psi(x)\right]^2 dx$, $m(x) := m$, $\psi(x) := x^2/L^2$

use examples near pg 289 in book

Lumped-mass systems

u becomes a vector, need shape vector



$\tilde{m}\ddot{z} + \tilde{k}\tilde{z} = -\tilde{L}\ddot{u}_g(t)$

$\tilde{m} = \sum_{j=1}^{N} m_j \psi_j^2$

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LUMPED-MASS SYSTEMS

Comparison of methods

Lumped (8.4) Discrete		Continuous (8.3)
$\underline{u}(t) = \underline{\psi} z(t)$		$u(x,t) = \psi(x) z(t)$
$\tilde{m}\ddot{z} + \tilde{k}z = -\tilde{L}\ddot{u}_g$		$\tilde{k} = \int_0^L EI(x)\left[\psi''(x)\right]^2 dx$
$\tilde{m} = \Sigma m_i \psi_i^2 = \underline{\psi}^T \underline{m} \underline{\psi}$		$\tilde{m} = \int_0^L m(x)\left[\psi(x)\right]^2 dx$
~~k~~ $\tilde{L} = \Sigma m_i \psi_i = \underline{\psi}^T \underline{m}\, \underline{1}$		$\tilde{L}$ same as mass, $\psi(x)$ not squared
$u_o = \psi_j \tilde{\Gamma} A/\omega_n^2$		$\tilde{\Gamma} = \tilde{L}/\tilde{m}$ = scale factor
(matrix, or —)		$u_o(x) = \psi(x) \tilde{\Gamma} A/\omega_n^2$
summation	vs.	integral

Forced movement (as opposed to earthquake)

$$\tilde{\Gamma} A/\omega_n^2 \longrightarrow R_d (z_{st})_o \,,\quad z_{st} = P_o/K$$

$$R_d \propto 2\pi/\omega,\ \omega_n$$

Lumped Mass

$$\tilde{k} = \Sigma K(\psi_j - \psi_{j-1})^2 \qquad \underset{\sim}{\tilde{k}} = \underline{\psi}^T \underline{k}\, \underline{\psi}$$

$\uparrow$ $\frac{12EI}{L^3}$, assumes fixed columns

equivalent static force

$$f_{s_j} = \tilde{\Gamma} m_j \psi_j A$$

choosing ψ-function

- function cannot be deemed "conservative," but the estimated ω_n will always overestimate the real one.
- thus, if a second function yields a lower ω_n, it is more exact.

Multi-degree of freedom systems

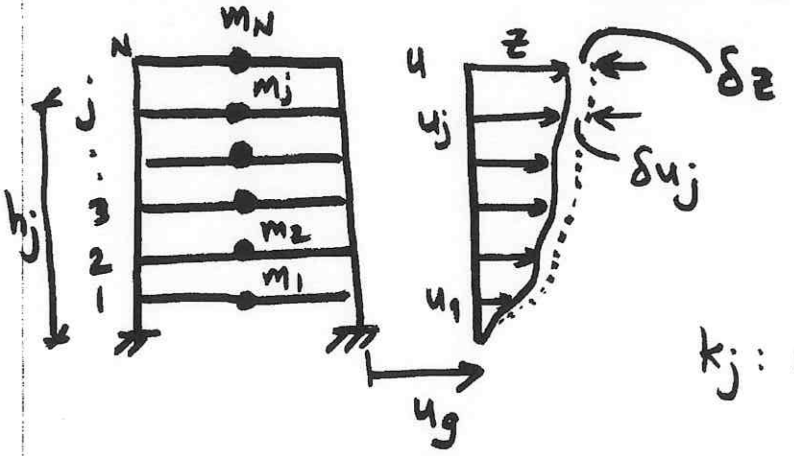
e.g. 2-story frame

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} f_{S1} \\ f_{S2} \end{bmatrix} + \begin{bmatrix} f_{D1} \\ f_{D2} \end{bmatrix} = \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}$$

or $p_j - f_{S_j} - f_{D_j} = m_j \ddot{u}_j \quad (j = 1, 2)$

Lumped-Mass Systems

Same idea as with systems with distributed mass & elasticity.



m_j : mass at the j^{th} floor

k_j : stiffness of story j

System has N degrees of freedom but will reduce it to a SDOF analysis

Assume $u_j(t) = \psi_j z(t) \quad j = 1, 2, \dots N$

i.e. assume a shape vector $\{\underline{\psi}\}$ [any reasonable shape]

$$\{\underline{u}(t)\} = \{\underline{\psi}\} \, z(t)$$

generalized coordinate

$\tilde{k} = \sum k_j (\psi_j - \psi_{j-1})^2$

$$u_j^t(t) = u_j(t) + u_g(t)$$

Assume beams are rigid axially & flexurally.

$\Rightarrow$ V_j (shear in j^{th} story) is related to story drift

$$\Delta_j = u_j - u_{j-1}$$

$$V_j = k_j \Delta_j = k_j (u_j - u_{j-1})$$

$$k_j = \sum_{\substack{\text{columns} \\ \text{on story } j}} \frac{12\,EI}{h^3}$$

D'Alembert's Principle:

$$f_{Ij} = -m_j \left[\ddot{u}_j(t) + \ddot{u}_g(t) \right]$$

as before, use

Principle of Virtual displacements

apply virtual displacements δu_j

$$\delta W_E = \delta W_I$$

where, $\delta W_E = \sum_{j=1}^{N} f_{Ij}(t)\,\delta u_j$

$$\delta W_I = \sum_{j=1}^{N} V_j(t) \left[\delta u_j - \delta u_{j-1} \right]$$

also, use the fact that:

$$\delta u_j = \psi_j\, \delta z \; ; \quad \delta \underset{\uparrow}{\underline{u}} = \underset{\uparrow}{\underline{\psi}}\, \delta z$$

vector

$$\Rightarrow \quad \boxed{\tilde{m}\,\ddot{z} + \tilde{k}\,z = -\tilde{L}\,\ddot{u}_g(t)} \quad \text{Ⓘ}$$

where,
$$\tilde{m} = \sum_{j=1}^{N} m_j \psi_j^2$$

$$\tilde{k} = \sum_{j=1}^{N} k_j \left[\psi_j - \psi_{j-1}\right]^2$$

$$\tilde{L} = \sum_{j=1}^{N} m_j \psi_j$$

or in vector notation:

$$\tilde{m} = \underbrace{\underline{\psi}^T}_{1\times N} \underbrace{\underline{m}}_{N\times N} \underbrace{\underline{\psi}}_{N\times 1}$$

$$\tilde{k} = \underbrace{\underline{\psi}^T}_{1\times N} \underbrace{\underline{k}}_{N\times N} \underbrace{\underline{\psi}}_{N\times 1}$$

$$\tilde{L} = \underbrace{\underline{\psi}^T}_{1\times N} \underbrace{\underline{m}}_{N\times N} \underbrace{\underline{1}}_{N\times 1} \qquad \underline{1} = \left\{\begin{matrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{matrix}\right\}$$

These are generalized mass, stiffness and scale factors of excitation.

$$\omega_n^2 = \frac{\underline{\psi}^T \underline{k} \underline{\psi}}{\underline{\psi}^T \underline{m} \underline{\psi}} \quad \text{or} \quad \frac{\tilde{k}}{\tilde{m}}$$

also, $$\boxed{\ddot{z} + \omega_n^2 z = -\tilde{\Gamma} \ddot{u}_g} \text{ — } \text{Ⅱ}$$

where $$\tilde{\Gamma} = \frac{\tilde{L}}{\tilde{m}} = \frac{\underline{\psi}^T \underline{m} \underline{1}}{\underline{\psi}^T \underline{m} \underline{\psi}}$$

Use I or II to solve for $z(t)$

$$\underline{u} = \underline{\psi} \, z(t)$$

or $u_j(t) = \psi_j \, z(t) \quad j = 1, 2, \ldots N$

Equivalent Static Forces

$$f_{s_j} = \tilde{\Gamma} m_j \psi_j A$$

Peak value of $z = z_0 = \tilde{\Gamma} D$

$\Rightarrow u_{jo} = \psi_j z_0 = \tilde{\Gamma} D \psi_j$ [Peak value of j^{th} story displacement]

At the i^{th} floor

shear Force. $$V_{io} = \sum_{j=i}^{N} f_{jo}$$

Sum all forces above that level

$$M_{io} = \sum_{j=i}^{N} (h_j - h_i) f_{jo}$$

height difference between two levels

At base,

$$V_{bo} = \sum_{j=1}^{N} f_{jo}$$

$$M_{bo} = \sum_{j=1}^{N} h_j f_{jo}$$

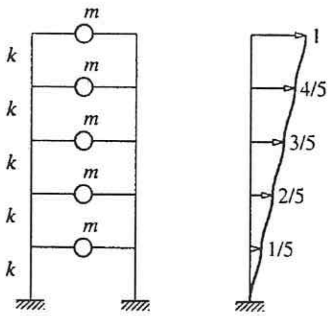
or $$V_{bo} = \tilde{L} \tilde{\Gamma} A$$

$$M_{bo} = \tilde{L}^{\theta} \tilde{\Gamma} A$$

where $$\tilde{L}^{\theta} = \sum_{j=1}^{N} h_j m_j \psi_j$$

Example 8.5

The uniform five-story frame with rigid beams shown in Fig. E8.5a is subjected to ground acceleration $\ddot{u}_g(t)$. All the floor masses are m, and all stories have the same height h and stiffness k. Assuming the displacements to increase linearly with height above the base (Fig. E8.5b), formulate the equation of motion for the system and determine its natural frequency.



(a) (b) **Figure E8.5**

Solution

1. *Determine the generalized properties.*

$$\tilde{m} = \sum_{j=1}^{5} m_j \psi_j^2 = m \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2}{5^2} = \frac{11}{5} m$$

$$\tilde{k} = \sum_{j=1}^{5} k_j (\psi_j - \psi_{j-1})^2 = k \frac{1^2 + 1^2 + 1^2 + 1^2 + 1^2}{5^2} = \frac{k}{5}$$

$$\tilde{L} = \sum_{j=1}^{5} m_j \psi_j = m \frac{1 + 2 + 3 + 4 + 5}{5} = 3m$$

2. *Formulate the equation of motion.* Substituting for $\tilde{m}$ and $\tilde{L}$ in Eq. (8.3.14) gives $\tilde{\Gamma} = \frac{15}{11}$ and Eq. (8.3.13b) becomes

$$\ddot{z} + \omega_n^2 z = -\tfrac{15}{11} \ddot{u}_g(t)$$

where z is the lateral displacement at the location where $\psi_j = 1$, in this case the top of the frame.

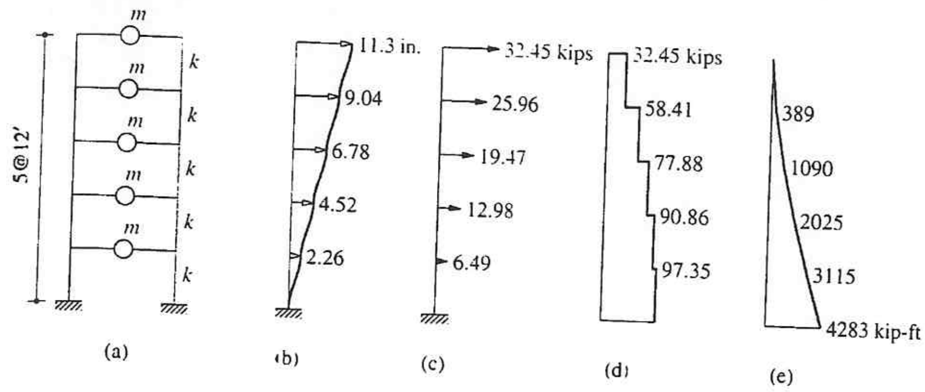
3. *Determine the natural vibration frequency.*

$$\omega_n = \sqrt{\frac{k/5}{11m/5}} = 0.302 \sqrt{\frac{k}{m}}$$

This is about 6% higher than $\omega_n = 0.285\sqrt{k/m}$, the exact frequency of the system determined in Chapter 12.

Example 8.6

Determine the peak displacements, story shears, and floor overturning moments for the frame of Example 8.5 with $m = 100$ kips/g, $k = 31.54$ kips/in., and $h = 12$ ft (Fig. E8.6a) due to the ground motion characterized by the design spectrum of Fig. 6.9.5 scaled to a peak ground acceleration of 0.25g.



(a) (b) (c) (d) (e)

Figure E8.6

Solution

1. *Compute the natural period.*

$$\omega_n = 0.302\sqrt{\frac{31.54}{100/386}} = 3.332$$

$$T_n = \frac{2\pi}{3.332} = 1.89 \text{ sec}$$

2. *Determine the peak value of $z(t)$.* For $T_n = 1.89$ sec and $\zeta = 0.05$, the design spectrum gives $A/g = 0.25(1.80/1.89) = 0.238$ and $D = A/\omega_n^2 = 8.28$ in. The peak value of $z(t)$ is

$$z_o = \tfrac{15}{11}D = \tfrac{15}{11}8.28 = 11.3 \text{ in.}$$

$\tilde{\Gamma} = \tilde{L}/\tilde{m} = 15/11$

3. *Determine the peak values u_{jo} of floor displacements.*

$$u_{jo} = \psi_j z_o \qquad \psi_j = \frac{j}{5}$$

Therefore, $u_{1o} = 2.26$, $u_{2o} = 4.52$, $u_{3o} = 6.78$, $u_{4o} = 9.04$, and $u_{5o} = 11.3$, all in inches (Fig. E8.6b).

4. *Determine the equivalent static forces.*

$$f_{jo} = \tilde{\Gamma} m_j \psi_j A = \tfrac{15}{11} m \psi_j (0.238g) = 32.45\psi_j \text{ kips}$$

These forces are shown in Fig. E8.6c.

5. *Compute the story shears and overturning moments.* Static analysis of the structure subjected to external floor forces f_{jo}, Eq. (8.4.16), gives the story shears (Fig. E8.6d) and overturning moments (Fig. E8.6e). If we were interested only in the forces at the base, they could be computed directly from Eq. (8.4.17). In particular, the base shear is

$$V_{bo} = \tilde{L}\tilde{\Gamma} A = (3m)\tfrac{15}{11}0.238g$$

$$= 0.195(5mg) = 97.35 \text{ kips}$$

This is 19.5% of the total weight of the building.

Rayleigh's Method for Determining Natural Frequency

Principle of conservation of energy basis.

We saw for SDOF system with mass m & stiffness k $\rightarrow \omega_n = \sqrt{k/m}$

Can derive this using conservation of energy.

Free Vibrations (Undamped case)

$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$$

change time variable to t' $\longrightarrow u(t') = u_0 \sin\omega_n t'$

$$\text{where } u_0 = \sqrt{u(0)^2 + \frac{\dot{u}(0)^2}{\omega_n^2}}$$

$$\text{also, } \dot{u}(t') = \omega_n u_0 \overset{\cos}{\cancel{\sin}}\,\omega_n t'$$

At $t' = T_n/4,\ 3T_n/4, \ldots$ Strain Energy is maximum since $u(t')$ is maximum

and $E_S(t) = \frac{1}{2} k[u(t)]^2$

Maximum: $E_{S_0} = \frac{1}{2} k u_0^2$ (also $\dot{u}(t') = 0$ $\Rightarrow E_k(t') = 0$)

At $t' = 0,\ T_n/2,\ 3T_n/2 \ldots$ Kinetic Energy is maximum since $\dot{u}(t')$ is maximum

and $E_k(t) = \frac{1}{2} m[\dot{u}(t)]^2$

$E_{k_0} = \frac{1}{2} m\dot{u}_0^2$ (also $u(t') = 0$ $\Rightarrow E_s(t') = 0$)

Equate energy at two times (one when $u(t')$ is max) other $\dot{u}(t')$ is max

$$E_{s_0} = E_{k_0} \Rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

Distributed mass & elasticity system	Lumped-Mass system
$u(x,t') = z_0 \sin \omega_n t' \psi(x)$	$\underline{u}(t') = z_0 \sin \omega_n t' \underline{\psi}$
$\dot{u}(x,t') = \omega_n z_0 \cos \omega_n t' \psi(x)$	$\dot{\underline{u}}(t') = \omega_n z_0 \cos \omega_n t' \underline{\psi}$
$E_{s_0} = \int_0^L \frac{1}{2} EI(x) \left[u_0''(x)\right]^2 dx$	$E_{s_0} = \sum_{j=1}^{N} \frac{1}{2} k_j \left(u_{j0} - u_{j-1,0}\right)^2$
$u_0(x) = z_0 \psi(x)$	$u_{j0} = z_0 \psi_j$
$E_{K_0} = \int_0^L \frac{1}{2} m(x) \left[\dot{u}_0(x)\right]^2 dx$	$E_{k_0} = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{u}_{j0}^2$
$\dot{u}_0(x) = \omega_n z_0 \psi(x)$	$\dot{u}_{j0} = \omega_n z_0 \psi_j$
$E_{s_0} = E_{k_0} \Rightarrow$	$E_{s_0} = E_{k_0} \Rightarrow$
$\omega_n^2 = \dfrac{\int_0^L EI(x)\left[\psi''(x)\right]^2 dx}{\int_0^L m(x)\left[\psi(x)\right]^2 dx}$	$\omega_n^2 = \dfrac{\sum_{j=1}^{N} k_j \left(\psi_j - \psi_{j-1}\right)^2}{\sum_{j=1}^{N} m_j \psi_j^2}$
same as we obtained before (not using conservation of energy)	or $\dfrac{\underline{\psi}^T \underline{k}\, \underline{\psi}}{\underline{\psi}^T \underset{\substack{\uparrow \\ \text{diagonal}}}{\underline{m}}\, \underline{\psi}}$

Estimating the "Exact" Fundamental Natural Frequency

Rayleigh's method useful.

Pick a shape function/vector

Use methods shown to get ω_n

Try a different shape function/vector

If ω_n is lower, that lower one
is more accurate

→ Rayleigh's method yields
~~"Upper Bound"~~ higher ω_n values always.

Rest of ch. 8 expands on selection of $\psi(x)$; $\underline{\psi}$ etc.

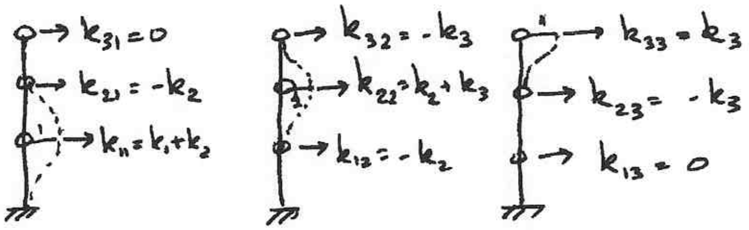
Clarification:

1. For lumped-mass systems, when we wrote

$$\tilde{k} = \underline{\psi}^T \, \underline{k} \, \underline{\psi} ,$$

the matrix $\underline{k}$ would need to be assembled using a direct stiffness formulation, for example.

Example:



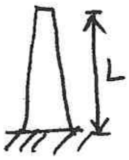
$$k_j = \text{story } j\text{'s stiffness} = \sum_{\substack{\text{no. of}\\ \text{cols.}}} \frac{12 \, EI_{c_j}}{h_j^3}$$

$$\Rightarrow \quad \underline{k} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

Note: $\underline{k}$ is not diagonal

2. The way we defined $\psi(x)$ and $Z(t)$, we had

$\psi(L) = 1 \quad \Rightarrow \quad u(L,t) = Z(t)$



i.e., $Z(t)$, our generalized coordinate was such that it directly represents the end displacement.

This is simply convenient. It is not necessary that $Z(t)$ represent the end displacement. If we select $\psi(x) = 2\,\frac{x^2}{L^2}$, we still have a valid shape. Then $Z(t) = \frac{1}{2} \cdot$ end displacement.

3. Rayleigh's Quotient (RQ)

$$\omega_n^2 = \frac{\int_0^L EI(x)\left[\psi''(x)\right]^2 dx}{\int_0^L m(x)\left[\psi(x)\right]^2 dx} \quad \text{OR} \quad \omega_n^2 = \frac{\underline{\psi}^T\, \underline{k}\, \underline{\psi}}{\underline{\psi}^T\, \underline{m}\, \underline{\psi}}$$

RQ has properties that

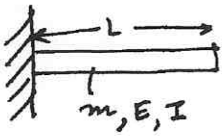
- Frequency estimated with an assumed $\underline{\psi}$ or $\psi(x)$ is <u>always greater than the exact fundamental frequency</u>.

★ As a result, the most accurate shape function is one that gives the lowest RQ and frequency – this is useful especially when the exact frequency is unknown

- it provides reasonable estimates of fundamental frequency even with a not-so-reasonable $\underline{\psi}$ or $\psi(x)$.

Example: Uniform Cantilever

$$\omega_n = \alpha_n \sqrt{\frac{EI}{mL^4}}$$



Shape Function, $\psi(x)$	α_n	% Error
(*) x^2/L^2	4.47	27
$1 - \cos\left(\frac{\pi x}{2L}\right)$	3.66	4
$\frac{3}{2}\left(\frac{x^2}{L^2}\right) - 2\left(\frac{x^3}{L^3}\right)$	3.57	1.5
(**) Exact	3.516	-

(*) Very poor choice of $\psi(x)$ since M(x), Bending moment is effectively constant along cantilever

(**) Fundamental mode shape is complex involving sines, cosines, sinh's, cosh's, etc.

TEST REVIEW

Chapter Coverage

Chapter 8

continuous	vs.	lumped-mass systems
– $u(x,t)=\psi(x)z(t)$		– $\underline{u}(t)=\underline{\psi}\,\underline{z}(t)$
– $\tilde{m}\ddot{z}+\tilde{k}z=\tilde{p}(t)=-\tilde{L}\ddot{u}_g(t)$		– (same general eq.)
– $\tilde{m},\tilde{k},\tilde{L},\tilde{p}(t)$		– $\tilde{m}$ as a Σ, $\tilde{k}$, $\tilde{L}$

Need $z(t)$ or z_0

$u_0(x)=\psi(x)z_0$

$\tilde{p}_0/\tilde{k}\,R_d=z_0$

$\tilde{p}_0=\int\psi p_j(t)$

the smaller the value of ω_n, the better the assumed $\psi(x)$ shape function.

- can use a $\psi(x)$, get $u(x)$ shape
- use that shape as new $\psi(x)$ for a better approximation

Chapter 6

peak ground accel = $\ddot{u}_{go}$

$A=u_0\omega_n^2$

$V_{bo}=mA$

add equations on eq. sheet

Chapter 5

decrease size of equations

add stability details

Questions:

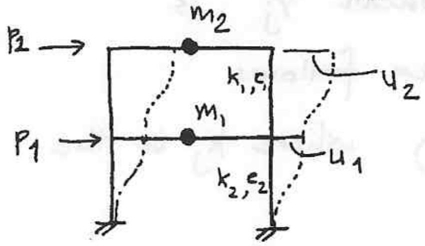
– $z(t)$ scales displacements (relationship defined by $\psi(x)$) through time. when $\psi(x)$ is normalized, $z(t)$ = roof displacement.

42 381 50 SHEETS 5 SQUARE
42 382 100 SHEETS 5 SQUARE
42 389 200 SHEETS 5 SQUARE
NATIONAL

Introduction to MDOF Systems

No. of independent displacements required to define the displaced position relative to the original equilibrium position = No. of Degrees of Freedom

e.g., Two-story frame

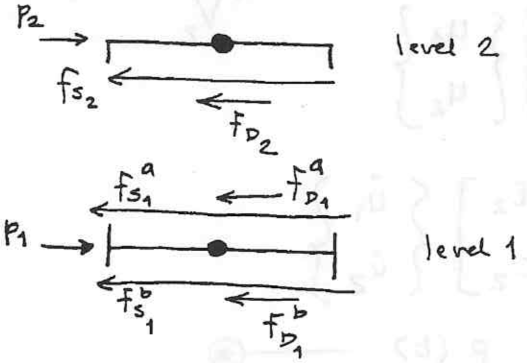


Assumptions:

- Beams / Floor systems rigid in flexure
- Beams / columns - no axial deformations
- No influence of axial loads on column stiffnesses (stability!)
- Masses concentrated at floor levels

Under the above assumptions, the 2-story frame above has TWO degrees of freedom: u_1 and u_2

Free Body Diagrams:



Let $f_{s_1} = f_{s_1}^a + f_{s_1}^b$

$f_{D_1} = f_{D_1}^a + f_{D_1}^b$

TWO Equations of motion : $p_j - f_{s_j} - f_{D_j} = m_j \ddot{u}_j \quad (j = 1, 2)$

OR
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{Bmatrix} f_{s_1} \\ f_{s_2} \end{Bmatrix} + \begin{Bmatrix} f_{D_1} \\ f_{D_2} \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

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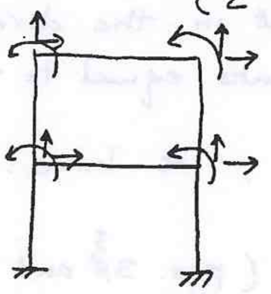
General Approach and Simplifications

For a 2-D frame, in general, we have:

3 degrees of freedom at each node
(2 displacements and 1 rotation)



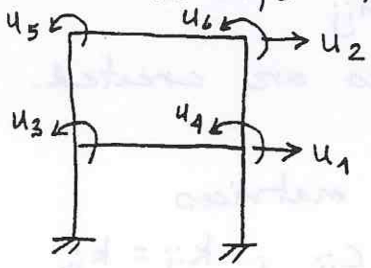
← 4 nodes

⇒ 12 degrees of freedom
$\begin{bmatrix} \text{8 translations} \\ \text{4 rotations} \end{bmatrix}$

For most buildings, we can neglect axial deformations of beams

For low-rise buildings, we can ignore axial deformations of columns as well.

⇒ 12 dof's reduced to 6 dof's



← 6 dof's
$\begin{bmatrix} \text{4 rotations} \\ \text{2 translations} \end{bmatrix}$

This would be a "typical" MDOF model for buildings.

External forces $p_j(t)$ for each j will be specified.

Usually, $p_j(t)$ for any rotation dof j is zero for buildings.

Getting the desired $\underline{m}$, $\underline{c}$, and $\underline{k}$ matrices requires some effort.

Various different methods may be employed.

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Once we have $\underline{m}$, $\underline{c}$, and $\underline{k}$, for any specified vector of external forces $\underline{p}(t)$, we have:

$$\underline{m}\,\ddot{\underline{u}} + \underline{c}\,\dot{\underline{u}} + \underline{k}\,\underline{u} = \underline{p}(t)$$

a system of N ordinary differential equations for the N-DOF system/structure.

These are, in general, coupled equations.

Static Condensation

Mass matrix has ZERO elements on diagonal for rotational DOF's.

Denote : $\underline{u}_0$: displacements for which $\underline{m}$ has ZERO elements on diagonal

$\underline{u}_t$: dynamic DOF's (i.e., $m_{jj} \neq 0$)

Partition all the matrices/vectors in our equations of motion. (ignore damping below)

$$\begin{bmatrix} \underline{m}_{tt} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \begin{Bmatrix} \ddot{\underline{u}}_t \\ \ddot{\underline{u}}_0 \end{Bmatrix} + \begin{bmatrix} \underline{k}_{tt} & \underline{k}_{t0} \\ \underline{k}_{0t} & \underline{k}_{00} \end{bmatrix} \begin{Bmatrix} \underline{u}_t \\ \underline{u}_0 \end{Bmatrix} = \begin{Bmatrix} \underline{p}_t(t) \\ \underline{0}(t) \end{Bmatrix}$$

generally, no external forces in rotational DOF's

Rewrite above as:

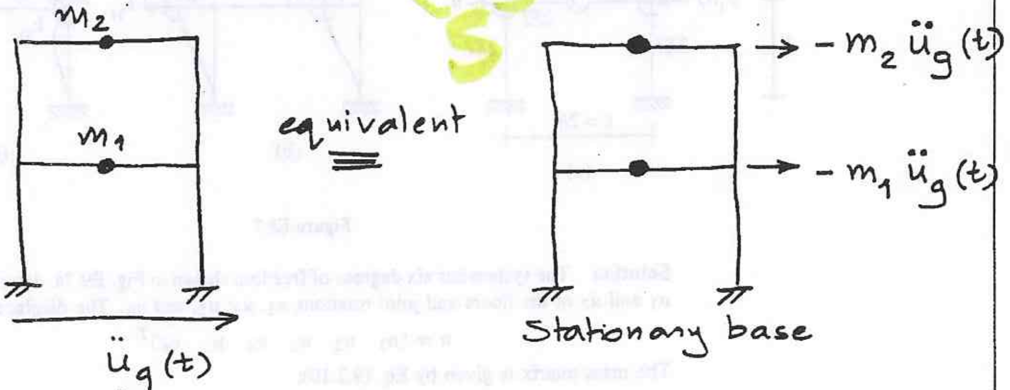
$$\underline{m}_{tt}\,\ddot{\underline{u}}_t + \underline{k}_{tt}\,\underline{u}_t + \underline{k}_{t0}\,\underline{u}_0 = \underline{p}_t(t) \quad -(1)$$

and

$$\underline{k}_{0t}\,\underline{u}_t + \underline{k}_{00}\,\underline{u}_0 = \underline{0} \quad -(2)$$

Thus, effective earthquake forces, $\underline{p}_{eff}(t)$ are such that :

$$\underline{p}_{eff}(t) = -\underline{m}\ \underline{1}\ \ddot{u}_g(t)$$



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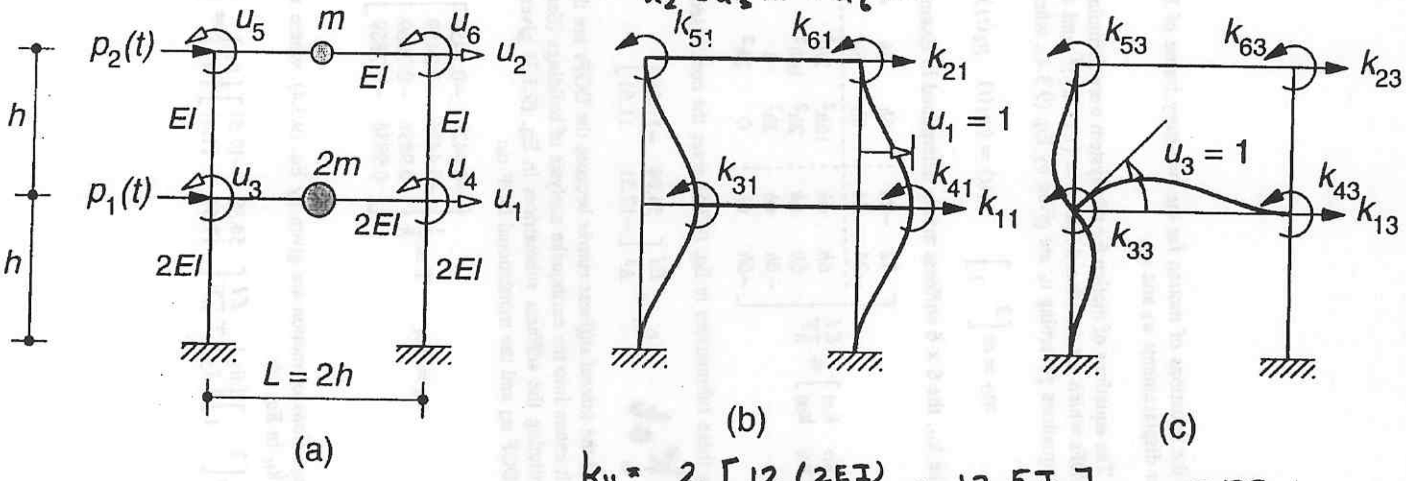
Example 9.7

$u_1 = 1$

$u_2 = u_3 \cdots = u_6 = 0$



$$k_{11} = \underset{\substack{\uparrow \\ \text{2 cols} \\ \text{per story}}}{2} \left[12 \underset{\substack{\uparrow \\ \text{1st} \\ \text{story}}}{\frac{(2EI)}{h^3}} + 12 \underset{\substack{\uparrow \\ \text{2nd} \\ \text{story}}}{\frac{EI}{h^3}} \right] = 72 \frac{EI}{h^2}$$

$$k_{21} = 2 \left[-12 \frac{EI}{h^3} \right] = -24 \frac{EI}{h^3}$$

Figure E9.7 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$$k_{31} = k_{41} = \frac{6\,(2EI)}{h^2} = \frac{12EI}{h^2}$$

$$k_{51} = k_{61} = -6 \frac{EI}{h^2}$$

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MULTI-DOF SYSTEMS

Equations of motion

$$\underline{m}\underline{\ddot{u}} + \underline{k}\underline{u} = \underline{p}(t)$$ (all matricies)

displacements

applied forces on each story

stiffness matrix

$$k = \frac{EI}{h^3}\left[\quad\right]$$

masses of the system, all zeros for non-diagonals

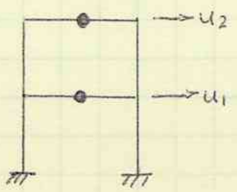
$$\hat{k}_{tt} = k_{tt} - k_{ot}^T k_{oo}^{-1} k_{ot}$$

$$T = -k_{oo}^{-1} k_{ot}$$

$$u_o = T u_t$$

introduction to MDOFs

standard: for an x-DOF system, there are x characteristic shapes



$$\phi_1 = \begin{bmatrix} \phi_{11} \\ \phi_{321} \end{bmatrix} \qquad \phi_2 = \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix}$$

u_2/u_1 is constant with time, equal to ϕ_{21}/ϕ_{11}

characteristic modeshapes and frequencies

$$\underline{u}(t) = \underline{q}_n(t)\,\underline{\phi}_n$$

simple harmonic with frequency ω_n

modeshape (constant)

$$q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t$$

$\underline{\Phi}$ modal matrix

$$\underline{k}\underline{\Phi} = \underline{m}\underline{\Phi}\underline{\Omega}^2$$

$\underline{\Omega}^2$ spectral matrix with ω_n^2 values on diagonal, 0 elsewhere

goal: uncouple system into N SDOF systems

$$\left.\begin{aligned} \underline{K} &= \underline{\Phi}^T \underline{k}\, \underline{\Phi} \\ M &= \underline{\Phi}^T \underline{m}\, \underline{\Phi} \end{aligned}\right\rangle \text{both diagonal matricies}$$

$$\underline{M}\ddot{q} + \underline{K}q = \underline{\Phi}^T \underline{p}(t)$$

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MDOF SYSTEMS

Undamped Systems

$\underline{m}\ddot{\underline{u}} + \underline{k}\underline{u} = \underline{0}$, $\dot{\underline{u}}(0), \underline{u}(0)$ given

coupled equation/variables

$$\underset{N\times 1}{\underline{u}(t)} = \underset{N\times N}{\underline{\Phi}}\,\underset{N\times 1}{\underline{q}(t)}$$

$\uparrow$ generalized coordinate

$\underline{\Phi}$ = matrix of eigenvectors = $\begin{bmatrix} \underline{\phi}_1 & \underline{\phi}_2 & \dots \end{bmatrix}$

$\underline{k}\underline{\phi}_n = \underline{\omega_n}^2 \underline{m}\underline{\phi}_n$

(solutions of eigenvalue problem)

using orthogonality:

$\phi_n^T m \phi_R = 0$

$\phi_n^T k \phi_n = 0$

$\underline{M} = \underline{\Phi}^T \underline{m}\,\underline{\Phi}$

uncoupled equations

$\underline{M}\ddot{\underline{q}}(t) + \underline{K}\,\underline{q}(t) = 0$

$q(t) = \begin{bmatrix} q_1 \\ q_2 \\ \dots \\ q_n \end{bmatrix}$

uncoupled set of N equations, as $\underline{M}$ and $\underline{K}$ are diagonal matricies

$M_n\ddot{q}_n + K_n q_n = 0$

$\phi_n = \omega_n^2$

Then, including damping

$M_n\ddot{q}_n + C_n q_n + k_n q_n = 0$

$\uparrow$

$\underline{C}$ is diagonal matrix,

$c_n = 2\zeta_n\omega_n M_n$

Normalization

$\phi_n = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$, $\phi_n = \begin{bmatrix} \alpha q_1 \\ \alpha q_2 \\ \alpha q_3 \end{bmatrix}$ — equivalent shapes, just scaled

so, scale mode shapes so that largest value = 1.0

or roof displacement is equal to 1.0

or, normalize M:

$\underline{M_n} = \underline{\phi}_n^T \underline{m}\,\underline{\phi}_n = \underline{1}$ $M_n = 1$, $K_n = \omega_n^2$, $\underline{K} = \underline{\Phi}^T \underline{k}\,\underline{\Phi} = \underline{\Omega}^2$

to solve for ϕ vectors, set one value equal to one, then calculate others

Free Vibrations – MDOF System

$$\underset{N\times N}{\underline{m}}\;\underset{N\times 1}{\ddot{\underline{u}}} + \underset{N\times N}{\underline{k}}\;\underset{N\times 1}{\underline{u}} = \underset{N\times 1}{\underline{0}}$$ [no damping] for a N-dof system

$\underline{u}(0)$, $\dot{\underline{u}}(0)$ given.

If $\underline{u}(0)$ is arbitrarily prescribed, we have the following difficulties in our solution :

(i) the motion of each degree of freedom is not simple harmonic,

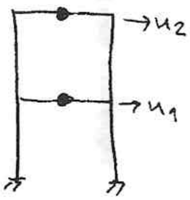
(ii) The frequency of the motion cant be defined,

(iii) The deflected shape is not preserved in time.

If instead $\underline{u}(0)$ is selected carefully in a characteristic mode / deflected shape, all 3 of the problems above are solved., i.e., the motion is simple harmonic with a characteristic frequency, and the deflected shape is unchanged with time.

e.g., For a 2-DOF system there are TWO characteristic shapes for which the above is true.

1st shape: No node (i.e., point of zero displacement)



2-DOF system

Frequency = ω_{n_1}

Deflected shape is such that $u_2 : u_1$ is constant with time (and equal to ϕ_{21}/ϕ_{11}).

2nd shape : One node

Frequency = ω_{n_2} ($\omega_{n_2} > \omega_{n_1}$)

Deflected shape : $\phi_{12} : \phi_{11} \equiv u_2 : u_1$ constant.

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Free vibration of undamped system: Arbitrary **u**(0)

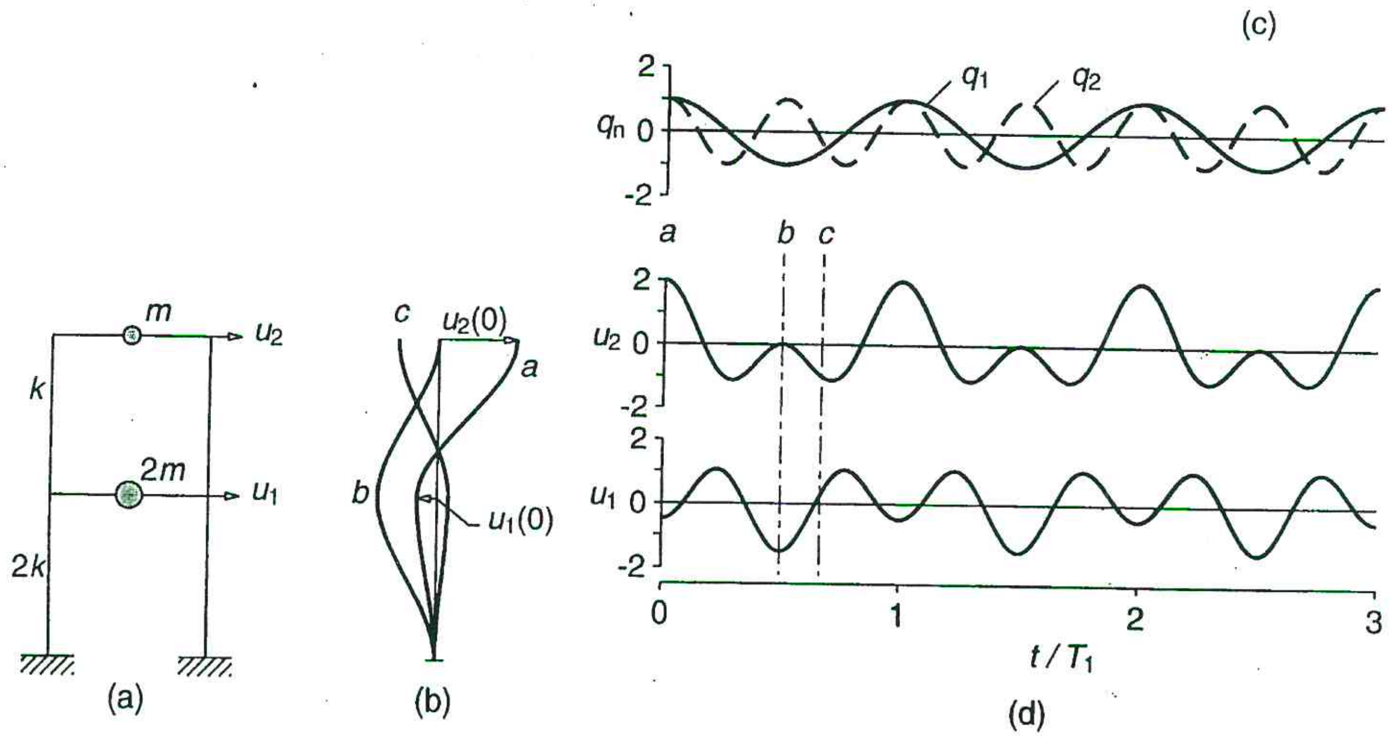

Figure 10.1.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Motion of $u_1(t)$ and $u_2(t)$

- not simple harmonic
- $u_2 : u_1$ not constant; shape changes
- more than one period is evident in histories

Free vibration of undamped system: $\mathbf{u}(0) = \phi_1$

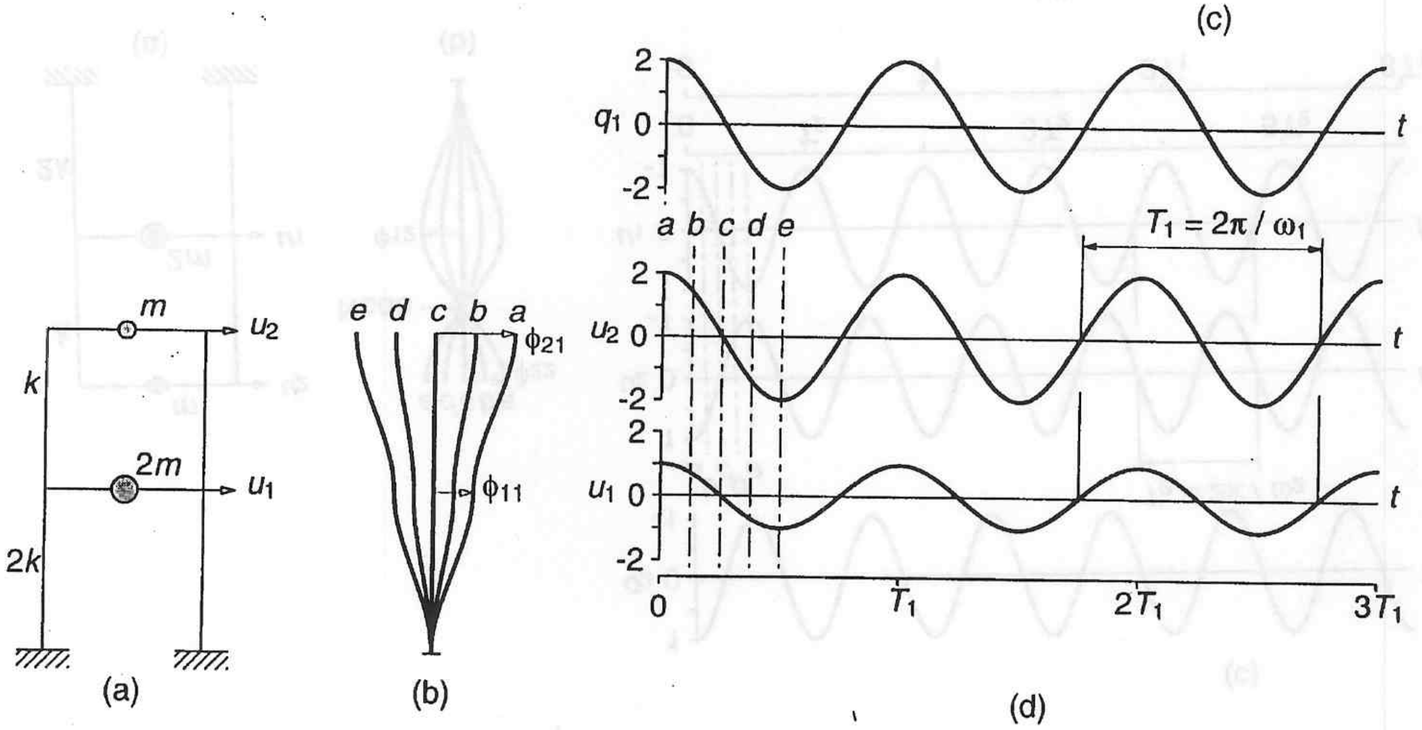

Figure10.1.2 Irom *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

- Single frequency (characteristic) in $u_1(t)$ & $u_2(t)$
- $u_2 : u_1$ constant ⇒ shape preserved

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Free vibration of undamped system: $\mathbf{u}(0) = \phi_2$

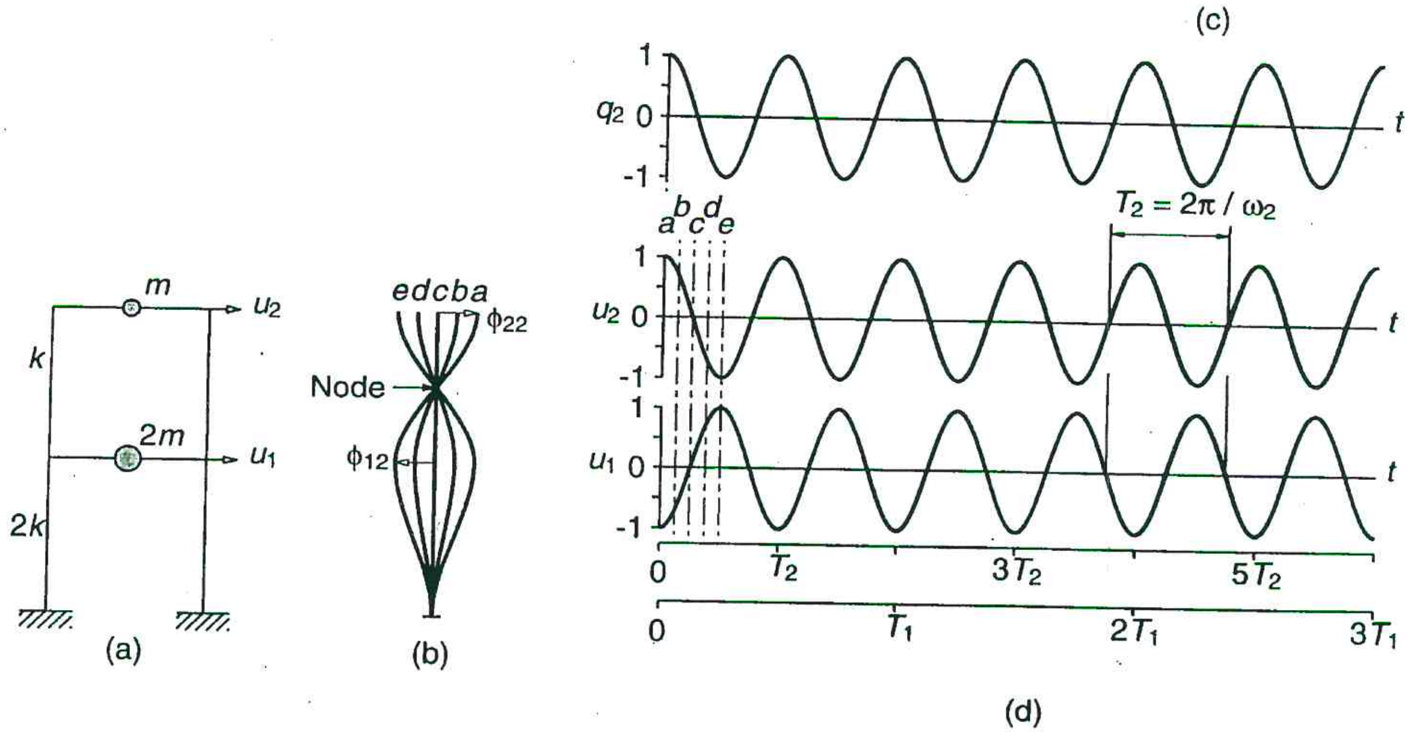

Figure 10.1.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

- Single frequency (characteristic) in $u_1(t)$ & $u_2(t)$
  $\omega_2 > \omega_1$
- $u_2 : u_1$ constant ⇒ shape preserved

Question: How do we obtain the characteristic frequencies and shapes ?

Frequencies and Mode shapes

In a characteristic "mode", motion is "simple harmonic" and shape is preserved.

$$\Rightarrow \quad \underline{u}(t) = \underset{\substack{\uparrow \\ \text{simple} \\ \text{harmonic} \\ \text{with} \\ \text{frequency, } \omega_n}}{q_n(t)} \; \underset{\substack{\uparrow \\ \text{mode} \\ \text{shape} \\ \text{(constant)}}}{\underline{\phi}_n}$$

[For motion in a characteristic mode]

$$q_n(t) = A_n \cos\omega_n t + B_n \sin\omega_n t$$

[since $q_n(t)$ is simple harmonic with frequency, ω_n]

$$\Rightarrow \quad u(t) = \underline{\phi}_n \left[A_n \cos\omega_n t + B_n \sin\omega_n t \right] \quad -(1)$$

Since $\underline{u}(t) = \underline{\phi}_n \, q_n(t)$

$$\underline{\ddot{u}}(t) = -\omega_n^2 \underline{\phi}_n \, q_n(t) \quad \text{from (1)}$$

Equation of motion: $\underline{m}\,\underline{\ddot{u}} + \underline{k}\,\underline{u} = \underline{0}$

$$\Rightarrow \quad \left[-\omega_n^2 \underline{m}\,\underline{\phi}_n + \underline{k}\,\underline{\phi}_n \right] q_n(t) = \underline{0}$$

$q_n(t) = 0$ trivial solution ; ignore.

$$\Rightarrow \quad \underline{k}\,\underline{\phi}_n = \omega_n^2 \underline{m}\,\underline{\phi}_n \quad \text{——} (2)$$

$$\text{or} \quad \left[\underline{k} - \omega_n^2 \underline{m} \right] \underline{\phi}_n = \underline{0}$$

A set of equations (N in number) for elements ϕ_{jn} ($j = 1, 2, \ldots N$). Solution exists if $\underline{\phi}_n = 0$ (trivial) OR if $\det\left[\underline{k} - \omega_n^2 \underline{m} \right] = 0$

Note: Both ω_n^2 and ϕ_n are unknown.

We have N equations in (N+1) unknowns

The unknowns are ω_n^2 and ϕ_{jn} (j = 1 to N).

The determinant of $[\underline{k} - \omega_n^2 \underline{m}]$ set equal to zero allows us to solve for ω_n^2.

A polynomial of order N is to be solved.

Then, we use the ω_n^2 determined and find (N-1) of the terms in ϕ_{jn} (j = 1 to N).

The last term can be adjusted as discussed later.

$\det[\underline{k} - \omega_n^2 \underline{m}] = 0$ is called the characteristic equation.

ω_n^2 will be real and positive because $\underline{k}$ and $\underline{m}$ are symmetric and positive definite

$\underline{\phi}_n$: characteristic vector
or eigenvector
or normal mode

ω_n : characteristic frequency
or eigenfrequency

For an N-DOF system, there will be
N characteristic frequencies and
N characteristic modes/eigenvectors.

The N distinct modes of an N-DOF system ($\underline{\phi}_1, \underline{\phi}_2, \ldots \underline{\phi}_N$) may be combined into a single matrix, $\underline{\Phi}$

$$\underline{\Phi} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix}$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$\underline{\phi}_1 \qquad \underline{\phi}_2 \qquad \underline{\phi}_N$$

[called the "Modal Matrix" and is of size N×N]

ω_n^2 from each of the N modes may be combined into a single (diagonal) matrix $\underline{\Omega}^2$

$$\underline{\Omega}^2 = \begin{bmatrix} \omega_1^2 & & & \bigcirc \\ & \omega_2^2 & & \\ & & \omega_3^2 & \\ \bigcirc & & \ddots & \omega_N^2 \end{bmatrix}$$

"Spectral Matrix" size: N × N

From (2), $\underline{k}\ \underline{\phi}_n = \omega_n^2\ \underline{m}\ \underline{\phi}_n$ for the n^{th} mode

Combining all modes and using $\underline{\Phi}$ and $\underline{\Omega}^2$,

$$\underset{N\times N}{\underline{k}}\ \underset{N\times N}{\underline{\Phi}} = \cancel{\Omega^2}\ \underset{N\times N}{\underline{m}}\ \underset{N\times N}{\underline{\Phi}}\ \underset{N\times N}{\underline{\Omega}^2}$$

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Orthogonality of Modes

This is a very important property that will be useful to take advantage of in MDOF system analysis.

$\Rightarrow \quad \underline{\phi}_n^T \underline{k} \underline{\phi}_r = 0 \quad ; \quad \underline{\phi}_n^T \underline{m} \underline{\phi}_r = 0 \quad$ if $\omega_n \neq \omega_r$

Proof: $\quad \underline{k} \underline{\phi}_n = \omega_n^2 \underline{m} \underline{\phi}_n \quad$ — (2)

$\Rightarrow \quad \underline{\phi}_r^T \underline{k} \underline{\phi}_n = \omega_n^2 \underline{\phi}_r^T \underline{m} \underline{\phi}_n \quad$ — (3)

[Using the property: $[A \cdot B \cdot C]^T = C^T B^T A^T$]

$\hookrightarrow \quad \underline{\phi}_n^T \underline{k} \underline{\phi}_r = \omega_n^2 \underline{\phi}_n^T \underline{m} \underline{\phi}_r \quad$ —(4)

also $\underline{k} = \underline{k}^T$; $\underline{m} = \underline{m}^T$ since $\underline{m}$ and $\underline{k}$ are symmetric

and $[A^T]^T = A$ $\Rightarrow$ both these properties were used in going from (3) to (4)

For the r^{th} mode, $\quad \underline{k} \underline{\phi}_r = \omega_r^2 \underline{m} \underline{\phi}_r \quad$ same as (2) but for r^{th} mode.

Multiply by $\underline{\phi}_n^T$; $\quad \underline{\phi}_n^T \underline{k} \underline{\phi}_r = \omega_r^2 \underline{\phi}_n^T \underline{m} \underline{\phi}_r \quad$ —(5)

Subtract (5) from (4) $\Rightarrow \quad (\omega_n^2 - \omega_r^2) \underline{\phi}_n^T \underline{m} \underline{\phi}_r = 0$

since $\omega_n^2 \neq \omega_r^2$; $\quad \underline{\phi}_n^T \underline{m} \underline{\phi}_r = 0$

From (4) ; $\quad \underline{\phi}_n^T \underline{k} \underline{\phi}_r = 0 \quad$ <u>proved</u>.

<u>Implication</u>: $\quad (\underline{f}_S)_n^T \underline{u}_r = 0 \quad ; \quad (\underline{f}_I)_n^T \underline{u}_r = 0 \quad [n \neq r]$

Work done by n^{th} mode's inertia / equivalent static forces in going through the r^{th} mode's displacement is <u>ZERO</u>.

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Orthogonality implies:

$$\underline{K} \equiv \underline{\Phi}^T \underline{k}\ \underline{\Phi} \quad \text{is diagonal}$$

$$\underline{M} = \underline{\Phi}^T \underline{m}\ \underline{\Phi} \quad \text{is diagonal}$$

since $K_n = \underline{\phi}_n^T \underline{k}\ \underline{\phi}_n$

$$= \underline{\phi}_n^T \omega_n^2 \underline{m}\ \underline{\phi}_n$$

$$= (\underline{\phi}_n^T \underline{m}\ \underline{\phi}_n)\omega_n^2$$

$$\Rightarrow \quad K_n = \omega_n^2 M_n$$

since $M_n = \underline{\phi}_n^T \underline{m}\ \underline{\phi}_n$

Normalization of Modes

$$\underline{\phi}_n = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{Nn} \end{Bmatrix} \text{ and } \begin{Bmatrix} \alpha\phi_{1n} \\ \alpha\phi_{2n} \\ \vdots \\ \alpha\phi_{Nn} \end{Bmatrix} \leftarrow \text{both are same shapes and, therefore, acceptable.}$$

Normalization is associated with this kind of scaling

e.g., scaling so that Largest element = 1.0 in $\underline{\phi}_n$

OR so that top floor of a building = 1.0 .

Mass-normalizing :

Normalize mode shapes so that $M_n = \underline{\phi}_n^T \underline{m} \underline{\phi}_n = \mathbf{1}$

$\underset{1 \times N}{} \; \underset{N \times N}{} \; \underset{N \times 1}{}$

M_n (scalar)

or $\underset{N\times N}{\underline{\Phi}^T} \; \underset{N\times N}{\underline{m}} \; \underset{N\times N}{\underline{\Phi}} = \underset{N\times N}{\underline{I}}$ (Identity Matrix)

This equation states that modes are
- orthogonal [since $\underline{\phi}_n^T \underline{m} \phi_r = 0$ if $n \neq r$)
- normalized with respect to $\underline{m}$

Orthogonality implies

$$\underline{K} = \underset{N\times N}{\underline{\Phi}^T} \; \underset{N\times N}{\underline{k}} \; \underset{N\times N}{\underline{\Phi}} \text{ is diagonal}$$

$$\underline{M} = \underline{\Phi}^T \underline{m} \underline{\Phi} \text{ is diagonal}$$

as we saw before

and we saw that $K_n = \omega_n^2 M_n$

If mass-normalized, $M_n = 1$; $K_n = \omega_n^2$, $\underline{K} = \underline{\Phi}^T \underline{k} \underline{\Phi} = \underline{\Omega}^2$

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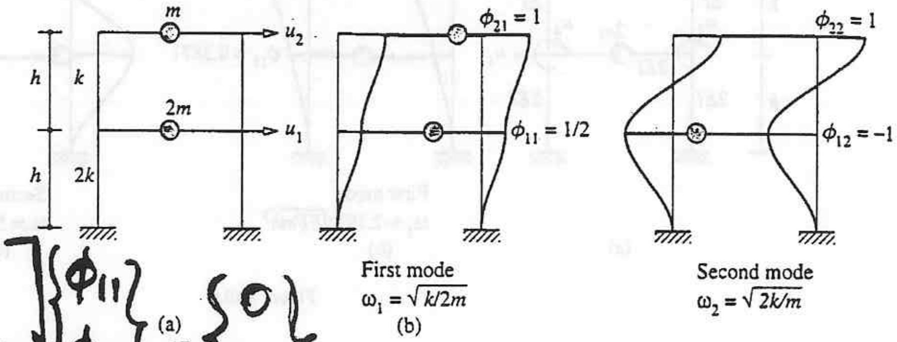
Example 10.4

Determine the natural frequencies and modes of the system shown in Fig. E10.4a and defined in Example E9.1, a two-story frame idealized as a shear building. Normalize the modes so that $M_n = 1$.

Solution The mass and stiffness matrices of the system, determined in Example 9.1, are

$$\mathbf{m} = \begin{bmatrix} 2m & \\ & m \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} \tag{a}$$



$$\begin{bmatrix} 3k - \omega_1^2(2m) & -k \\ -k & k - \omega_1^2(m) \end{bmatrix} \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Figure E10.4

where $k = 24EI_c/h^3$. The frequency equation is Eq. (10.2.6), which, after substituting for **m** and **k** and evaluating the determinant, can be written as

$$(2m^2)\omega^4 + (-5km)\omega^2 + 2k^2 = 0 \tag{b}$$

The two roots are $\omega_1^2 = k/2m$ and $\omega_2^2 = 2k/m$, and the two natural frequencies are

$$\omega_1 = \sqrt{\frac{k}{2m}} \qquad \omega_2 = \sqrt{\frac{2k}{m}} \tag{c}$$

Substituting for k gives

$$\omega_1 = 3.464\sqrt{\frac{EI_c}{mh^3}} \qquad \omega_2 = 6.928\sqrt{\frac{EI_c}{mh^3}} \tag{d}$$

Can set $\phi_{21} = 1$ arbitrarily and find $\phi_{11} = 1/2$; similarly set $\phi_{22} = 1$, find $\phi_{12} = -1$

The natural modes are determined from Eq. (10.2.5) following the procedure used in Example 10.1 to obtain

$$\phi_1 = \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \tag{e}$$

These natural modes are shown in Fig. E10.4b, and c.

To normalize the first mode, M_1 is calculated using Eq. (10.4.6), with ϕ_1 given by Eq. (e):

$$M_1 = \phi_1^T \mathbf{m} \phi_1 = m \left(\tfrac{1}{2} \;\; 1 \right) \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} = \frac{3}{2} m$$

To make $M_1 = 1$, divide ϕ_1 of Eq. (e) by $\sqrt{3m/2}$ to obtain the normalized mode

$$\phi_1 = \frac{1}{\sqrt{6m}} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\phi_2 = \frac{1}{\sqrt{3m}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For this ϕ_1 it can be verified that $M_1 = 1$. The second mode can be normalized similarly.

ϕ_1 obtained by dividing by $\sqrt{M_1}$

$$m\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & \\ & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 3m$$

$M_2 = 3m$; Divide ϕ_2 by $\sqrt{3m}$

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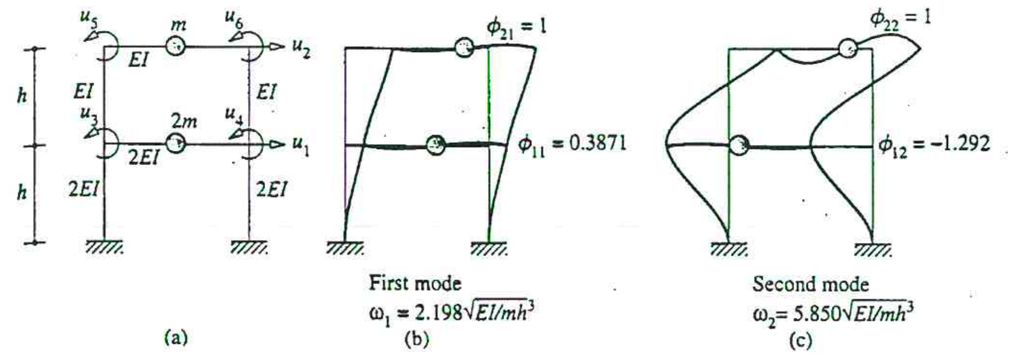
Example 10.5

Determine the natural frequencies and modes of the system shown in Fig. E10.5a and defined earlier in Example 9.9. The story height $h = 10$ ft.



(a)

First mode
$\omega_1 = 2.198\sqrt{EI/mh^3}$
(b)

Second mode
$\omega_2 = 5.850\sqrt{EI/mh^3}$
(c)

Figure E10.5

Solution With reference to the lateral displacements u_1 and u_2 of the two floors as the two DOFs, the mass matrix and the condensed stiffness matrix were determined in Example 9.9:

$$\mathbf{m}_{tt} = m\begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \qquad \hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 54.88 & -17.51 \\ -17.51 & 11.61 \end{bmatrix} \tag{a}$$

The frequency equation is

$$\det(\hat{\mathbf{k}}_{tt} - \omega^2 \mathbf{m}_{tt}) = 0 \tag{b}$$

Substituting for $\mathbf{m}_{tt}$ and $\hat{\mathbf{k}}_{tt}$, evaluating the determinant, and obtaining the two roots just as in Example 10.4 leads to

$$\omega_1 = 2.198\sqrt{\frac{EI}{mh^3}} \qquad \omega_2 = 5.850\sqrt{\frac{EI}{mh^3}} \tag{c}$$

It is of interest to compare these frequencies for a frame with flexible beams with those for the frame with flexurally rigid beams, determined in Example 10.4. It is clear that beam flexibility has the effect of lowering the frequencies, consistent with intuition.

The natural modes are determined by solving

$$(\hat{\mathbf{k}}_{tt} - \omega_n^2 \mathbf{m}_{tt})\phi_n = \mathbf{0} \tag{d}$$

with ω_1 and ω_2 substituted successively from Eq. (c) to obtain

$$\phi_1 = \begin{Bmatrix} 0.3871 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1.292 \\ 1 \end{Bmatrix} \tag{e}$$

These vectors define the lateral displacements of each floor. ~~They~~ They are shown in Fig. E10.5b and c together with the joint rotations. The joint rotations associated with the first mode are determined by substituting $\mathbf{u}_t = \phi_1$ from Eq. (e) in Eq. (d) of Example 9.9:

$$\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{h}\begin{bmatrix} -0.4426 & -0.2459 \\ -0.4426 & -0.2459 \\ 0.9836 & -0.7869 \\ 0.9836 & -0.7869 \end{bmatrix}\begin{Bmatrix} 0.3871 \\ 1.0000 \end{Bmatrix} = \frac{1}{h}\begin{Bmatrix} -0.4172 \\ -0.4172 \\ -0.4061 \\ -0.4061 \end{Bmatrix} \tag{f}$$

Similarly, the joint rotations associated with the second mode are obtained by substituting $\mathbf{u}_t = \phi_2$ from Eq. (e) in Eq. (d) of Example 9.9:

$$\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{h}\begin{Bmatrix} 0.3258 \\ 0.3258 \\ -2.0573 \\ -2.0573 \end{Bmatrix} \tag{g}$$

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Modal Expansion of Displacements

The natural modes ($\underline{\phi}_n \}$ for $n = 1$ to N) represent a basis for representing $\underline{u}$

$$\underline{u} = \sum_{r=1}^{N} \underset{\substack{\uparrow \\ \text{mode} \\ \text{shape}}}{\underline{\phi}_r}\; \underset{\nwarrow \text{normal coordinates or modal coordinates}}{q_r} \qquad -(6)$$

or
$$\underset{N\times1}{\underline{u}} = \underset{N\times N}{\underline{\Phi}}\; \underset{N\times1}{\underline{q}}$$

Multiply Eq. 6 by $\underline{\phi}_n^T \underline{m}$

$$\Rightarrow \underline{\phi}_n^T \underline{m}\, \underline{u} = \sum_{r=1}^{N} \underline{\phi}_n^T \underline{m}\, \underline{\phi}_r\, q_r$$

Right hand side sum is zero except when $r = n$ due to orthogonality

$$\Rightarrow \underset{1\times N}{\underline{\phi}_n^T}\; \underset{N\times N}{\underline{m}}\; \underset{N\times1}{\underline{u}} = \left(\underset{1\times N}{\underline{\phi}_n^T}\; \underset{N\times N}{\underline{m}}\; \underset{N\times1}{\underline{\phi}_n} \right) \underset{\nwarrow 1\times1}{q_n}$$

Scalars on both sides of equation:

$$\Rightarrow q_n = \frac{\underline{\phi}_n^T \underline{m}\, \underline{u}}{\underline{\phi}_n^T \underline{m}\, \underline{\phi}_n} = \frac{\underline{\phi}_n^T \underline{m}\, \underline{u}}{M_n} \qquad -(7)$$

Equation (7) can be used to obtain modal expansions of any specified physical displacement vector $\underline{u}(t)$. Thus $q_1, q_2 \ldots q_N$ can be obtained. [See Ex. 10.8]

In chapter 12, we'll actually solve for q_j ($j = 1$ to N) and then obtain $\underline{u}(t)$.

Example 10.8

For the two-story shear frame of Example 10.4, determine the modal expansion of the displacement vector $\mathbf{u} = \langle 1 \quad 1 \rangle^T$.

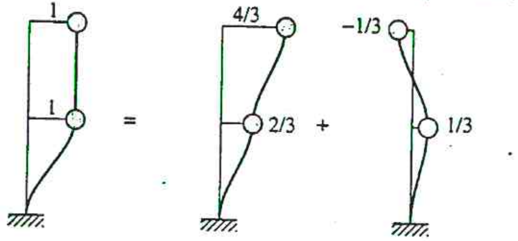
Solution The displacement $\mathbf{u}$ is substituted in Eq. (10.7.2) together with $\phi_1 = \langle \frac{1}{2} \quad 1 \rangle^T$ and $\phi_2 = \langle -1 \quad 1 \rangle^T$, from Example 10.4, to obtain

$$q_1 = \frac{\langle \frac{1}{2} \quad 1 \rangle \begin{bmatrix} 2m & \\ & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}{\langle \frac{1}{2} \quad 1 \rangle \begin{bmatrix} 2m & \\ & m \end{bmatrix} \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix}} = \frac{2m}{3m/2} = \frac{4}{3}$$

$$q_2 = \frac{\langle -1 \quad 1 \rangle \begin{bmatrix} 2m & \\ & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}{\langle -1 \quad 1 \rangle \begin{bmatrix} 2m & \\ & m \end{bmatrix} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}} = \frac{-m}{3m} = -\frac{1}{3}$$

Substituting q_n in Eq. (10.7.1) gives the desired modal expansion, which is shown in Fig. E10.8.

1st mode 2nd mode



$\mathbf{u} \quad = \quad (4/3)\phi_1 \quad + \quad (-1/3)\phi_2$ **Figure E10.8**

$$q_n = \frac{\underline{\phi}_n^T \, \underline{m} \, \underline{u}}{\underline{\phi}_n^T \, \underline{m} \, \underline{\phi}_n}$$

Free Vibrations of MDOF Systems

Undamped systems only:

Given $\underline{m}$, $\underline{k}$ → obtain $\underline{\Omega}^2$, $\underline{\Phi}$. [Eigenvalue Analysis]

Given $\underline{u}(0)$, $\underline{\dot{u}}(0)$ → obtain $q_1(0), q_2(0) \ldots q_N(0)$

$\dot{q}_1(0), \dot{q}_2(0) \ldots \dot{q}_N(0)$

$$\text{eg. } q_i(0) = \frac{\underline{\phi}_i^T \, \underline{m} \, \underline{u}(0)}{\underline{\phi}_i^T \, \underline{m} \, \underline{\phi}_i}$$

We saw that $\underline{u}(t)$ can be expressed in terms of $\underline{\phi}_n$ and ω_n as follows:

$$\underline{u}(t) = \sum_{n=1}^{N} \underline{\phi}_n \left[A_n \cos \omega_n t + B_n \sin \omega_n t \right]$$

Question: What are A_n and B_n?

$$\text{Setting } t=0 \,; \quad \underline{\dot{u}}(0) = \sum_{n=1}^{N} \underline{\phi}_n \, \omega_n \, B_n \quad — 1$$

$$\underline{u}(0) = \sum_{n=1}^{N} \underline{\phi}_n \, A_n \quad - 2$$

$$\text{Also,} \quad \underline{u}(t) = \sum_{n=1}^{N} \underline{\phi}_n \, q_n(t)$$

$$\Rightarrow \underline{u}(0) = \sum_{n=1}^{N} \underline{\phi}_n \, q_n(0) \quad — 3$$

$$\underline{\dot{u}}(0) = \sum_{n=1}^{N} \underline{\phi}_n \, \dot{q}_n(0) \quad - 4$$

From Eqs 2 & 3, $A_n = q_n(0)$

From Eqs 1 & 4, $B_n = \dfrac{\dot{q}_n(0)}{\omega_n}$

Thus, $$\underline{u}(t) = \sum_{n=1}^{N} \underline{\phi}_n \left[q_n(0) \cos\omega_n t + \frac{\dot{q}_n(0)}{\omega_n} \sin\omega_n t \right]$$

↑ Free vibrations solution (similar to SDOF systems)

Note: $$q_n(0) = \frac{\underline{\phi}_n^T \, \underline{m} \, \underline{u}(0)}{\underline{\phi}_n^T \, \underline{m} \, \underline{\phi}_n}$$

$$\dot{q}_n(0) = \frac{\underline{\phi}_n^T \, \underline{m} \, \dot{\underline{u}}(0)}{\underline{\phi}_n^T \, \underline{m} \, \underline{\phi}_n}$$

(See Section 10.8)

National Brand
13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.

Damping Matrix in MDOF systems

Analagous to our definitions of $\underline{K}$ and $\underline{M}$, we'd like to write:

$$\underline{C} = \underline{\Phi}^T \underline{c}\, \underline{\Phi} \qquad \textbf{(Refer to § 11.4.2)}$$

where $\underline{C}$ is also diagonal as were $\underline{K}$ and $\underline{M}$

If further, $C_n = 2\xi_n M_n \omega_n$ $\left[\begin{array}{l}\text{specified } \xi_n\\ \text{in mode } n\end{array}\right]$

we can find $\underline{c}$: $\quad \underline{c} = (\underline{\Phi}^T)^{-1}\underline{C}\,\underline{\Phi}^{-1}$

$$\Big[\text{ since } \underline{\Phi}^T \underline{m}\, \underline{\Phi} = \underline{M},$$
$$\underline{\Phi}^{-1} = \underline{M}^{-1}\underline{\Phi}^T \underline{m} \;;\; (\underline{\Phi}^T)^{-1} = \underline{m}\,\underline{\Phi}\,\underline{M}^{-1} \Big]$$

$$\Rightarrow \quad \underline{c} = (\underline{m}\,\underline{\Phi}\,\underline{M}^{-1})\,\underline{C}\,(\underline{M}^{-1}\underline{\Phi}^T \underline{m})$$

$$\text{or} \quad \boxed{\underset{N\times N}{\underline{c}} = \underset{N\times N}{\underline{m}}\left[\sum_{n=1}^{N} \overset{\text{scalar}}{\left(\frac{2\xi_n \omega_n}{M_n}\right)} \underset{N\times 1}{\phi_n}\,\underset{1\times N}{\phi_n^T}\right]\underset{N\times N}{\underline{m}}}$$

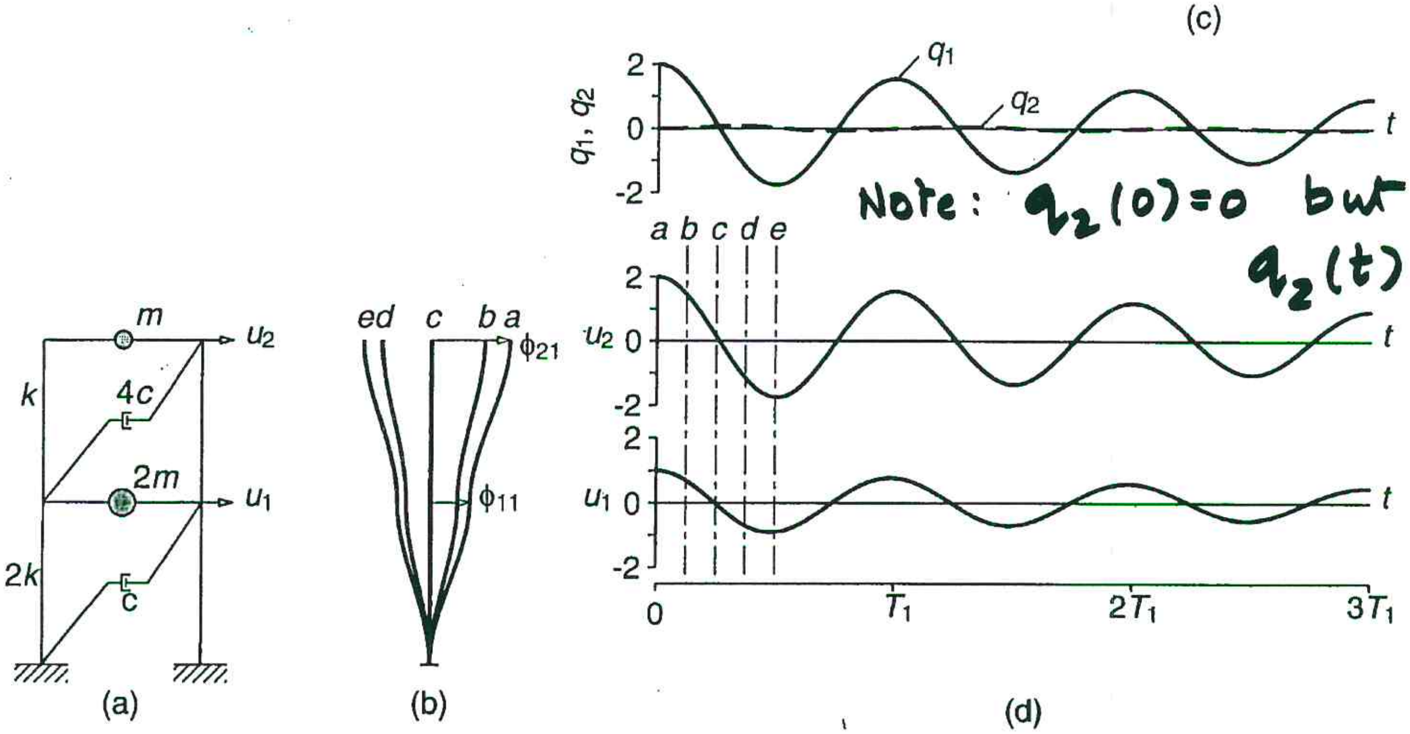
Given ξ_n, ω_n, ϕ_n, this is how damping matrix $\underline{c}$ can be created.

Key Guarantees desired (specified) damping in each mode.

→ Note: $\underline{C}$ is diagonal. This is, then, classical damping.

$\underline{c} = a_0 \underline{m} + a_1 \underline{k}$ (Rayleigh damping) ~ **See Sect. 11.4.1**
also guarantees that $\underline{C}$ is diagonal
but only 2 modes can be assumed to have specified ξ_n's.

Free vibration of nonclassically damped system: $\mathbf{u}(0) = \phi_1 = \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}$

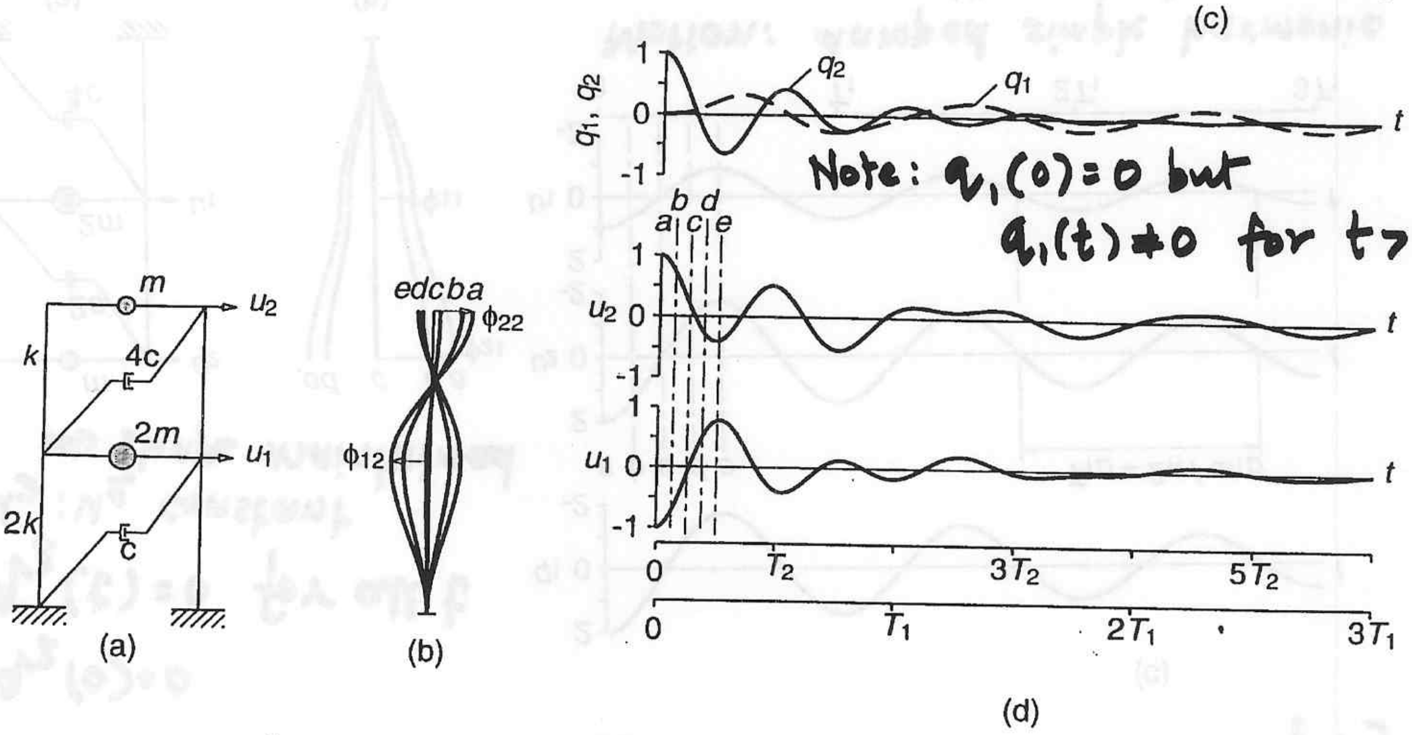


Note: $q_2(0) = 0$ but $q_2(t) \neq 0$ for $t > 0$

Figure 10.9.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Shape not maintained ⇒ $u_2 : u_1$ not const.
Motion is not damped simple harmonic with single frequency

Free vibration of nonclassically damped system: $\mathbf{u}(0) = \phi_2 = \left\{ \begin{matrix} -1 \\ 1 \end{matrix} \right\}$



Note: $q_1(0) = 0$ but $q_1(t) \neq 0$ for $t > 0$

Figure 10.9.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Shape not maintained ⇒ $u_2 : u_1$ not const.
Motion is not damped simple harmonic
with single frequency

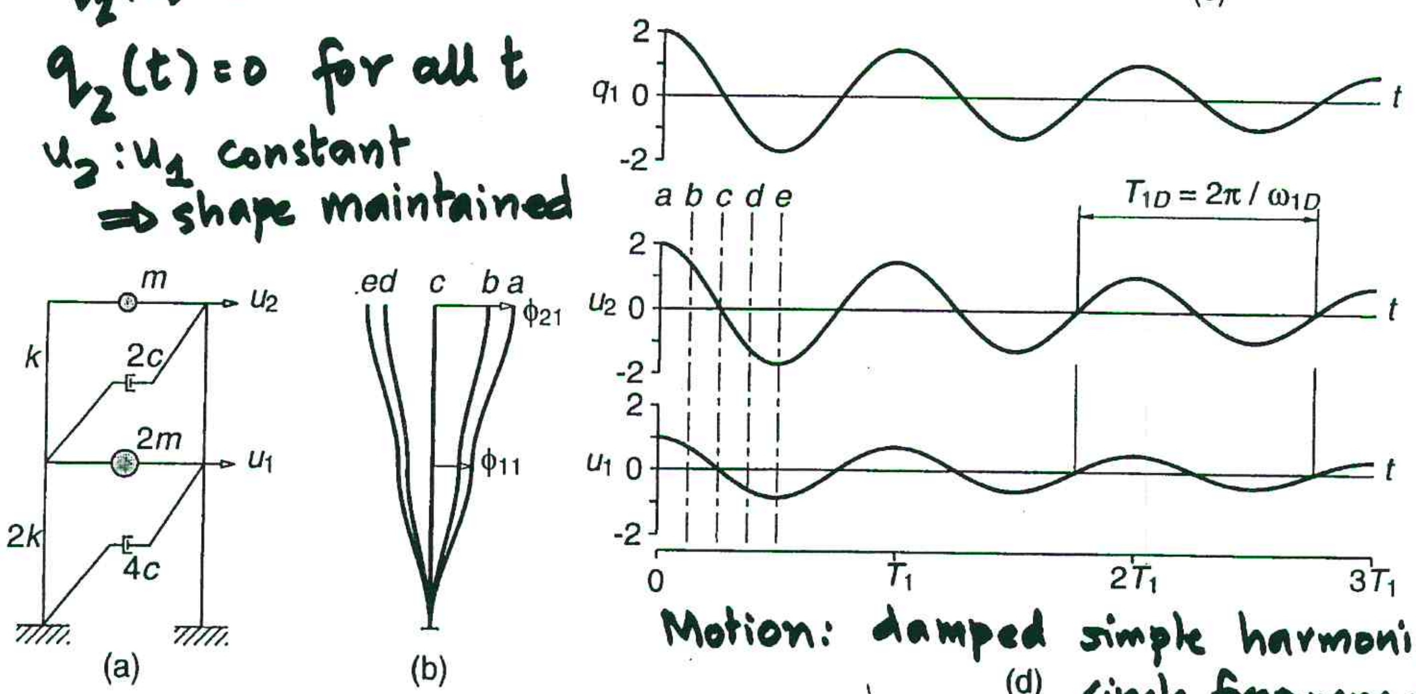
Free vibration of classically damped system: $\mathbf{u}(0) = \phi_1 : \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}$

$q_2(0)=0$

$q_2(t)=0$ for all t

$u_2 : u_1$ constant

$\Rightarrow$ shape maintained



Motion: damped simple harmonic

single frequency = ω_{1D}

Figure 10.9.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

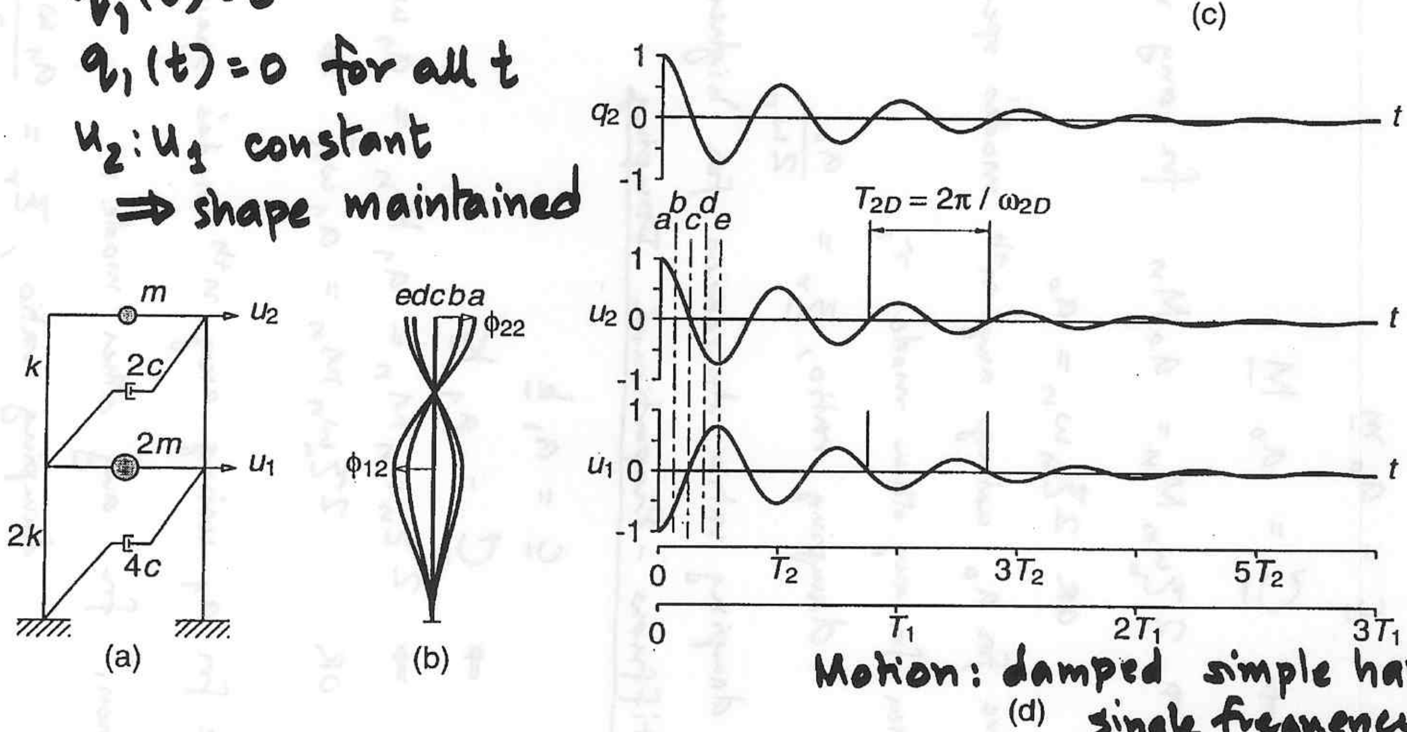
Free vibration of classically damped system: $\mathbf{u}(0) = \phi_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

$q_1(0) = 0$

$q_1(t) = 0$ for all t

$u_2 : u_1$ constant

$\Rightarrow$ shape maintained



Motion: damped simple harmonic

single frequency $= \omega_{2D}$

Figure 10.9.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$\omega_{2D} > \omega_{1D}$

STAEDTLER

Mass - Proportional Damping

$$\underline{c} = a_0 \underline{m}$$

$$\Rightarrow \underline{C} = a_0 \underline{M}$$

$$\Rightarrow 2\xi_n\omega_n M_n = a_0 M_n \quad \text{for any } n$$

$$\text{OR} \quad 2\xi_n\omega_n = a_0$$

Solve for a_0 using any n^{th} mode's specified damping.

Then for any other mode r,

$$\text{damping ratio, } \xi_r = \frac{a_0}{2\omega_r}$$

OR damping ratio decreases for higher modes

Stiffness - Proportional Damping

$$\underline{c} = a_1 \underline{k}$$

$$\Rightarrow \underline{C} = a_1 \underline{K}$$

$$\Rightarrow 2\xi_n\omega_n M_n = a_1 K_n = a_1\omega_n^2 M_n \quad \text{for any } n$$

$$\text{OR} \quad 2\xi_n\omega_n M_n = a_1\omega_n^2 \Rightarrow \frac{2\xi_n}{\omega_n} = a_1$$

Solve for a_1 using any n^{th} mode's specifed damping.

Then, for any other mode r,

$$\text{damping ratio}, \xi_r = \frac{a_1\omega_r}{2}$$

OR damping ratio increases for higher modes

STAEDTLER No 937 811E Engineer's Computation Pad

Rayleigh Damping

$$\underline{c} = a_0 \underline{m} + a_1 \underline{k}$$

$$\Rightarrow \quad \underline{C} = a_0 \underline{M} + a_1 \underline{K}$$

$$\Rightarrow \quad 2 \xi_n \omega_n M_n = a_0 M_n + a_1 K_n$$

$$= a_0 M_n + a_1 \omega_n^2 M_n$$

OR $$\xi_n = \frac{a_0}{2\omega_n} + \frac{a_1 \omega_n}{2}$$

Solve for a_0 and a_1 using any TWO modes' specified damping (say modes i and j)

Solve:
$$\frac{1}{2} \begin{bmatrix} 1/\omega_i & \omega_i \\ 1/\omega_j & \omega_j \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} \xi_i \\ \xi_j \end{Bmatrix} \quad -(\mathrm{I})$$

For any other mode (than i or j),

$$\xi_r = \frac{a_0}{2\omega_r} + \frac{a_1 \omega_n}{2} \quad \text{for } r^{th} \text{ mode}$$

when a_0 & a_1 are obtained by solving (I)

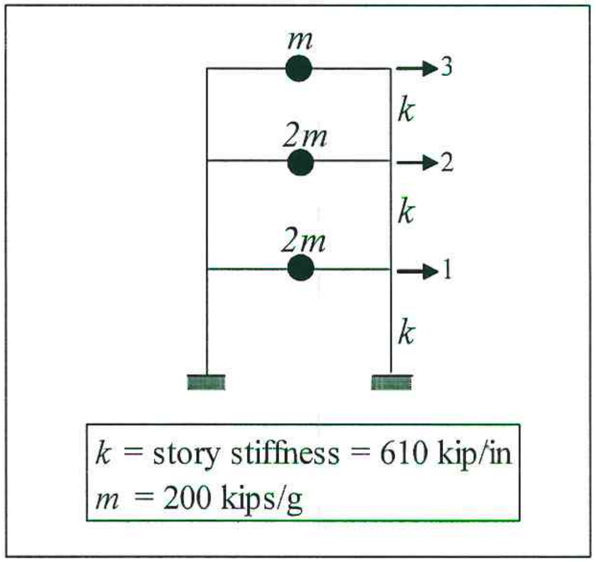
[Solution for (I) in special case when $\xi_i = \xi_j = \xi$

$$a_0 = \xi \frac{2\omega_i \omega_j}{\omega_i + \omega_j} \quad ; \quad a_1 = \xi \frac{2}{\omega_i + \omega_j}$$]

NOTES ON PRACTICAL ISSUES IN CONSTRUCTING CLASSICAL DAMPING MATRICES

Refer to Example 11.1

$$k := 610 \cdot \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad m := \frac{1}{32.2 \cdot 12} \cdot \begin{pmatrix} 400 & 0 & 0 \\ 0 & 400 & 0 \\ 0 & 0 & 200 \end{pmatrix}$$



k = story stiffness = 610 kip/in
m = 200 kips/g

$$\text{genvals}(k, m) = \begin{pmatrix} 2.199 \times 10^{3} \\ 1.179 \times 10^{3} \\ 1.579 \times 10^{2} \end{pmatrix} \qquad \text{genvecs}(k, m) = \begin{pmatrix} 0.354 & 0.707 & 0.354 \\ -0.612 & 0 & 0.612 \\ 0.707 & -0.707 & 0.707 \end{pmatrix}$$

$$\omega_1 := \sqrt{\text{genvals}(k, m)_3} \qquad \omega_1 = 12.565$$

$$\omega_2 := \sqrt{\text{genvals}(k, m)_2} \qquad \omega_2 = 34.33$$

$$\omega_3 := \sqrt{\text{genvals}(k, m)_1} \qquad \omega_3 = 46.895$$

$$\Phi^{\langle 1 \rangle} := \text{genvecs}(k, m)^{\langle 3 \rangle} \qquad \Phi^{\langle 2 \rangle} := \text{genvecs}(k, m)^{\langle 2 \rangle} \qquad \Phi^{\langle 3 \rangle} := \text{genvecs}(k, m)^{\langle 1 \rangle}$$

$$\Phi = \begin{pmatrix} 0.354 & 0.707 & 0.354 \\ 0.612 & 0.000 & -0.612 \\ 0.707 & -0.707 & 0.707 \end{pmatrix}$$

Mass-proportional damping $\zeta_1 := 0.05$

$$a_0 := 2 \cdot \zeta_1 \cdot \omega_1$$

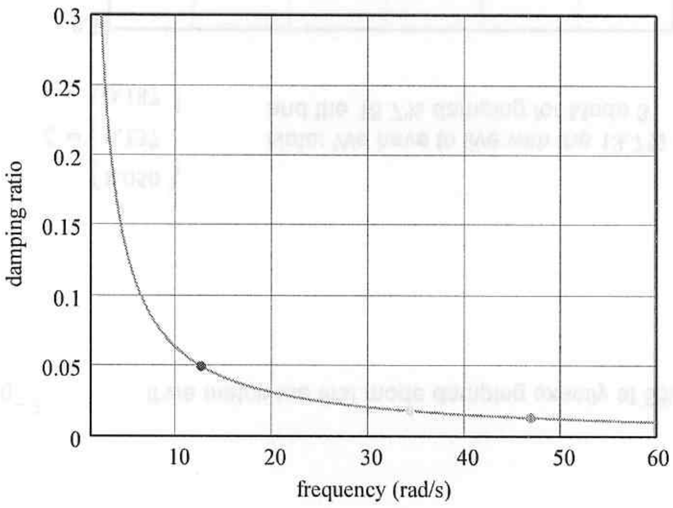
$a_0 = 1.257$ if we match the first mode damping exactly at 5%

$$j := 2 .. 3$$

$$\zeta_j := \frac{a_0}{2 \cdot \omega_j}$$

$$\zeta = \begin{pmatrix} 0.050 \\ 0.018 \\ 0.013 \end{pmatrix}$$

Note: We have to live with the 1.8% damping for Mode 2 and the 1.3% damping for Mode 3



22-26

Stiffness-proportional damping $\zeta_1 := 0.05$

$$a_1 := \frac{2 \cdot \zeta_1}{\omega_1}$$

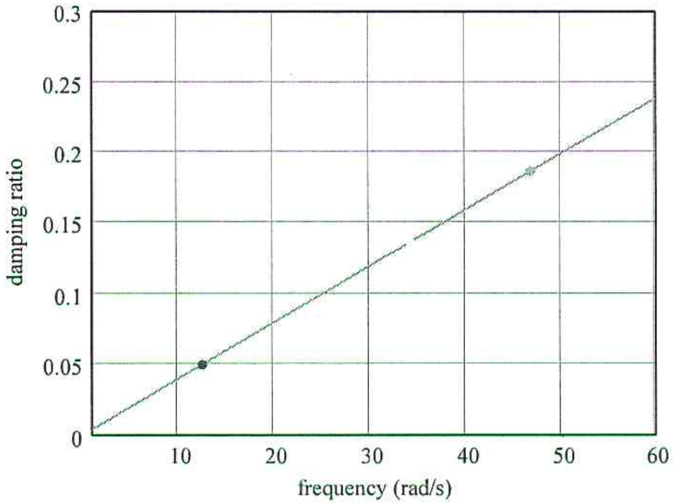
$a_1 = 7.958 \times 10^{-3}$ if we match the first mode damping exactly at 5%

$j := 2..3$

$$\zeta_j := \frac{a_1 \cdot \omega_j}{2}$$

$$\zeta = \begin{pmatrix} 0.050 \\ 0.137 \\ 0.187 \end{pmatrix}$$

Note: We have to live with the 13.7% damping for Mode 2 and the 18.7% damping for Mode 3



Rayleigh damping $\zeta_1 := 0.05$ $\zeta_2 := 0.05$

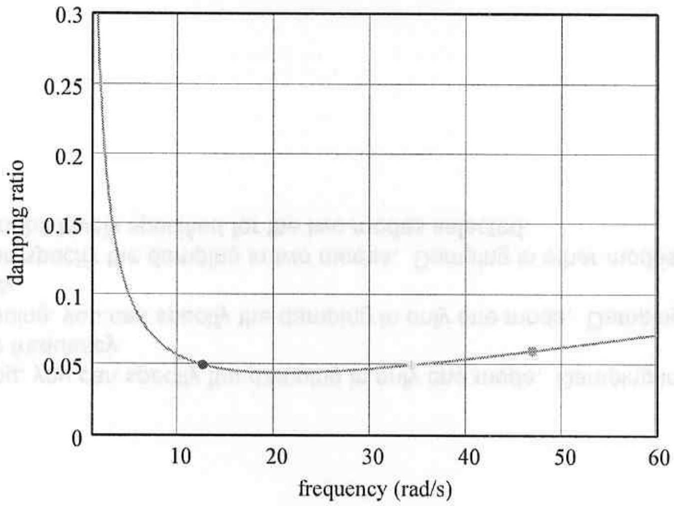
$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} := \begin{pmatrix} \frac{1}{2 \cdot \omega_1} & \frac{\omega_1}{2} \\ \frac{1}{2 \cdot \omega_2} & \frac{\omega_2}{2} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

$a_0 = 0.920$ $a_1 = 2.132 \times 10^{-3}$ if we match the first mode damping exactly at 5%

$$\zeta_3 := \frac{a_0}{2} \cdot \frac{1}{\omega_3} + \frac{a_1}{2} \cdot \omega_3$$

$$\zeta = \begin{pmatrix} 0.050 \\ 0.050 \\ 0.060 \end{pmatrix}$$

Note: We have to live with the 6.0% damping for Mode 3.



SUMMARY

1. With mass-proportional damping, you can specify the damping in only one mode. Damping in the various modes is inversely proportional to frequency.
2. With stiffness-proportional damping, you can specify the damping in only one mode. Damping in the various modes is proportional to frequency.
3. With Rayleigh damping, you can specify the damping in two modes. Damping in other modes may be either higher or lower than the levels specified for the two modes selected.

FREE VIBRATION AND THE EIGENVALUE PROBLEM

For an N-degree-of-freedom system,

$$\underset{\sim}{m}\,\ddot{\underset{\sim}{u}} + \underset{\sim}{k}\,\underset{\sim}{u} = \underset{\sim}{0} \quad -(\text{I})$$

given $\underset{\sim}{u}(0)$ and $\dot{\underset{\sim}{u}}(0)$

is a statement of the free vibration problem.

Eqn (I) is a coupled set of equations because $\underset{\sim}{m}$ and $\underset{\sim}{k}$ are non-diagonal in general (often $\underset{\sim}{m}$ is diagonal; but $\underset{\sim}{k}$ is rarely diagonal).

Problems:

1. Not trivial to solve (I)
2. If one solves (I) in its coupled form (which is possible), no physical insights are gained from the solution.

Modal Analysis is an answer to both these problems.

In a natural mode, the system will vibrate with a characteristic frequency, ω_n, and will maintain a characteristic shape, ϕ_n

i.e., in the n^{th} mode

$$\underset{\sim}{u}(t) = \underset{\sim}{\phi}_n\, q_n(t)$$

where $q_n(t) = A_n \cos\omega_n t + B_n \sin\omega_n t$ (nth modal coordinate)

For the N-degree-of-freedom system, there are N such characteristic modes (with associated frequencies and shapes). $\underset{\sim}{u}(t)$ is, in general, a combination of all of these characteristic modes.

$$\underset{N\times 1}{\underline{u}}(t) = \sum_{n=1}^{N} \underset{N\times 1}{\underline{\phi}_n}\ \underset{1\times 1}{q_n(t)}$$

or $$\underset{N\times 1}{\underline{u}(t)} = \underset{N\times N}{\underline{\Phi}}\ \underset{N\times 1}{\underline{q}(t)} \qquad \text{— (II)}$$

or $$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{N1} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{N2} \\ \vdots & \vdots & & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_N(t) \end{Bmatrix}$$

$$\underbrace{\overset{\uparrow}{\underline{\phi}_1} \quad \overset{\uparrow}{\underline{\phi}_2} \quad \cdots \quad \overset{\uparrow}{\underline{\phi}_N}}_{\text{N mode shapes as columns of matrix, } \underline{\Phi}}$$

$\underline{\phi}_n$ and ω_n for each mode are obtained by solving the eigenvalue problem

$$\underline{k}\,\underline{\phi}_n = \underline{m}\,\underline{\phi}_n\,\omega_n^2$$

$$\underset{N\times N}{\underline{k}}\ \underset{N\times N}{\underline{\Phi}} = \underset{N\times N}{\underline{m}}\ \underset{N\times N}{\underline{\Phi}}\ \underset{N\times N}{\underline{\Omega}^2}$$

Modes are orthogonal (w.r.t. $\underline{k}$ and $\underline{m}$)

$$\Rightarrow \left.\begin{aligned} \underline{\phi}_n^T\,\underline{m}\,\underline{\phi}_r &= 0 \\ \underline{\phi}_n^T\,\underline{k}\,\underline{\phi}_r &= 0 \end{aligned}\right\} \text{ if } r \neq n$$

Because of this property of orthogonality, we can simplify (and uncouple) Eqn.(I).

$$\underline{m}\,\underline{\Phi}\,\ddot{\underline{q}} + \underline{k}\,\underline{\Phi}\,\underline{q} = \underline{0}$$

$$\Rightarrow \underline{\Phi}^T\,\underline{m}\,\underline{\Phi}\,\ddot{\underline{q}} + \underline{\Phi}^T\,\underline{k}\,\underline{\Phi}\,\underline{q} = \underline{0}$$

or $$\underset{-}{M}\,\underset{-}{\ddot{q}} + \underset{-}{K}\,\underset{-}{q} = \underset{-}{0} \quad \text{—(III)}$$

where $$\underset{-}{M} = \underset{-}{\Phi}^T\,\underset{-}{m}\,\underset{-}{\Phi}$$

& $$\underset{-}{K} = \underset{-}{\Phi}^T\,\underset{-}{k}\,\underset{-}{\Phi}$$

are diagonal matrices because of orthogonality

What about the initial conditions?

$\underset{-}{u}(0)$ & $\underset{-}{\dot{u}}(0)$ are given

but we'll need $\underset{-}{q}(0)$ and $\underset{-}{\dot{q}}(0)$ to solve Eqn. (III)

$$\underset{-}{q}(0) = \begin{Bmatrix} q_1(0) \\ q_2(0) \\ \vdots \\ q_N(0) \end{Bmatrix} ; \quad \underset{-}{\dot{q}}(0) = \begin{Bmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \\ \vdots \\ \dot{q}_N(0) \end{Bmatrix}$$ for the N modes

For the n^{th} mode, initial conditions $q_n(0)$ & $\dot{q}_n(0)$ can be obtained using

$$\underset{-}{u} = \sum_{n=1}^{N} \phi_n\, q_n(t)$$

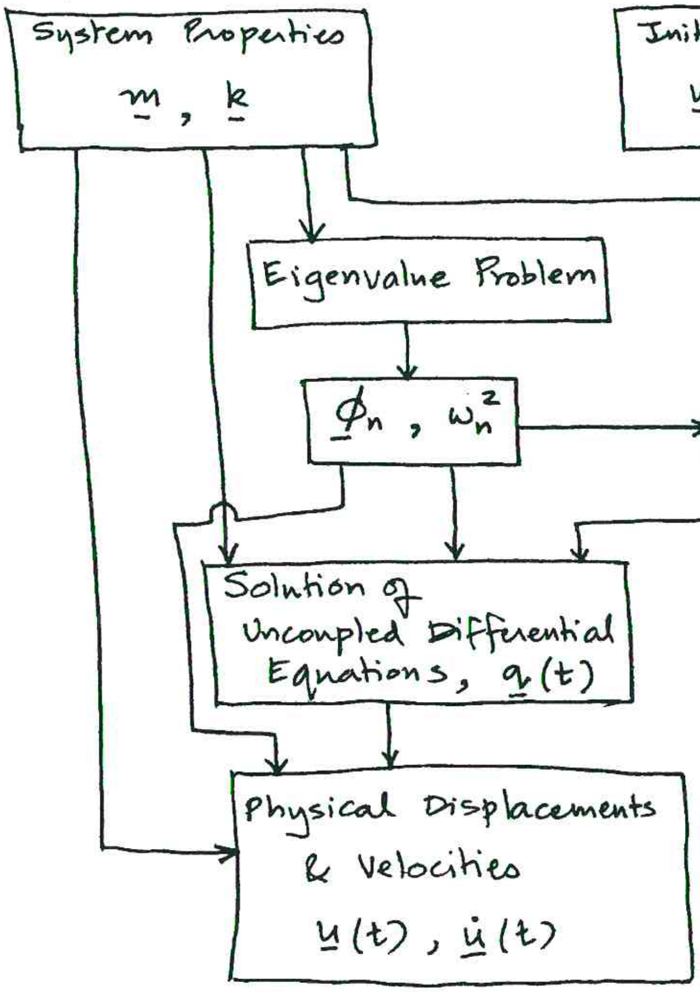
and $$\underset{-}{\phi}_n^T\,\underset{-}{m}\,\underset{-}{\phi}_r = 0 \quad \text{if } n \neq r$$

$$q_n(0) = \frac{\underset{-}{\phi}_n^T\,\underset{-}{m}\,\underset{-}{u}(0)}{\underset{-}{\phi}_n^T\,\underset{-}{m}\,\underset{-}{\phi}_n} \; ; \quad \dot{q}_n(0) = \frac{\underset{-}{\phi}_n^T\,\underset{-}{m}\,\underset{-}{\dot{u}}(0)}{\underset{-}{\phi}_n^T\,\underset{-}{m}\,\underset{-}{\phi}_n} \quad \text{-(IV)}$$

for $n = 1$ to N

Eqns. III & IV can be used to solve for $q_n(t)$ ($n = 1$ to N). Then Eqn II can be used to obtain $\underset{-}{u}(t)$

Summary


System Properties
$\underline{m}$, $\underline{k}$
Initial Conditions
$\underline{u}(0)$, $\dot{\underline{u}}(0)$
Eigenvalue Problem
$\underline{\phi}_n$, ω_n^2
$\underline{q}(0)$, $\dot{\underline{q}}(0)$
Solution of Uncoupled Differential Equations, $\underline{q}(t)$
Physical Displacements & Velocities
$\underline{u}(t)$, $\dot{\underline{u}}(t)$

Free Vibration with Classical Damping

Natural modes are unaffected by damping (if classical)

For n^{th} mode:

$$\ddot{q}_n + 2\xi_n \omega_n \dot{q}_n + \omega_n^2 q_n = 0 \quad -(A)$$

as always, $$\underline{u}(t) = \sum_{n=1}^{N} \underline{\Phi}_n q_n(t) \quad -(B)$$

Solution of Eqn. (A):-

$$q_n(t) = e^{-\xi_n \omega_n t}\left[q_n(0)\cos\omega_{nD} t + \frac{\dot{q}_n(0) + \xi_n\omega_n q_n(0)}{\omega_{nD}}\sin\omega_{nD} t\right] -(C)$$

where $\omega_{nD} = \omega_n\sqrt{1-\xi_n^2}$

Substituting (C) into (B) gives solution $\underline{u}(t)$.

Note, as we saw for SDOF systems,

$\omega_{nD} \simeq \omega_n$ if $\xi_n < 20\%$ (usually the case for most structures)

As n increases, ω_n increases

$\Rightarrow$ higher modes have higher frequencies

From (C), decay of higher modes will be faster due to $e^{-\xi_n\omega_n t}$ term.

$\Rightarrow$ In free vibration, after some time has elapsed, only lower modes are represented.

Damping Issues in Chapter 11

→ Good discussion of damping and natural period measurements from earthquake response of buildings in Section 11.1

→ Estimated Damping Ratios for different materials as well as for different limit states is discussed in Section 11.2

Table 11.2.1 good reference for suggested damping ratios

→ Constructing Damping Matrices [Classical]

- Rayleigh Damping

$$\underline{c} = a_0 \underline{m} + a_1 \underline{k}$$

(solve for a_0 & a_1 using 2 modes)

- Caughey Damping (a bit more general)

$$\underline{c} = \underline{m} \sum_{l=0}^{N-1} a_l \left[\underline{m}^{-1} \underline{k}\right]^l$$

N = no. of degrees of freedom

can include all N degrees of freedom or a reduced number, J

$$\underline{c} = \underline{m} \sum_{l=0}^{J-1} a_l \left[\underline{m}^{-1} \underline{k}\right]^l$$

Can solve for a_l ($l = 0$ to $J-1$)

- Superposition of Modal Damping Matrices

Start by recognizing that $\underline{C}$ (modal damping matrix) is diagonal if modal analysis (uncoupled) is to be possible

Find that $$\underset{(N\times N)}{\underline{c}} \doteq \underset{(N\times N)}{\underline{m}} \left[\sum_{n=1}^{N} \frac{2\xi_n \omega_n}{M_n} \underset{(N\times 1)}{\underline{\phi}_n} \underset{(1\times N)}{\underline{\phi}_n^T}\right] \underset{(N\times N)}{\underline{m}}$$

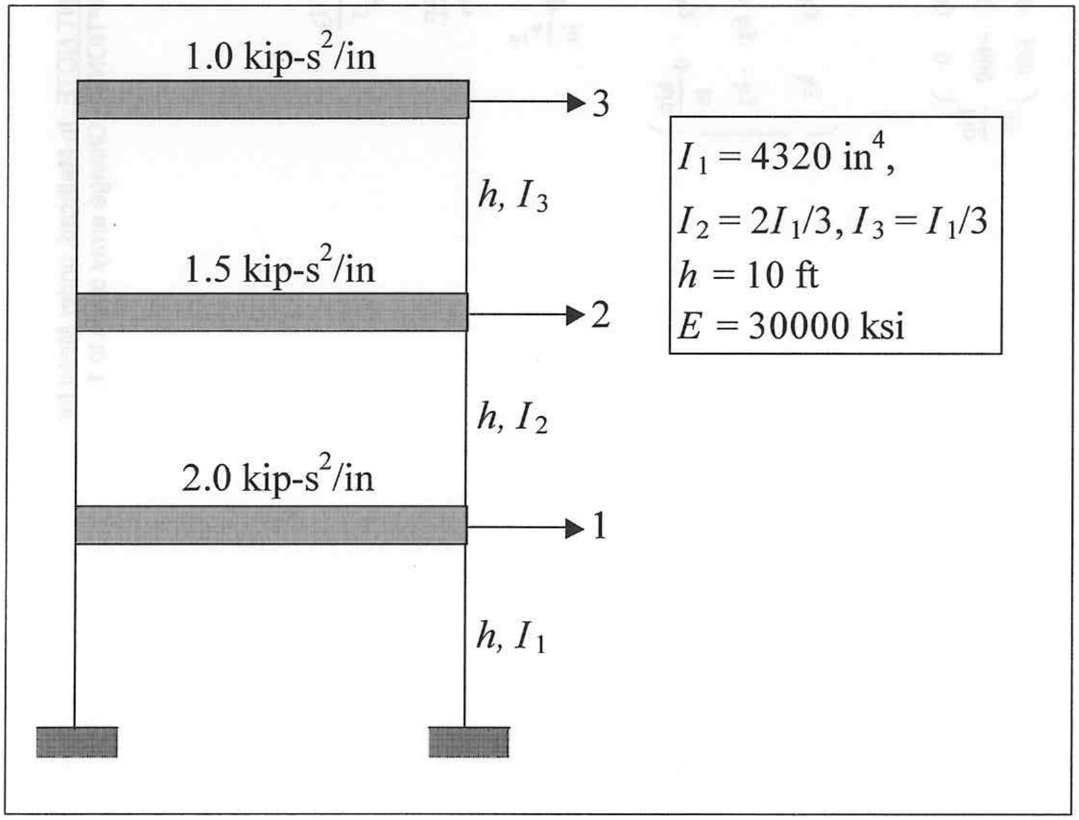
Exercises with MDOF systems – Eigenvalue Problem, Free Vibration

(a) Given the three-story structure shown in the figure below, determine the vibration mode shapes and frequencies. Assume that damping may be neglected.

(b) If the initial conditions are such that the first, second, and third story displacements are 1.0, 2.0, and 3.0 inches, respectively, and the velocities are zero, write expressions for the three deformations as functions of time. Which mode dominates the response? Would you have expected this?

(c) Repeat the calculations for free vibrations for the same initial displacements and velocities as in Part (b) except for the second story displacement which should be assumed to have an initial displacement equal to –2.0 in.

(d) Assume that classical damping is valid and that a 5-percent damping ration may be appropriate for each mode, solve the free vibration problem with the same initial conditions as in (b) and (c) above. Comment on the relative contributions of the three modes as a function of time.



$I_1 = 4320$ in^4,

$I_2 = 2I_1/3, I_3 = I_1/3$

$h = 10$ ft

$E = 30000$ ksi

$kip := 1000 \cdot lbf$ $\quad ksi := \dfrac{kip}{in^2}$

IMPORTANT NOTE: In Mathcad, under Menu for MATH > OPTIONS > Change array origin to 1.

$E := 30000 \cdot ksi$ $\quad h := 10 \cdot ft$

$I_1 := 4320 \cdot in^4$ $\quad I_2 := \dfrac{2 \cdot I_1}{3}$ $\quad I_3 := \dfrac{I_1}{3}$

$k_1 := 24 \cdot \dfrac{E \cdot I_1}{h^3}$ $\quad k_2 := 24 \cdot \dfrac{E \cdot I_2}{h^3}$ $\quad k_3 := 24 \cdot \dfrac{E \cdot I_3}{h^3}$

$k_1 = 1800 \dfrac{kip}{in}$ $\quad k_2 = 1200 \dfrac{kip}{in}$ $\quad k_3 = 600 \dfrac{kip}{in}$

$m_1 := 2.0 \cdot kip \cdot \dfrac{s^2}{in}$ $\quad m_2 := 1.5 \cdot kip \cdot \dfrac{s^2}{in}$ $\quad m_3 := 1.0 \cdot kip \cdot \dfrac{s^2}{in}$

$$m := \begin{pmatrix} m_1 & 0 \cdot kip \cdot \dfrac{s^2}{in} & 0 \cdot kip \cdot \dfrac{s^2}{in} \\ 0 \cdot kip \cdot \dfrac{s^2}{in} & m_2 & 0 \cdot kip \cdot \dfrac{s^2}{in} \\ 0 \cdot kip \cdot \dfrac{s^2}{in} & 0 \cdot kip \cdot \dfrac{s^2}{in} & m_3 \end{pmatrix} \quad k := \begin{pmatrix} k_1 + k_2 & -k_2 & 0 \cdot \dfrac{kip}{in} \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 \cdot \dfrac{kip}{in} & -k_3 & k_3 \end{pmatrix}$$

$$m = \begin{pmatrix} 2.000 & 0.000 & 0.000 \\ 0.000 & 1.500 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{pmatrix} kip \cdot \frac{s^2}{in} \quad k = \begin{pmatrix} 3000 & -1200 & 0 \\ -1200 & 1800 & -600 \\ 0 & -600 & 600 \end{pmatrix} \frac{kip}{in}$$

(a) Eigenvalues and eigenvectors (modal frequencies and mode shapes)

$$\mathrm{genvals}(k,m) = \begin{pmatrix} 2125.162 \\ 963.959 \\ 210.879 \end{pmatrix} \frac{\mathrm{rad}}{\mathrm{s}^2}$$

$$\lambda := \sqrt{\mathrm{genvals}(k,m)}$$

$$\lambda = \begin{pmatrix} 46.099 \\ 31.048 \\ 14.522 \end{pmatrix} \frac{\mathrm{rad}}{\mathrm{s}} \leftarrow \text{1st mode}$$

Note: These are ordered from highest to lowest frequency. Hence, the first mode is at the bottom.

$$j := 1 .. 3$$

$$\omega_j := \lambda_{4-j}$$

Change order of eigenvalues so that the smallest frequency is first. There are other (more general) ways of doing this.

$$\omega = \begin{pmatrix} 14.522 \\ 31.048 \\ 46.099 \end{pmatrix} \frac{\mathrm{rad}}{\mathrm{sec}} \qquad T_n := \frac{2\cdot\pi}{\omega} \qquad T_n = \begin{pmatrix} 0.433 \\ 0.202 \\ 0.136 \end{pmatrix} \mathrm{s}$$

Modal frequencies and periods

$$\mathrm{genvecs}(k,m) = \begin{pmatrix} 0.666 & -0.502 & -0.246 \\ -0.694 & -0.449 & -0.527 \\ 0.273 & 0.739 & -0.813 \end{pmatrix}$$

Mode shapes. Note the number of "nodes" in each column of this matrix of mode shapes. You can tell that the last column is the first mode (it has NO node).

$$\Phi1 := \mathrm{genvecs}(k,m)$$

$$\Phi1 = \begin{pmatrix} 0.666 & -0.502 & -0.246 \\ -0.694 & -0.449 & -0.527 \\ 0.273 & 0.739 & -0.813 \end{pmatrix}$$

24-4

$\Phi^{\langle j\rangle} := \Phi1^{\langle 4-j\rangle}$ Reorder to match the re-ordering of frequencies

$$\Phi = \begin{pmatrix} -0.246 & -0.502 & 0.666 \\ -0.527 & -0.449 & -0.694 \\ -0.813 & 0.739 & 0.273 \end{pmatrix}$$

$$\Phi^{\langle j\rangle} := \frac{\Phi^{\langle j\rangle}}{\left(\Phi^{\langle j\rangle}\right)_3}$$

$$\Phi = \begin{pmatrix} 0.302 & -0.679 & 2.440 \\ 0.649 & -0.607 & -2.542 \\ 1.000 & 1.000 & 1.000 \end{pmatrix}$$

Normalized so that roof displacement = 1

$n_{floor} := 3$

Sketch mode shapes

$i := 1 .. n_{floor}$

$ii := 1 .. n_{floor} + 1$

$ht_i := h$

$$H_{ii} := \begin{vmatrix} 0\cdot ft \ \ \text{if } ii = 1 \\ \displaystyle\sum_{j=1}^{ii-1} ht_j \ \ \text{otherwise} \end{vmatrix}$$

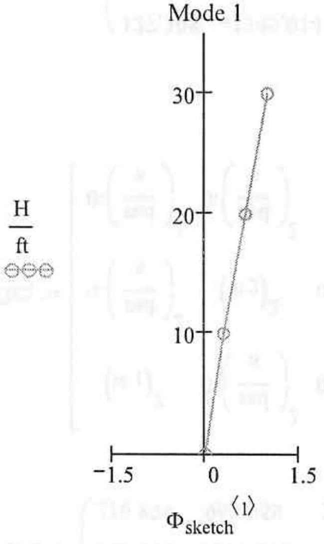
$zeroes := (0 \ \ 0 \ \ 0)$

$\Phi_{sketch} := stack(zeroes, \Phi)$

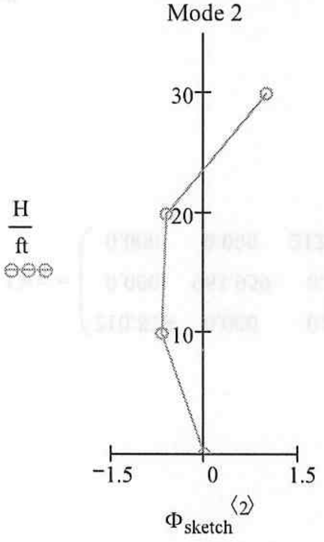
24-5

$$\Phi_{sketch} = \begin{pmatrix} 0.000 & 0.000 & 0.000 \\ 0.302 & -0.679 & 2.440 \\ 0.649 & -0.607 & -2.542 \\ 1.000 & 1.000 & 1.000 \end{pmatrix} \qquad H = \begin{pmatrix} 0 \\ 10 \\ 20 \\ 30 \end{pmatrix} \text{ft}$$

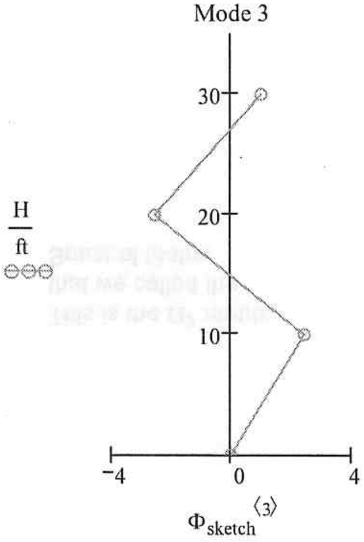
$$\Phi^{\langle 1 \rangle} = \begin{pmatrix} 0.302 \\ 0.649 \\ 1.000 \end{pmatrix} \qquad \Phi^{\langle 2 \rangle} = \begin{pmatrix} -0.679 \\ -0.607 \\ 1.000 \end{pmatrix} \qquad \Phi^{\langle 3 \rangle} = \begin{pmatrix} 2.440 \\ -2.542 \\ 1.000 \end{pmatrix}$$



$$\omega_1 = 14.522 \, \frac{rad}{s}$$



$$\omega_2 = 31.048 \, \frac{rad}{s}$$



$$\omega_3 = 46.099 \, \frac{rad}{s}$$

24-6

$$k\cdot\Phi = \begin{pmatrix} 127.308 & -1309.014 & 10369.206 \\ 205.144 & -877.105 & -8103.038 \\ 210.879 & 963.959 & 2125.162 \end{pmatrix}\frac{\text{kip}}{\text{in}}$$

$$\Omega sq := \begin{bmatrix} (\omega_1)^2 & 0\cdot\left(\frac{\text{rad}}{\text{s}}\right)^2 & 0\cdot\left(\frac{\text{rad}}{\text{s}}\right)^2 \\ 0\cdot\left(\frac{\text{rad}}{\text{s}}\right)^2 & (\omega_2)^2 & 0\cdot\left(\frac{\text{rad}}{\text{s}}\right)^2 \\ 0\cdot\left(\frac{\text{rad}}{\text{s}}\right)^2 & 0\cdot\left(\frac{\text{rad}}{\text{s}}\right)^2 & (\omega_3)^2 \end{bmatrix}$$

$$\Omega sq = \begin{pmatrix} 210.879 & 0.000 & 0.000 \\ 0.000 & 963.959 & 0.000 \\ 0.000 & 0.000 & 2125.162 \end{pmatrix}\left(\frac{\text{rad}}{\text{s}}\right)^2$$

This is the Ω^2 matrix
that we called the
Spectral Matrix

$$m\cdot\Phi\cdot\Omega sq = \begin{pmatrix} 127.308 & -1309.014 & 10369.206 \\ 205.144 & -877.105 & -8103.038 \\ 210.879 & 963.959 & 2125.162 \end{pmatrix}\frac{\text{kip}}{\text{in}}$$

Same as $\underline{k}*\underline{\Phi}$ above, i.e., $\underline{k}\ \underline{\Phi} = \underline{m}\ \underline{\Phi}\ \underline{\Omega sq}$

(b) Free Vibration

$$u0 := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot in \qquad ud0 := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{in}{s}$$

$$q0_j := \frac{\left(\left(\Phi^{\langle j \rangle}\right)^T \cdot m \cdot u0\right)_1}{\left(\left(\Phi^{\langle j \rangle}\right)^T \cdot m \cdot \Phi^{\langle j \rangle}\right)_1} \qquad qd0_j := \frac{\left(\left(\Phi^{\langle j \rangle}\right)^T \cdot m \cdot ud0\right)_1}{\left(\left(\Phi^{\langle j \rangle}\right)^T \cdot m \cdot \Phi^{\langle j \rangle}\right)_1} \qquad \boxed{q_j(t) = \frac{\underline{\phi}_j^T \, \underline{m} \, \underline{u}(t)}{\underline{\phi}_j^T \, \underline{m} \, \underline{\phi}_j}} \qquad \boxed{\dot{q}_j(t) = \frac{\underline{\phi}_j^T \, \underline{m} \, \dot{\underline{u}}(t)}{\underline{\phi}_j^T \, \underline{m} \, \underline{\phi}_j}}$$

$$q0 = \begin{pmatrix} 3.061 \\ -0.072 \\ 0.011 \end{pmatrix} in \qquad qd0 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \frac{in}{s}$$

Note that $q0_1$ is the largest of the three values in the matrix, q0. This indicates that the initial conditions are such that the first mode has a large contribution to the inital shape.

$$i := 1 .. 3 \qquad j := 1 .. 3$$

$$U_{i,j} := \Phi_{i,j} \cdot q0_j$$

$$\Phi = \begin{pmatrix} 0.302 & -0.679 & 2.440 \\ 0.649 & -0.607 & -2.542 \\ 1.000 & 1.000 & 1.000 \end{pmatrix} \qquad U = \begin{pmatrix} 0.924 & 0.049 & 0.027 \\ 1.985 & 0.044 & -0.029 \\ 3.061 & -0.072 & 0.011 \end{pmatrix} in$$

First column here is contribution of mode 1 to displacements. It is clearly the largest on all floors. On the 3rd floor in particular, the first mode has the largest portion of the 3.0 inches.

$$\Phi \cdot q0 = \begin{pmatrix} 1.000 \\ 2.000 \\ 3.000 \end{pmatrix} in$$

Check (same as u0)

$$\Phi \cdot qd0 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \frac{in}{s}$$

Check (same as ud0)

24-7

24-8

$\Delta t := 0.01 \cdot s$

$$NT := \frac{5 \cdot \frac{2 \cdot \pi}{\omega_1}}{\Delta t}$$

We'll plot 5 cycles of the response

$NT = 216.338$

$i := 1, 2 .. \text{floor}(NT)$

Expressions for response below are for undamped free vibration of a SDOF system

$$\boxed{\underline{u} = \underline{\Phi} \underline{q} = \sum_{j=1}^{3} \phi_j q_j(t)}$$

$$\boxed{q_j(t) = q_j(0) \cos \varpi_j t + \frac{\dot{q}_j(0)}{\varpi_j} \sin \varpi_j t}$$

$$u_i := \sum_{j=1}^{3} \Phi^{\langle j \rangle} \cdot \left(q0_j \cdot \cos(\omega_j \cdot i \cdot \Delta t) + \frac{qd0_j}{\omega_j} \cdot \sin(\omega_j \cdot i \cdot \Delta t) \right)$$

Summing contributions of all modes to the displacements of all floors at time, i*Δt. u is a 3X1 vector of the three floor displacements including all three modes.

$$u1_i := \Phi^{\langle 1 \rangle} \cdot \left(q0_1 \cdot \cos(\omega_1 \cdot i \cdot \Delta t) + \frac{qd0_1}{\omega_1} \cdot \sin(\omega_1 \cdot i \cdot \Delta t) \right)$$

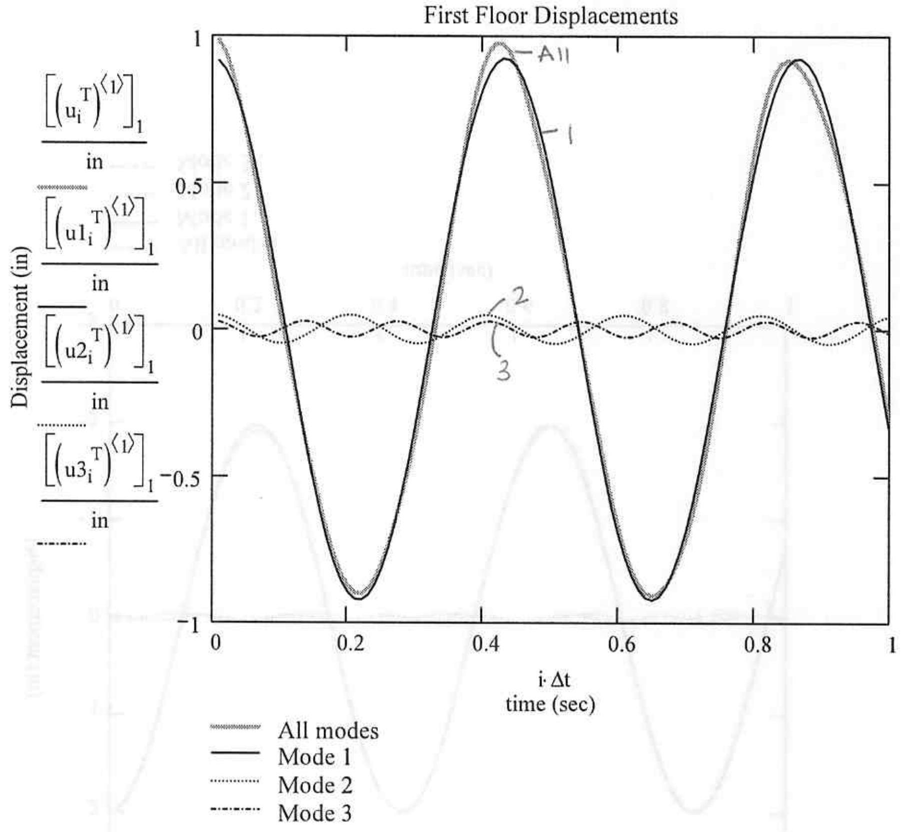
This is mode 1's contribution to the displacements of all floors at time, i*Δt. u1 is a 3X1 vector of the three floor displacements for mode 1.

$$u2_i := \Phi^{\langle 2 \rangle} \cdot \left(q0_2 \cdot \cos(\omega_2 \cdot i \cdot \Delta t) + \frac{qd0_2}{\omega_2} \cdot \sin(\omega_2 \cdot i \cdot \Delta t) \right)$$

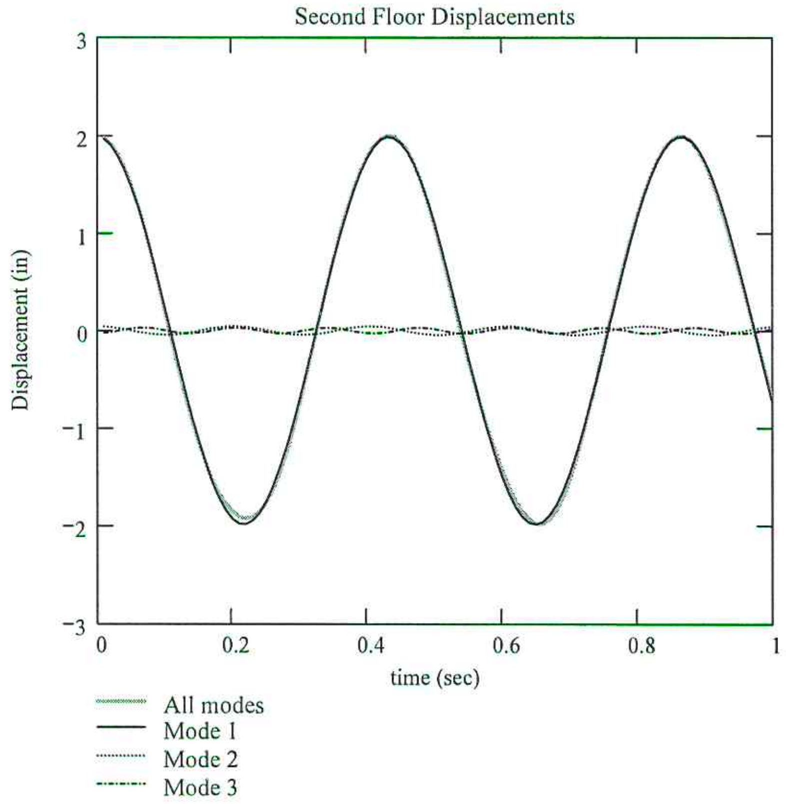
This is mode 2's contribution to the displacements of all floors at time, i*Δt. u2 is a 3X1 vector of the three floor displacements for mode 2.

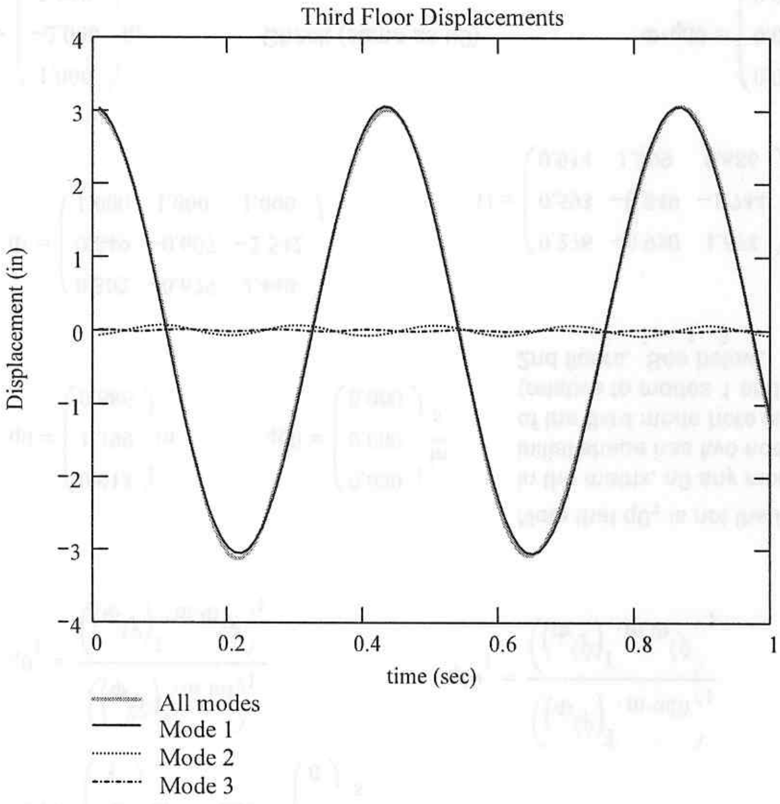
$$u3_i := \Phi^{\langle 3 \rangle} \cdot \left(q0_3 \cdot \cos(\omega_3 \cdot i \cdot \Delta t) + \frac{qd0_3}{\omega_3} \cdot \sin(\omega_3 \cdot i \cdot \Delta t) \right)$$

This is mode 3's contribution to the displacements of all floors at time, i*Δt. u3 is a 3X1 vector of the three floor displacements for mode 3.



<u>Note</u>: Very small contributions from modes 2 and 3.


Second Floor Displacements



At top story, the influence of modes 2 and 3 is even smaller than it was on the lower stories. This was already evident even at t = 0.

24-12

(c) Free Vibration with different initial conditions

$u0 := \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot in \qquad ud0 := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{in}{s}$

$$q0_j := \frac{\left(\left(\Phi^{\langle j\rangle}\right)^T \cdot m \cdot u0\right)_1}{\left(\left(\Phi^{\langle j\rangle}\right)^T \cdot m \cdot \Phi^{\langle j\rangle}\right)_1} \qquad qd0_j := \frac{\left(\left(\Phi^{\langle j\rangle}\right)^T \cdot m \cdot ud0\right)_1}{\left(\left(\Phi^{\langle j\rangle}\right)^T \cdot m \cdot \Phi^{\langle j\rangle}\right)_1}$$

$q0 = \begin{pmatrix} 0.914 \\ 1.399 \\ 0.686 \end{pmatrix} in \qquad qd0 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \frac{in}{s}$

Note that $q0_1$ is not the largest of the three values in the matrix, q0 any more. Rather, because the initial shape has two nodes in it, the contribution of the third mode here is more important (relative to modes 1 and 2) at least on the 1st and 2nd floors. See below.

$i := 1..3 \qquad j := 1..3$

$\Phi = \begin{pmatrix} 0.302 & -0.679 & 2.440 \\ 0.649 & -0.607 & -2.542 \\ 1.000 & 1.000 & 1.000 \end{pmatrix}$

$U_{i,j} := \Phi_{i,j} \cdot q0_j$

$U = \begin{pmatrix} 0.276 & -0.950 & 1.674 \\ 0.593 & -0.849 & -1.744 \\ 0.914 & 1.399 & 0.686 \end{pmatrix} in$

Third column here is contribution of mode 3 to displacements. It is clearly the largest except on the third floor where mode 2 has the largest contribution.

$\Phi \cdot q0 = \begin{pmatrix} 1.000 \\ -2.000 \\ 3.000 \end{pmatrix} in$ Check (same as u0)

$\Phi \cdot qd0 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \frac{in}{s}$ Check (same as ud0)

$i := 1, 2 .. \mathrm{floor}(NT)$

$$u_i := \sum_{j=1}^{3} \Phi^{\langle j \rangle} \cdot \left(q0_j \cdot \cos(\omega_j \cdot i \cdot \Delta t) + \frac{qd0_j}{\omega_j} \cdot \sin(\omega_j \cdot i \cdot \Delta t) \right)$$

Summing contributions of <u>all modes</u> to the displacements of all floors at time, i*Δt. u is a 3X1 vector of the three floor displacements including all three modes.

$$u1_i := \Phi^{\langle 1 \rangle} \cdot \left(q0_1 \cdot \cos(\omega_1 \cdot i \cdot \Delta t) + \frac{qd0_1}{\omega_1} \cdot \sin(\omega_1 \cdot i \cdot \Delta t) \right)$$

This is mode 1's contribution to the displacements of all floors at time, i*Δt. u1 is a 3X1 vector of the three floor displacements for mode 1.

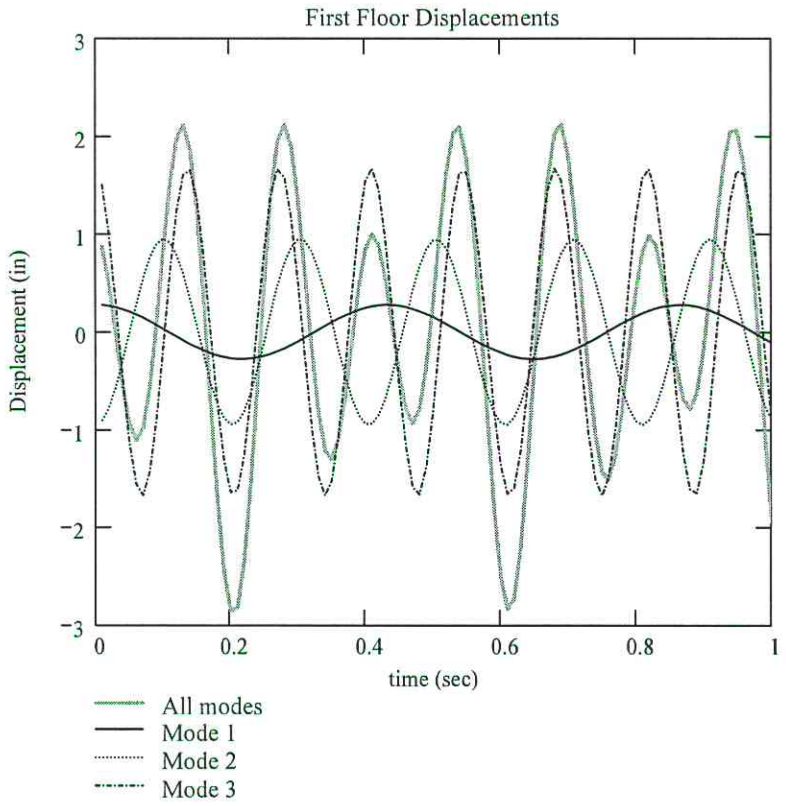
$$u2_i := \Phi^{\langle 2 \rangle} \cdot \left(q0_2 \cdot \cos(\omega_2 \cdot i \cdot \Delta t) + \frac{qd0_2}{\omega_2} \cdot \sin(\omega_2 \cdot i \cdot \Delta t) \right)$$

This is mode 2's contribution to the displacements of all floors at time, i*Δt. u2 is a 3X1 vector of the three floor displacements for mode 2.

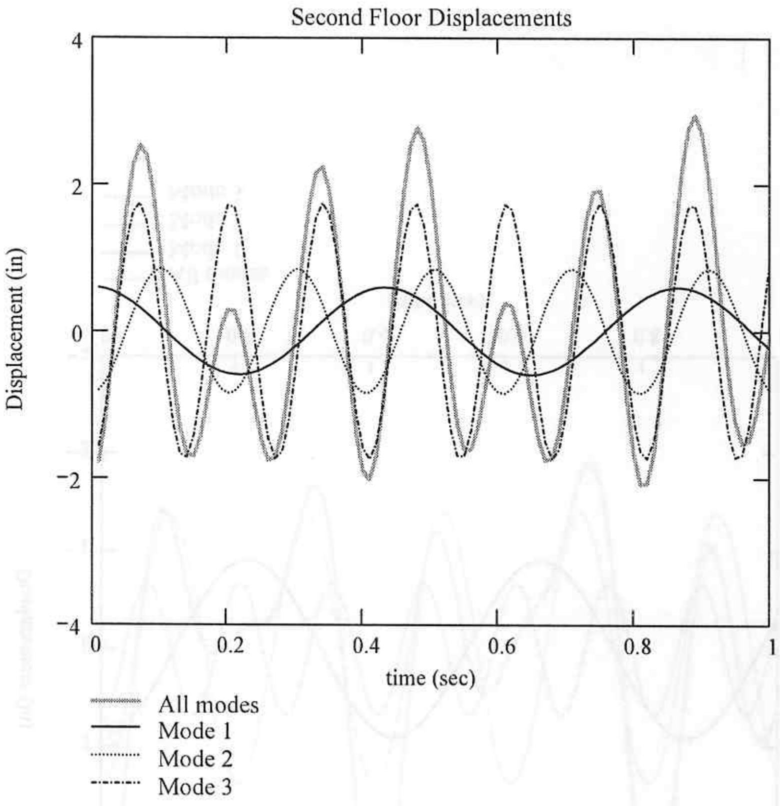
$$u3_i := \Phi^{\langle 3 \rangle} \cdot \left(q0_3 \cdot \cos(\omega_3 \cdot i \cdot \Delta t) + \frac{qd0_3}{\omega_3} \cdot \sin(\omega_3 \cdot i \cdot \Delta t) \right)$$

This is mode 3's contribution to the displacements of all floors at time, i*Δt. u3 is a 3X1 vector of the three floor displacements for mode 3.

24-14

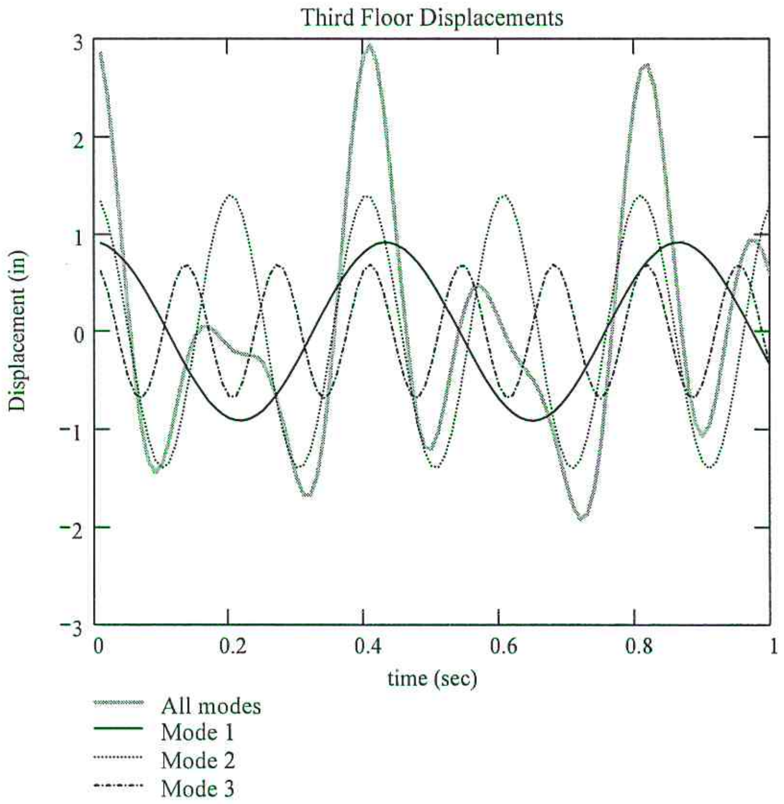


Notice that in this case, the first mode is relatively less important (for first floor displacments) compared to modes 2 and 3. As expected, mode 3 has the largest contribution based on the initial conditions.



At the second floor level, modes 1 and 2 are almost comparable in amplitude. Mode 3 is still the largest contributor.

24-16



The contribution of modes 1 and 2 have grown at the third floor level. Mode 2, in fact, is now the biggest contributor - this could have been anticipated based on mode 2's contribution at t = 0 (see matrix U on pg. 11).

(d) Damped Free Vibration - compare with part (b) above.

$\zeta = 5\%$

$$u0 := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot in \qquad ud0 := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{in}{s}$$

$j := 1 .. 3$

$$q0_j := \frac{\left(\left(\Phi^{\langle j\rangle}\right)^T \cdot m \cdot u0\right)_1}{\left(\left(\Phi^{\langle j\rangle}\right)^T \cdot m \cdot \Phi^{\langle j\rangle}\right)_1} \qquad qd0_j := \frac{\left(\left(\Phi^{\langle j\rangle}\right)^T \cdot m \cdot ud0\right)_1}{\left(\left(\Phi^{\langle j\rangle}\right)^T \cdot m \cdot \Phi^{\langle j\rangle}\right)_1}$$

$\zeta_j := 0.05$

$$\omega_{D_j} := \omega_j \cdot \sqrt{1 - \left(\zeta_j\right)^2}$$

$$q0 = \begin{pmatrix} 3.061 \\ -0.072 \\ 0.011 \end{pmatrix} in \qquad qd0 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \frac{in}{s}$$

Same comments on mode 1's greater contribution to initial displacement of all floors as in part (a). See pg. 6.

$$\Phi = \begin{pmatrix} 0.302 & -0.679 & 2.440 \\ 0.649 & -0.607 & -2.542 \\ 1.000 & 1.000 & 1.000 \end{pmatrix}$$

$$\Phi \cdot q0 = \begin{pmatrix} 1.000 \\ 2.000 \\ 3.000 \end{pmatrix} in$$

Check (same as u0)

$$\Phi \cdot qd0 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \frac{in}{s}$$

Check (same as ud0)

Expressions for response below are for <u>damped</u> free vibration of a SDOF system

$i := 1, 2 .. \mathrm{floor}(NT)$

$$\boxed{\underline{u} = \underline{\Phi}\underline{q} = \sum_{j=1}^{3} \phi_j q_j(t)}$$

$$\boxed{q_j(t) = \exp(-\zeta_j \varpi_j t)\left[q_j(0)\cos\varpi_{Dj}t + \frac{\dot{q}_j(0) + \zeta_j \varpi_j q_j(0)}{\varpi_{Dj}}\sin\varpi_{Dj}t\right]}$$

$$u_i := \sum_{j=1}^{3} \Phi^{\langle j \rangle} \cdot \exp(-\zeta_j \cdot \omega_j \cdot i \cdot \Delta t) \cdot \left(q0_j \cdot \cos(\omega_{D_j} \cdot i \cdot \Delta t) + \frac{qd0_j + \zeta_j \cdot \omega_j \cdot q0_j}{\omega_{D_j}} \cdot \sin(\omega_{D_j} \cdot i \cdot \Delta t) \right)$$

Summing contributions of <u>all modes</u> to the displacements of all floors at time, i*Δt. u is a 3X1 vector of the three floor displacements including all three modes.

$$u1_i := \Phi^{\langle 1 \rangle} \cdot \exp(-\zeta_1 \cdot \omega_1 \cdot i \cdot \Delta t) \cdot \left(q0_1 \cdot \cos(\omega_{D_1} \cdot i \cdot \Delta t) + \frac{qd0_1 + \zeta_1 \cdot \omega_1 \cdot q0_1}{\omega_{D_1}} \cdot \sin(\omega_{D_1} \cdot i \cdot \Delta t) \right)$$

This is mode 1's contribution to the displacements of all floors at time, i*Δt. u1 is a 3X1 vector of the three floor displacements for mode 1.

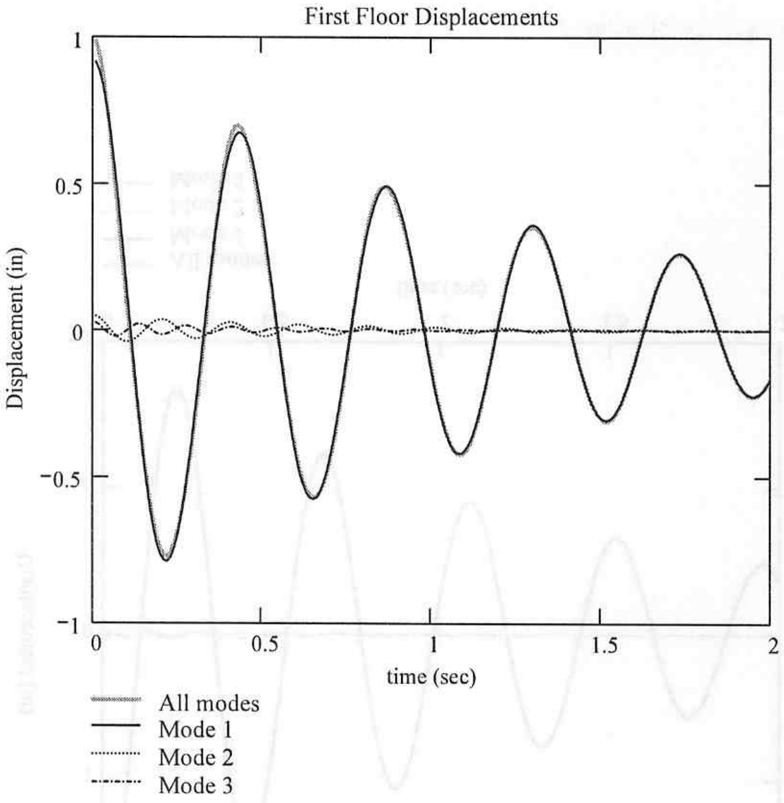
$$u2_i := \Phi^{\langle 2 \rangle} \cdot \exp(-\zeta_2 \cdot \omega_2 \cdot i \cdot \Delta t) \cdot \left(q0_2 \cdot \cos(\omega_{D_2} \cdot i \cdot \Delta t) + \frac{qd0_2 + \zeta_2 \cdot \omega_2 \cdot q0_2}{\omega_{D_2}} \cdot \sin(\omega_{D_2} \cdot i \cdot \Delta t) \right)$$

This is mode 2's contribution to the displacements of all floors at time, i*Δt. u2 is a 3X1 vector of the three floor displacements for mode 2.

$$u3_i := \Phi^{\langle 3 \rangle} \cdot \exp(-\zeta_3 \cdot \omega_3 \cdot i \cdot \Delta t) \cdot \left(q0_3 \cdot \cos(\omega_{D_3} \cdot i \cdot \Delta t) + \frac{qd0_3 + \zeta_3 \cdot \omega_3 \cdot q0_3}{\omega_{D_3}} \cdot \sin(\omega_{D_3} \cdot i \cdot \Delta t) \right)$$

This is mode 3's contribution to the displacements of all floors at time, i*Δt. u3 is a 3X1 vector of the three floor displacements for mode 3.

It is important to note that the exponential term will cause faster decay of the higher modes because $\omega_3 > \omega_2 > \omega_1$. Thus, over time, we should expect that the higher modes will become less important.


First Floor Displacements

Note: As we'd expect for the initial shape given, the first mode dominates at the beginning but note how over time, the relative importance of the first mode gets even larger because the higher modes damp out (decay) faster.

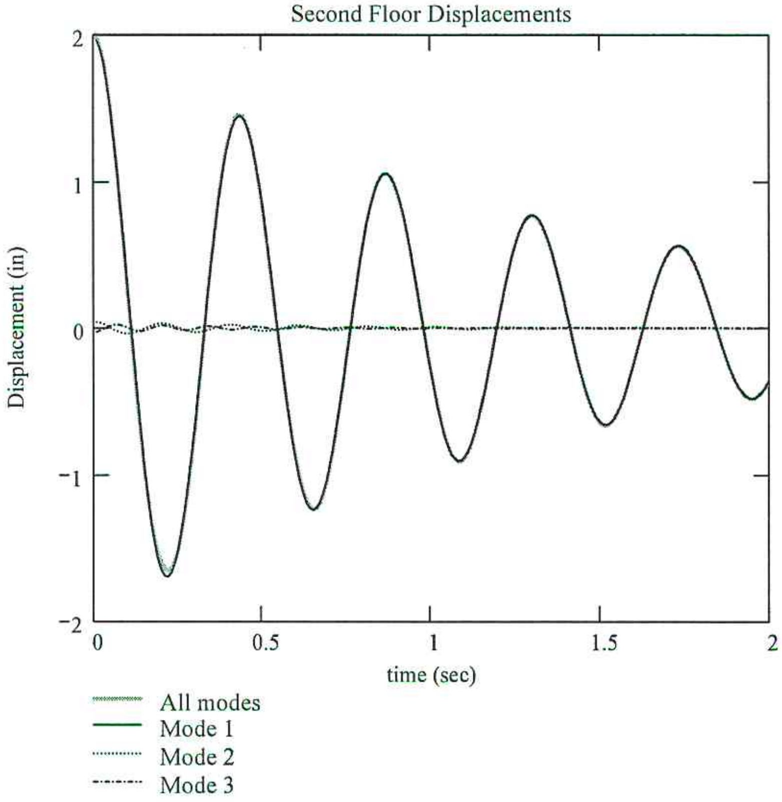
Consider t = 1.7 seconds

$$\frac{1.7}{\frac{\Delta t}{\text{sec}}} = 170$$

$\left(u1_{170}\right)_1 = 0.233\ \text{in}$ Mode 1

$\left(u2_{170}\right)_1 = -0.003\ \text{in}$ Mode 2

$\left(u3_{170}\right)_1 = -0.001\ \text{in}$ Mode 3



Mode 1's relative contribution is even greater on the second floor than it is on the first.

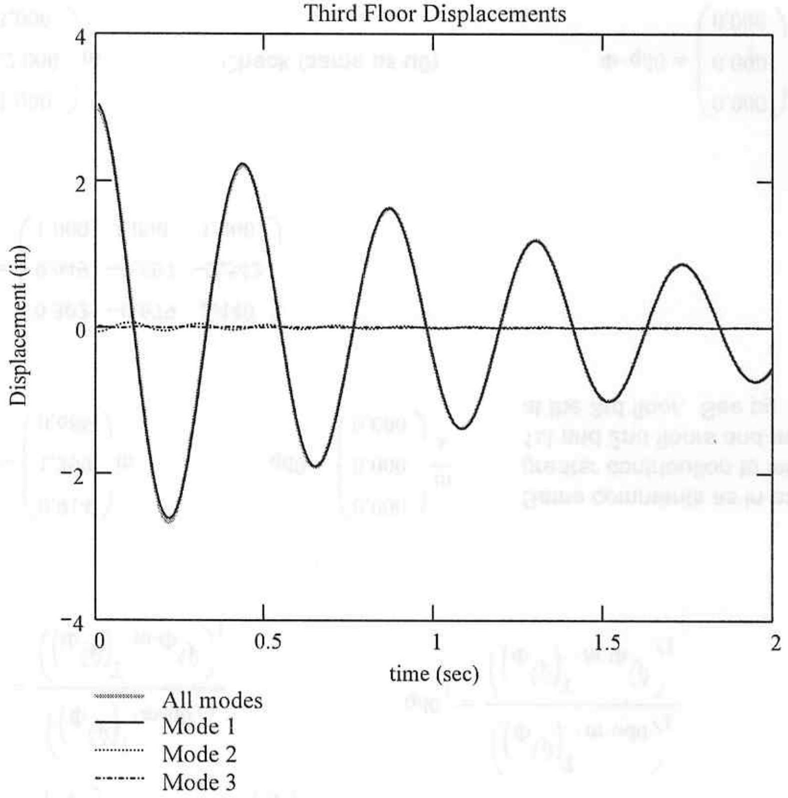
Consider t = 1.7 seconds

$$\frac{1.7}{\frac{\Delta t}{\text{sec}}} = 170$$

$\left(u1_{170}\right)_2 = 0.500\text{ in}$ Mode 1

$\left(u2_{170}\right)_2 = -0.002\text{ in}$ Mode 2

$\left(u3_{170}\right)_2 = 0.001\text{ in}$ Mode 3



Again at the third floor, the first mode is more important than at both lower floors. Also, over time, the higher modes become virtually unimportant.

Consider t = 1.7 seconds

$$\frac{1.7}{\frac{\Delta t}{\text{sec}}} = 170$$

$\left(u1_{170}\right)_3 = 0.771 \text{ in}$ Mode 1

$\left(u2_{170}\right)_3 = 0.004 \text{ in}$ Mode 2

$\left(u3_{170}\right)_3 = -0.000 \text{ in}$ Mode 3

24-21

24-22

(d) Damped Free Vibration - compare with part (c) above

$u0 := \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot in \qquad ud0 := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{in}{s}$

$j := 1..3$

$\zeta_j := 0.05$

$\omega D_j := \omega_j \cdot \sqrt{1 - (\zeta_j)^2}$

$$q0_j := \frac{\left(\left(\Phi^{\langle j \rangle}\right)^T \cdot m \cdot u0\right)_1}{\left(\left(\Phi^{\langle j \rangle}\right)^T \cdot m \cdot \Phi^{\langle j \rangle}\right)_1} \qquad qd0_j := \frac{\left(\left(\Phi^{\langle j \rangle}\right)^T \cdot m \cdot ud0\right)_1}{\left(\left(\Phi^{\langle j \rangle}\right)^T \cdot m \cdot \Phi^{\langle j \rangle}\right)_1}$$

$q0 = \begin{pmatrix} 0.914 \\ 1.399 \\ 0.686 \end{pmatrix} in \qquad qd0 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \frac{in}{s}$

Same comments as in part (b) regarding mode 3's greater contribution to initial displacement at the 1st and 2nd floors and mode 2's greater contribution at the 3rd floor. See pg. 11.

$$\Phi = \begin{pmatrix} 0.302 & -0.679 & 2.440 \\ 0.649 & -0.607 & -2.542 \\ 1.000 & 1.000 & 1.000 \end{pmatrix}$$

$\Phi \cdot q0 = \begin{pmatrix} 1.000 \\ -2.000 \\ 3.000 \end{pmatrix} in$ Check (same as u0)

$\Phi \cdot qd0 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \frac{in}{s}$ Check (same as ud0)

$$i := 1, 2 .. \mathrm{floor}(NT)$$

$$u_i := \sum_{j=1}^{3} \Phi^{\langle j \rangle} \cdot \exp\left(-\zeta_j \cdot \omega_j \cdot i \cdot \Delta t\right) \cdot \left(q0_j \cdot \cos\left(\omega_{D_j} \cdot i \cdot \Delta t\right) + \frac{qd0_j + \zeta_j \cdot \omega_j \cdot q0_j}{\omega_{D_j}} \cdot \sin\left(\omega_{D_j} \cdot i \cdot \Delta t\right) \right)$$

imming contributions of <u>all modes</u> to the displacements all floors at time, i*Δt. u is a 3X1 vector of the three or displacements including all three modes.

$$u1_i := \Phi^{\langle 1 \rangle} \cdot \exp\left(-\zeta_1 \cdot \omega_1 \cdot i \cdot \Delta t\right) \cdot \left(q0_1 \cdot \cos\left(\omega_{D_1} \cdot i \cdot \Delta t\right) + \frac{qd0_1 + \zeta_1 \cdot \omega_1 \cdot q0_1}{\omega_{D_1}} \cdot \sin\left(\omega_{D_1} \cdot i \cdot \Delta t\right) \right)$$

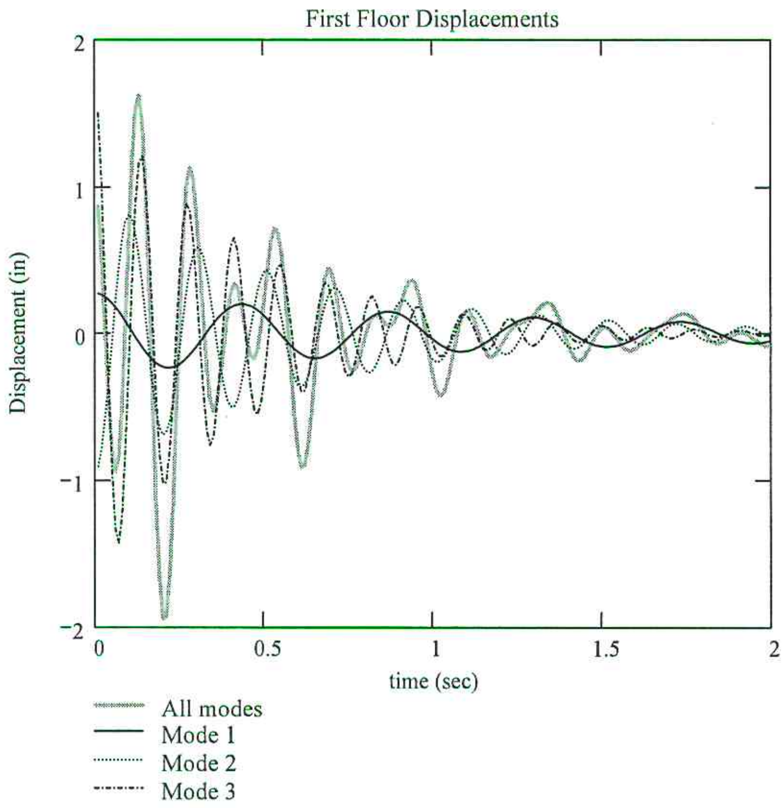
is is mode 1's contribution to the displacements all floors at time, i*Δt. u1 is a 3X1 vector of the three or displacements for mode 1.

$$u2_i := \Phi^{\langle 2 \rangle} \cdot \exp\left(-\zeta_2 \cdot \omega_2 \cdot i \cdot \Delta t\right) \cdot \left(q0_2 \cdot \cos\left(\omega_{D_2} \cdot i \cdot \Delta t\right) + \frac{qd0_2 + \zeta_2 \cdot \omega_2 \cdot q0_2}{\omega_{D_2}} \cdot \sin\left(\omega_{D_2} \cdot i \cdot \Delta t\right) \right)$$

is is mode 2's contribution to the displacements all floors at time, i*Δt. u2 is a 3X1 vector of the three or displacements for mode 2.

$$u3_i := \Phi^{\langle 3 \rangle} \cdot \exp\left(-\zeta_3 \cdot \omega_3 \cdot i \cdot \Delta t\right) \cdot \left(q0_3 \cdot \cos\left(\omega_{D_3} \cdot i \cdot \Delta t\right) + \frac{qd0_3 + \zeta_3 \cdot \omega_3 \cdot q0_3}{\omega_{D_3}} \cdot \sin\left(\omega_{D_3} \cdot i \cdot \Delta t\right) \right)$$

is is mode 3's contribution to the displacements all floors at time, i*Δt. u3 is a 3X1 vector of the three or displacements for mode 3.



Here we start to see how even though mode 3 was the biggest contributor to the initial shape, it becomes less important over time. For example, at t = 1.7 seconds, mode 1 seems to be largest. Modes 2 and 3 damp out faster than mode 1.

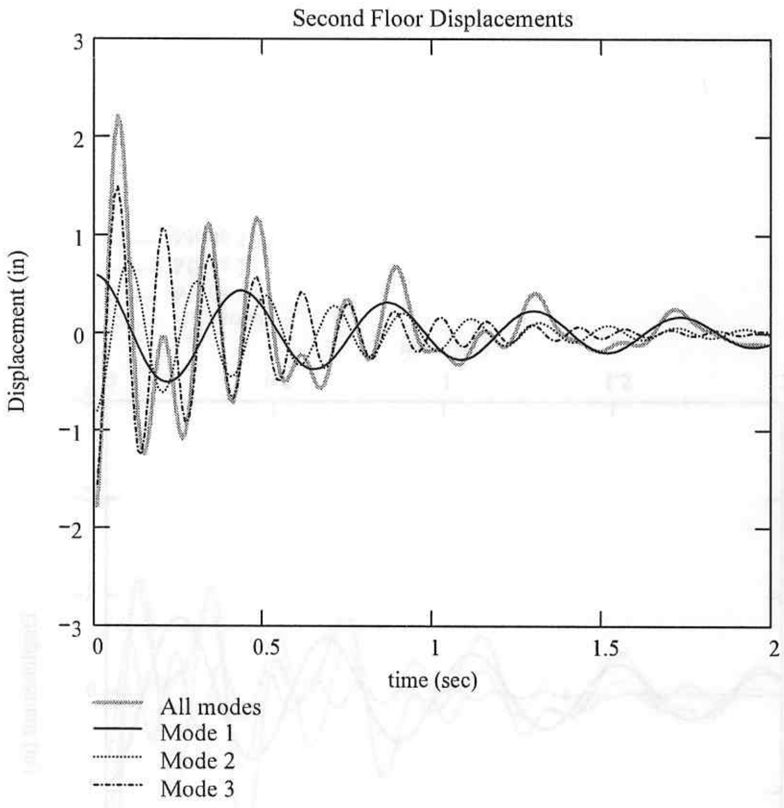
Consider t = 1.7 seconds

$$\frac{1.7}{\frac{\Delta t}{\text{sec}}} = 170$$

$\left(u1_{170}\right)_1 = 0.070\ \text{in}$ Mode 1

$\left(u2_{170}\right)_1 = 0.050\ \text{in}$ Mode 2

$\left(u3_{170}\right)_1 = -0.032\ \text{in}$ Mode 3



Second Floor Displacements

Effect of mode 1 even more important than on first floor.

Consider t = 1.7 seconds

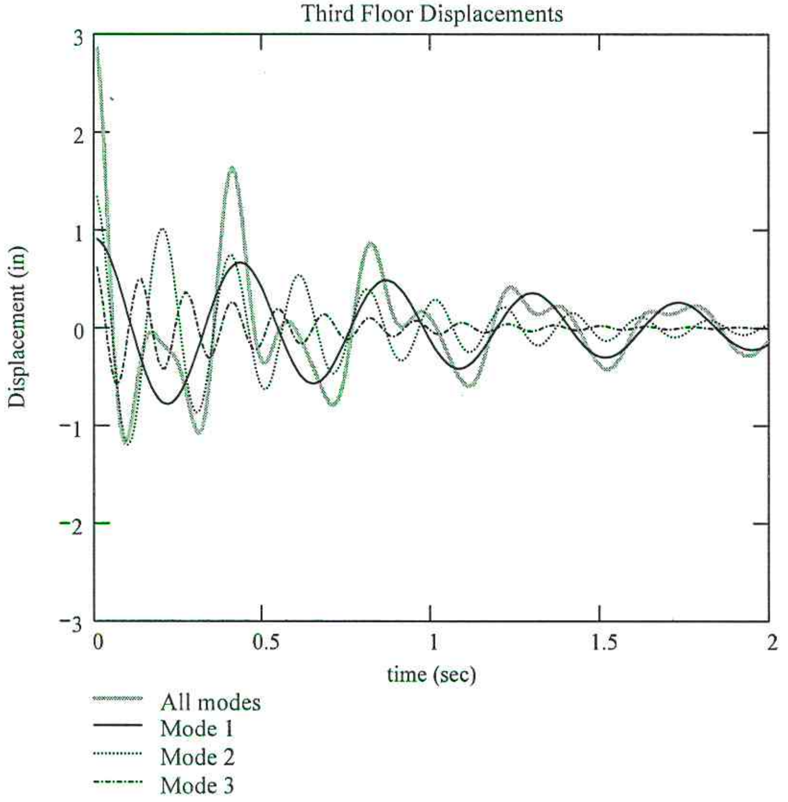
$$\frac{1.7}{\frac{\Delta t}{\text{sec}}} = 170$$

$\left(u1_{170}\right)_2 = 0.149 \text{ in}$ Mode 1

$\left(u2_{170}\right)_2 = 0.045 \text{ in}$ Mode 2

$\left(u3_{170}\right)_2 = 0.033 \text{ in}$ Mode 3

24-26



Again, at the third floor level, the effect of mode 1 is seen to be more important than at lower floors. Also, the importance of mode 1 increases with time due to faster damping (decay) of higher modes.

Consider t = 1.7 seconds

$$\frac{1.7}{\frac{\Delta t}{\text{sec}}} = 170$$

$\left(u1_{170}\right)_3 = 0.230 \text{ in}$ Mode 1

$\left(u2_{170}\right)_3 = -0.074 \text{ in}$ Mode 2

$\left(u3_{170}\right)_3 = -0.013 \text{ in}$ Mode 3

MDOF Responses

BigAssHW Assignment

Problem One – K, m matricies

solve for ω_ns, mode shapes

then normalize

Problem Two – solve for $q_n(0), \dot{q}_n(0)$

use $\underline{c}$ matrix, too

use to calculate total displacements

$\sum \phi_n q_n(t)$, $\underline{u}(t) = \underline{\Phi}\, q$ ← scalar value

jth story peak:

$u_{jo} = \left[\sum u_{jn_o}^2 \right]^{1/2}$ max displacement in each mode on that level

Base shear:

$\underline{f} = \underline{k}\,\underline{u}$, $V = \sum f h$

static forces needed to create same response as dynamic system

Problem Three

Add force vector, solve for displacements

$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = P_n(t)$ — SDOF equation, one mode

$\uparrow$

diagonal matrix

$2\xi_n \omega_n M_n$

$\xi = 0.05$

$P_n(t) = \phi_n^T \cdot p(t)$

find P_n for each mode shape

$= \phi_{3n} P_3(t)$

solving for q_n

$$q_n(t) = \frac{p_{no}}{K_n} \sin(\omega t - \theta_n) R_{dn}$$

each mode has ω_n,

use with ω to calculate R_{dn}

order ω_ns from low to high

$M_n = \phi_n^T \underline{m}\, \phi_n$

$\theta_n = \tan^{-1}\left[\frac{2\xi r_n}{1 - r_n^2} \right]$

Problem Four

$P_n(t) = \phi_{1n} P_1(t)$ — calc. R_d using rectangular pulse response

if $t_d/T_n > 1/2$, $R_d = 2.0$

4.7.2, 4.7.3 equations can be modified

also, use for t=0.5s calcs

time-varying eq – just above 4.7.5 (pg 138)

Problem Five

equation of motion $= \Gamma_n M_n \ddot{u}_g(t)$, $\Gamma_n = L_n / M_n$

$L_n = \phi_n^T \underline{m}\, \underline{1}$

look in chapt. 13

each mode has floor max. at different times

$$u_{jo} = \left[\sum_{n=1}^{N} u_{jn_o}^2 \right]^{1/2}$$

$$u_o = \frac{A(\omega_n, \xi)}{\omega_n^2}$$

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CONTINUED WRAP-UP

Handouts

24 – 3-story, free vibration problem in mathcad
25 – general procedure for any p(t)
26 – $\underline{p}(t) = \underline{s}\,p(t)$
$\quad\uparrow$ spacial pattern, non time-varying

page seven:

$$s_n = \Gamma_n \underline{m}\, \underline{\phi}_n$$

27 – Mathcad example, using $\underline{s}$ method
28 – Earthquake problem
similar to $\underline{s}$ approach
use Γ_n and spacial approach; book example (chapt. 13)
29 – EQ response spectrum
5-story mathcad example (using SRSS, etc.)
30 – Summary of MDOF response for $\underline{p}(t)$ and $\ddot{u}_g(t)$

Test comments

1. $\tilde{p}_o = \psi_i p_i$ (summation)

$$\tilde{p}_o = 2/3\; p_o = 2.0\text{ K}$$

$V_j = k_j(u_j - u_{j-1}) = k_j(\psi_j - \psi_{j-1})z_o$, or $\underline{f} = \underline{k}\,\underline{u}$

k_j = story stiffness, 4k – summation of column ks on floor, not to be confused with any part of the global K matrix.

$$f_j = \omega_n^2 m_j \psi_j z_o$$

change I to $\tilde{I}$, reduce by ψ again

Handout 25 – summation (or, summary), pg 25-4

Modal force = $\phi_n^T p(t)$

$q_n(t) \rightarrow$ uncoupled equations to solve

Adding damping

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2 q_n = \frac{p_n(t)}{M_n}$$ – linear systems, classical damping

$$\underset{\text{mode 1}}{u_j} = \underset{\substack{\uparrow\ \ \uparrow\\ \text{story}\ \ \text{mode}}}{\phi_{j1}}\, q_1$$

$$u_j(t) = \Sigma\, u_{jn}(t) = \Sigma\, \phi_{jn} q_n(t)$$

Forces

$$V_{b_n}(t) = \Sigma\, f_{jn}(t)$$

base shear from a specific mode

$$\underline{f}_{sn} = \underline{k}\,\underline{u}_n(t)$$

$\uparrow$ only displacements from the mode in question – not summation.

Modal Analysis

Equations of motion: $\underline{m}\,\underline{\ddot{u}} + \underline{k}\,\underline{u} = \underline{p}(t)$ $\left[\begin{array}{l}\text{Coupled}\\ \text{Set of}\\ \text{Equations}\end{array}\right]$

$\underline{u}(t)$ can be expressed in terms of modal contributions.

$$\underline{u}(t) = \sum_{r=1}^{N} \underline{\phi}_r\, q_r(t) = \underset{\substack{N\times N\\ \uparrow\\ \text{modal}\\ \text{matrix}}}{\underline{\Phi}}\; \underset{\substack{N\times 1\\ \wedge\ \text{modal / normal}\\ \text{coordinates}}}{\underline{q}}(t)$$

$$\Rightarrow \sum_{r=1}^{N} \underline{m}\,\underline{\phi}_r\, \ddot{q}_r(t) + \sum_{r=1}^{N} \underline{k}\,\underline{\phi}_r\, q_r(t) = \underline{p}(t)$$

Multiply by ϕ_n^T:

$$\sum_{r=1}^{N} \underline{\phi}_n^T\, \underline{m}\, \underline{\phi}_r\, \ddot{q}_r(t) + \sum_{r=1}^{N} \underline{\phi}_n^T\, \underline{k}\, \underline{\phi}_r\, q_r(t) = \underline{\phi}_n^T\, \underline{p}(t)$$

since $\underline{\phi}_n^T\, \underline{m}\, \phi_r = 0$; $\underline{\phi}_n^T\, \underline{k}\, \underline{\phi}_r = 0$ if $n \neq r$

$$\underline{\phi}_n^T\, \underline{m}\, \underline{\phi}_n\, \ddot{q}_n(t) + \underline{\phi}_n^T\, \underline{k}\, \underline{\phi}_n\, q_n = \underline{\phi}_n^T\, \underline{p}(t)$$

$$\Rightarrow \quad M_n\, \ddot{q}_n(t) + K_n\, q_n(t) = P_n(t) \quad \text{—(I)}$$

where $M_n = \underline{\phi}_n^T\, \underline{m}\, \underline{\phi}_n$ (generalized mass for the n^{th} mode)

$K_n = \underline{\phi}_n^T\, \underline{k}\, \underline{\phi}_n$ (generalized stiffness for the n^{th} mode

$P_n(t) = \underline{\phi}_n^T\, \underline{p}(t)$ (generalized force for the n^{th} mode)

Equation (I) is an equation for $q_n(t)$ which we can solve if we know <u>only the n^{th} mode</u>.

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Dividing by M_n:

$$\ddot{q}_n(t) + \omega_n^2\, q_n(t) = \frac{P_n(t)}{M_n} \qquad \text{— (II)}$$

N such equations; $n = 1$ to N
[Uncoupled now versus coupled equations when written in terms of $\underline{u}(t)$.]

(I) can be re-written in matrix form as:

$$\underset{\substack{\uparrow \\ \text{diagonal} \\ \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_N \end{bmatrix}}}{\underline{M}}\,\ddot{\underline{q}} + \underset{\substack{\uparrow \\ \text{diagonal} \\ \begin{bmatrix} K_1 & & & \\ & K_2 & & \\ & & \ddots & \\ & & & K_N \end{bmatrix}}}{\underline{K}}\,\underline{q} = \underset{\substack{\nwarrow \text{ column vector} \\ \begin{bmatrix} \underline{\phi}_1^T\, \underline{p}(t) \\ \vdots \\ \underline{\phi}_N^T\, \underline{p}(t) \end{bmatrix}}}{\underline{P}(t)}$$

For classical damping situations, easy to extend.

$$\ddot{q}_n + 2\underset{\substack{\uparrow \\ \text{damping} \\ \text{ratio for } n^{th} \text{ mode}}}{\xi_n}\,\omega_n\,\dot{q}_n + \omega_n^2\, q_n = \frac{P_n(t)}{M_n} \qquad \text{— (III)}$$

Thus, we can solve (II) or (III) for the modal coordinates, $q_n(t)$ for each mode n ($n = 1$ to N).
The contribution of the n^{th} mode to the displacement, $\underline{u}(t)$ is

$$\underline{u}_n(t) = \underline{\phi}_n\, q_n(t)$$

since we have solved for $q_n(t)$ and know $\underline{\phi}_n$, we can find $\underline{u}_n(t)$

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Combining all the modal contributions, gives us

$$\underline{u}(t) = \sum_{n=1}^{N} \underline{u}_n(t)$$

$$= \sum_{n=1}^{N} \underline{\phi}_n \, q_n(t) \quad \left[\text{or} \quad \underset{N\times1}{\underline{u}(t)} = \underset{N\times N}{\underline{\Phi}} \; \underset{N\times1}{\underline{q}(t)} \right]$$

This is CLASSICAL MODAL ANALYSIS

only works for Linear systems with classical damping.

Element Forces

Combine modal contributions as with displacements.

e.g., Unknown element force, $r(t)$

$$r(t) = \sum_{n=1}^{N} r_n(t)$$

where $r_n(t)$ is n^{th} modes contribution to the element force.

$r_n(t)$ is obtained using force-displacement relations.

Alternatively: First find $\underline{f}_{s_n}(t) = \underline{k} \, \underline{u}_n(t)$, equivalent static force associated with n^{th} mode.

OR $\underline{f}_{s_n}(t) = \omega_n^2 \, \underline{m} \, \underline{\phi}_n \, q_n(t)$

Apply these forces statically to find $r_n(t)$

Then, combine to get $r(t)$

MODAL ANALYSIS - Steps in Process

1. obtain $\underline{m}$, $\underline{k}$ (physical mass & stiffness matrices)

2. Estimate Damping ratios (guidelines in Table 11.2.1, for example)

3. Determine ω_n, $\underline{\phi}_n$ [frequencies, modes]

4. Response calculation
   a) For each mode n, find $q_n(t)$
   b) Determine $\underline{u}_n(t) = \phi_n q_n(t)$
   c) Determine $r_n(t)$

5. ~~Repeat~~ Sum responses in 4b) and 4c) for all modes to determine $\underline{u}(t)$ and $r(t)$.

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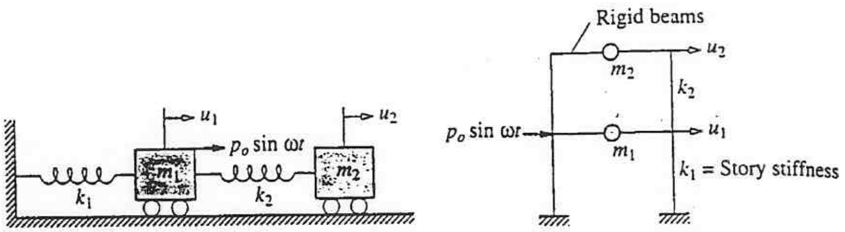

Figure 12.1.1 Two-degree-of-freedom systems.

$$\underline{m} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

Example 12.2

Consider the systems and excitation of Example 12.1. By modal analysis determine the steady-state response of the system.

Solution The natural vibration frequencies and modes of this system were determined in Example 10.4, from which the generalized masses and stiffnesses are calculated using Eq. (12.3.4). These results are summarized next:

$$\underline{k} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\omega_1 = \sqrt{\frac{k}{2m}} \qquad \omega_2 = \sqrt{\frac{2k}{m}}$$

$$\phi_1 = \langle \tfrac{1}{2} \quad 1 \rangle^T \qquad \phi_2 = \langle -1 \quad 1 \rangle^T$$

$$M_1 = \frac{3m}{2} \qquad M_2 = 3m$$

$$K_1 = \frac{3k}{4} \qquad K_2 = 6k$$

1. *Compute the generalized forces.*

$$P_1(t) = \phi_1^T \, \mathbf{p}(t) = \underbrace{(p_o/2)}_{P_{1o}} \sin \omega t \qquad P_2(t) = \phi_2^T \, \mathbf{p}(t) = \underbrace{-p_o}_{P_{2o}} \sin \omega t \tag{a}$$

2. *Set up the modal equations.*

[No damping]

$$M_n \ddot{q}_n + K_n q_n = P_{no} \sin \omega t \tag{b}$$

3. *Solve the modal equations.* To solve Eq. (b) we draw upon the solution presented in Eq. (3.1.7) for a SDF system subjected to harmonic force. The governing equation is

$$m\ddot{u} + ku = p_o \sin \omega t \tag{c}$$

and its steady-state solution is

$$u(t) = \frac{p_o}{k} C \sin \omega t \qquad C = \frac{1}{1 - (\omega/\omega_n)^2} \tag{d}$$

where $\omega_n = \sqrt{k/m}$. Comparing Eqs. (c) and (b), the solution for Eq. (b) is

$$q_n(t) = \frac{P_{no}}{K_n} C_n \sin \omega t \tag{e}$$

where C_n is given by Eq. (d) with ω_n interpreted as the natural frequency of the nth mode. Substituting for P_{no} and K_n for $n = 1$ and 2 gives

$$q_1(t) = \frac{2p_o}{3k} C_1 \sin \omega t \qquad q_2(t) = -\frac{p_o}{6k} C_2 \sin \omega t \tag{f}$$

4. *Determine the modal responses.* The nth mode contribution to displacements—from Eq. (12.3.2)—is $\mathbf{u}_n(t) = \phi_n q_n(t)$. Substituting Eq. (f) gives the displacement response due to the two modes:

$$\mathbf{u}_1(t) = \phi_1 \frac{2p_o}{3k} C_1 \sin \omega t \qquad \mathbf{u}_2(t) = \phi_2 \frac{-p_o}{6k} C_2 \sin \omega t \tag{g}$$

5. *Combine the modal responses.*

jth story displ. (scalar)

$$(2\times1)\ \mathbf{u}(t) = \mathbf{u}_1(t) + \mathbf{u}_2(t) \quad \text{or} \quad u_j(t) = u_{j1}(t) + u_{j2}(t) \qquad j = 1, 2 \tag{h}$$

Substituting Eq. (g) and for ϕ_1 and ϕ_2 gives

$$u_1(t) = \frac{p_o}{6k} (2C_1 + C_2) \sin \omega t \qquad u_2(t) = \frac{p_o}{6k} (4C_1 - C_2) \sin \omega t \tag{i}$$

These results are equivalent to those obtained in Example 12.1 by solving the coupled equations (12.3.1) of motion.

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Example 12.3

Consider the systems and excitation of Example 12.1. Determine the spring forces $V_j(t)$ for the system of Fig. 12.1.1a, or story shears $V_j(t)$ in the system of Fig. 12.1.1b, without introducing equivalent static forces. Consider only the steady-state response.

Solution Steps 1, 2, 3a, and 3b of the analysis summary of Section 12.7 have already been completed in Example 12.2.

Step 3c: The spring forces in the system of Fig. 12.1.1a or the story shears in the system of Fig. 12.1.1b are

$$V_{1n}(t) = k_1 u_{1n}(t) = k_1 \phi_{1n} q_n(t) \tag{a}$$

$$V_{2n}(t) = k_2 [u_{2n}(t) - u_{1n}(t)] = k_2 (\phi_{2n} - \phi_{1n}) q_n(t) \tag{b}$$

Substituting Eq. (f) of Example 12.2 in Eqs. (a) and (b) with $n = 1$, $k_1 = 2k$, $k_2 = k$, $\phi_{11} = \frac{1}{2}$, and $\phi_{21} = 1$ gives the forces due to the first mode:

$$V_{11}(t) = \frac{2p_o}{3} C_1 \sin \omega t \qquad V_{21}(t) = \frac{p_o}{3} C_1 \sin \omega t \tag{c}$$

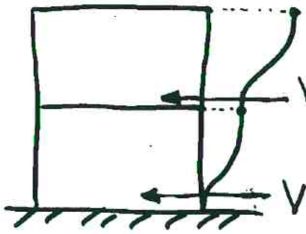
Substituting Eq. (f) of Example 12.2 in Eqs. (a) and (b) with $n = 2$, $\phi_{12} = -1$, and $\phi_{22} = 1$ gives the second-mode forces:

$$V_{12}(t) = \frac{p_o}{3} C_2 \sin \omega t \qquad V_{22}(t) = -\frac{p_o}{3} C_2 \sin \omega t \tag{d}$$

Step 4b: Substituting Eqs. (c) and (d) in $V_j(t) = V_{j1}(t) + V_{j2}(t)$ gives

$$V_1(t) = \frac{p_o}{3} (2C_1 + C_2) \sin \omega t \qquad V_2(t) = \frac{p_o}{3} (C_1 - C_2) \sin \omega t \tag{e}$$

Equation (e) gives the time variation of spring forces and story shears. For a given p_o and ω and the ω_n already determined, all quantities on the right side of these equations are known; thus $V_j(t)$ can be computed.



$V_{2n} = k_2 (u_{2n} - u_{1n})$

$V_{1n} = k_1 (u_{1n} - 0)$

Set $n = 1$ first; then, $n = 2$

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Example 12.4

Repeat Example 12.3 using equivalent static forces.

Solution From Eq. (12.6.2), for a lumped mass system the equivalent static force in the jth DOF due to the nth mode is

$$f_{jn}(t) = \omega_n^2 m_j \phi_{jn} q_n(t) \tag{a}$$

Step 3c: In Eq. (a) with $n = 1$, substitute $m_1 = 2m$, $m_2 = m$, $\phi_{11} = \frac{1}{2}$, $\phi_{21} = 1$, $\omega_1^2 = k/2m$, and $q_1(t)$ from Eq. (f) of Example 12.2 to obtain

$$f_{11}(t) = \frac{p_o}{3} C_1 \sin \omega t \qquad f_{21}(t) = \frac{p_o}{3} C_1 \sin \omega t \tag{b}$$

In Eq. (a) with $n = 2$, substituting $m_1 = 2m$, $m_2 = m$, $\phi_{12} = -1$, $\phi_{22} = 1$, $\omega_1^2 = 2k/m$, and $q_2(t)$ from Eq. (f) of Example 12.2 gives

$$f_{12}(t) = \frac{2p_o}{3} C_2 \sin \omega t \qquad f_{22}(t) = -\frac{p_o}{3} C_2 \sin \omega t \tag{c}$$

Static analysis of the systems of Fig. E12.4 subjected to forces $f_{jn}(t)$ gives the two spring forces and story shears due to the nth mode:

$$V_{1n}(t) = f_{1n}(t) + f_{2n}(t) \qquad V_{2n}(t) = f_{2n}(t) \tag{d}$$

Substituting Eq. (b) in Eq. (d) with $n = 1$ gives the first mode forces that are identical to Eq. (c) of Example 12.3. Similarly, substituting Eq. (c) in Eq. (d) with $n = 2$ gives the second-mode results that are identical to Eq. (d) of Example 12.3.

Step 4: Proceed as in step 4b of Example 12.3.

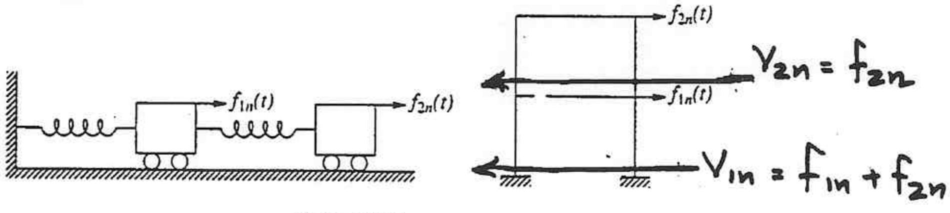

Figure E12.4

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MDOF Systems - Forced Vibration

Given an applied force vector : $\underline{p}(t)$ (N×1 vector)

Suppose $\underline{p}(t)$ has a spatial distribution that does not change with time, and $\underline{s}$ defines the spatial distribution.

Then,

$$\underline{p}(t) = \underset{\substack{\uparrow \\ N\times 1 \\ \text{vector} \\ (\underline{\text{not}}\text{ a function} \\ \text{of time})}}{\underline{s}}\; p(t) \;\leftarrow \text{scalar}$$

Define $\underline{s} = \sum_{r=1}^{N} \underline{s}_r \leftarrow$ spatial pattern contributed to by the r^{th} mode

<u>How do we find $\underline{s}_r$</u> ?

Imagine that $\underline{s}_r$ is a scaled version of the shape, $\underline{\phi}_r$ (actually of $\underline{m}\,\underline{\phi}_r$)

i.e., assume that $\underset{N\times1}{\underline{s}_r} = \underset{\substack{\downarrow \\ (\text{scalar})}}{\Gamma_r} \cdot \underset{N\times N}{\underline{m}}\; \underset{N\times1}{\underline{\phi}_r}$

Since $\underline{s} = \sum_{r=1}^{N} \underline{s}_r = \sum_{r=1}^{N} \Gamma_r \cdot \underline{m} \cdot \underline{\phi}_r$, we can pre-multiply by $\underline{\phi}_n^T$ and use orthogonality.

$$\underline{\phi}_n^T \underline{s} = \sum_{r=1}^{N} \Gamma_r \cdot \underline{\phi}_n^T \cdot \underline{m} \cdot \underline{\phi}_r = \Gamma_n \cdot \underline{\phi}_n^T\, \underline{m}\, \underline{\phi}_n$$

or $\boxed{\Gamma_n = \dfrac{\underline{\phi}_n^T \underline{s}}{M_n}}$ and $\boxed{\underline{s}_n = \Gamma_n\, \underline{m}\, \underline{\phi}_n}$

Generalized Force, $P_n(t) = \underline{\phi}_n^T\, \underline{p}(t) = \underline{\phi}_n^T\, \underline{s}\; p(t)$

OR $\underline{\underline{P_n(t) = \Gamma_n M_n\, p(t)}}$

Thus, equation of motion for the n^{th} mode is now:

Classical Damping can also be included here. $\rightarrow$

$$M_n \ddot{q}_n(t) + K_n q_n(t) = P_n(t) \qquad -(A)$$

where, as before, $M_n = \underline{\phi}_n^T \underline{m} \underline{\phi}_n$

$$K_n = \underline{\phi}_n^T \underline{k} \underline{\phi}_n$$

and $P_n(t) = \Gamma_n M_n p(t) \quad -(B)$

where original load vector $\underline{P}(t) = \underline{s}\, p(t)$

and $\Gamma_n = \dfrac{\underline{\phi}_n^T \underline{s}}{M_n}$

Inserting (B) into (A); we have:

$$M_n \ddot{q}_n(t) + K_n q_n(t) = \Gamma_n M_n p(t)$$

or $\ddot{q}_n(t) + \omega_n^2 q_n(t) = \Gamma_n p(t)$ for the n^{th} mode

For a SDOF system with force $p(t)$, we have:

$$\ddot{D}_n(t) + \omega_n^2 D_n(t) = p(t) \qquad \text{for mass} = 1$$

Thus, $q_n(t) = \Gamma_n D_n(t) \quad —(C)$

Therefore, if we can solve for $D_n(t)$ as we've done in SDOF response analysis (Chs. 1-8), we can use (C) to obtain $q_n(t)$.

Finally, $\underline{u}(t) = \sum_{n=1}^{N} \underline{\phi}_n q_n(t)$.

n^{th} mode contribution to $\underline{u}(t)$ $= \underline{u}_n(t) = \underline{\phi}_n\, q_n(t)$

$$= \underline{\phi}_n \Gamma_n D_n(t)$$

n^{th} mode contribution to equivalent static force, $\underline{f}_s(t)$ $= \underline{f}_{s,n}(t) = \underline{k}\; \underline{u}_n(t)$

$$= \underline{k}\; \underline{\phi}_n \Gamma_n D_n(t)$$

$$= \Gamma_n \left(\omega_n^2\, \underline{m}\, \underline{\phi}_n \right) D_n(t)$$

$$= \omega_n^2\, \underline{s}_n D_n(t)$$

$$\underline{f}_{s,n}(t) = \underline{s}_n \left[\omega_n^2 D_n(t) \right]$$

Assume response from n^{th} mode due to external forces, $\underline{s}_n$ $= r_n^{st}$

Then, dynamic response from nth mode $= r_n(t) = r_n^{st}\left[\omega_n^2 D_n(t)\right]$

Total Response including all N modes $= r(t) = \sum_{n=1}^{N} r_n(t)$

$$r(t) = \sum_{n=1}^{N} r_n^{st} \left[\omega_n^2 D_n(t) \right].$$

Can now determine each mode's contribution to the total response.

Define $\bar{r}_n = \dfrac{r_n^{st}}{r^{st}} = \dfrac{\text{Static response from } n^{th} \text{ mode due to } \underline{s}_n}{\text{Static response due to } \underline{s} \text{ (i.e., all modes)}}$

$\left(\text{Note: } \sum_{n=1}^{N} \bar{r}_n = 1\right)$

We have seen: $r(t) = \sum_{n=1}^{N} r_n^{st} \left[\omega_n^2 D_n(t)\right]$

and $r_n(t) = r_n^{st} \left[\omega_n^2 D_n(t)\right]$

$$\Rightarrow \quad r_n(t) = r^{st}\, \bar{r}_n\, \omega_n^2 D_n(t)$$

Peak value of $r_n(t) = r_{no} = r^{st}\, \bar{r}_n\, \omega_n^2 D_{no}$ — (D)

Recall that $\underbrace{D_{no}}_{\text{Peak Dynamic deformation}} = \underbrace{R_{dn}}_{\text{Deformation Response factor (SDOF)}} \cdot \underbrace{(D_{n,st})_o}_{\text{Peak static deformation}}$

Also, for external force $p(t)$ and mass $= 1$,

$$(D_{n,st})_o = \frac{p_o}{\omega_n^2} \quad \text{since} \quad D_{n,st}(t) = \frac{p(t)}{\omega_n^2}$$

Thus, Eqn (D) reduces to: $r_{no} = p_o \cdot r^{st} \cdot \bar{r}_n \cdot R_{dn}$

r_{no} : Peak n^{th} modal response

r_{no} is made up of 4 parts :

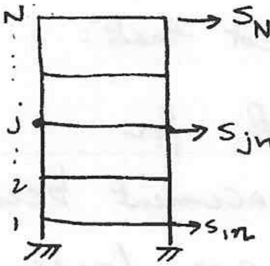
(1) R_{dn} ; <u>dimensionless</u> dynamic ~~deformation~~ response factor for SDOF system w/ force $p(t)$ and w/ ω_n, ξ_n

(2) $\bar{r}_n$; <u>dimensionless</u> modal contribution factor (static)

(3) r_{st} ; static value of response r due to forces $\underline{s}$

(4) p_o ; peak value of $p(t)$.

Modal Contribution



$s_{jn} = \Gamma_n \, m_j \, \phi_{jn}$ (at j^{th} floor)

(I) $\underline{s}_n$ n^{th} mode excitation

<u>Base shear</u> due to n^{th} mode static forces, $\underline{s}_n$ $= V_{bn}^{st}$ $\boxed{r = V_b}$

BASE SHEAR $\quad V_{bn}^{st} = \sum_{j=1}^{N} s_{jn} = \Gamma_n \sum_{j=1}^{N} m_j \, \phi_{jn}$

$$V_{bn} = p_o \cdot V_b^{st} \cdot \bar{V}_{bn} \cdot R_{dn}$$

Finally, $V_b = \sum_{n=1}^{N} V_{bn}$ (combining all modes)

(II) <u>Roof displacement</u> due to n^{th} mode static forces, $\underline{s}_n$ $= u_{Nn}^{st}$ $\boxed{r = u_N}$

Roof ⇒ N^{th} story.

Since $\underline{k}\,\underline{u}_n^{st} = \underline{s}_n$; $\underline{s}_n = \Gamma_n \, \underline{m} \, \underline{\phi}_n = \Gamma_n \, \underline{k} \, \underline{\phi}_n / \omega_n^2$,

$$\underline{u}_n^{st} = \left(\frac{\Gamma_n}{\omega_n^2}\right) \underline{\phi}_n$$

and $\quad u_{Nn}^{st} = \left(\frac{\Gamma_n}{\omega_n^2}\right) \phi_{Nn}$ $\quad$ ROOF DISPLACEMENT

$$u_{Nn} = p_o \cdot u_N^{st} \cdot \bar{u}_{Nn} \cdot R_{dn}$$

$$u_N = \sum_{n=1}^{N} u_{Nn}$$

Modal contribution studies suggest that:

→ more modes need to be retained for base shear than for roof displacement because modal contribution factors ($\bar{r}_n$) are larger for base shear than for roof displacement

→ the number of modes to be retained for sufficient accuracy will depend on the force distribution, $\underline{s}$.

What about R_{dn}?

This factor R_{dn} will depend on modal frequency ω_n and on damping.

It is as important as $\bar{r}_n$

$\bar{r}_n$ depends on spatial distribution of forces, $\underline{s}$

R_{dn} depends on time variation of forces, $p(t)$

If both $\bar{r}_n$ & R_{dn} are small for some modes, those modes can be ignored.

Recap for MDOF Forced Vibration Analysis when $\underline{p}(t) = \underline{s}\, p(t)$

STEP 1. Perform Eigenvalue Analysis & Estimate Damping

Obtain ω_n, $\underline{\phi}_n$; Estimate ζ_n

STEP 2. If $\underline{p}(t) = \underline{s}\, p(t)$,

Obtain $\underline{s}_n = \Gamma_n \underline{m}\, \underline{\phi}_n$

where $\Gamma_n = \dfrac{\underline{\phi}_n^T \underline{s}}{M_n}$ and $M_n = \underline{\phi}_n^T \underline{m}\, \underline{\phi}_n$

Note : $\underline{s} = \sum_{n=1}^{N} \underline{s}_n$ (N = no. of modes)

i.e., Break up shape/distribution $\underline{s}$ of loads into parts that will be defined for each mode

STEP 3 : Perform ONE static analysis with loads, $\underline{s}$

Obtain response, $\boxed{r^{st}}$

STEP 4 : Perform N static analyses with loads, $\underline{s}_n$ for all N modes (i.e, n = 1 to N)

Obtain response, $\boxed{r_n^{st}}$ OR $\boxed{\bar{r}_n}$ where $\bar{r}_n = \dfrac{r_n^{st}}{r^{st}}$

STEP 5 : Obtain SDOF dynamic response ($D_n(t)$ ≡ instantaneous, or D_{n_0} ≡ peak) for each of the N modes (i.e., n = 1 to N). The dynamic response is due to a load p(t) on a SDOF system with mass = 1, and with properties ω_n, ζ_n

n^{th} mode Dynamic Response: $r_n(t) = r_n^{st} \cdot [\omega_n^2 D_n(t)]$

TOTAL RESPONSE ≡ $r(t) = \sum_{n=1}^{N} r_n(t)$

Schematic Summary for obtaining Instantaneous Response, $r(t)$ or Peak Response, r_o

Mode n

STATIC

Static Loads, $\underline{s}_n$

$$\Big\downarrow \text{STEP 4}$$

Response, r_n^{st}

DYNAMIC

Equivalent static Loads, $\underline{f}_n$

$$\underline{f}_n(t) = \underline{k}\,\underline{u}_n(t) = \underline{k}\,\underline{\phi}_n q_n(t) = \omega_n^2\,\underline{m}\,\underline{\phi}_n \cdot \Gamma_n D_n(t)$$

or $$\underline{f}_n(t) = \underline{s}_n\left[\omega_n^2 D_n(t)\right]$$

$$\Big\downarrow \text{scaling relative to STEP 4}$$

$$\boxed{\text{Response} \equiv r_n^{st}\left[\omega_n^2 D_n(t)\right]} \leftarrow r_n(t)$$

where,

$D_n(t)$ = response of SDOF system (mass = 1, frequency = ω_n, damping ratio = ξ_n) when excitation = $p(t)$.

and $q_n(t) = \Gamma_n D_n(t)$ because

$$\ddot{q}_n + 2\xi_n\omega_n\dot{q}_n + \omega_n^2 q_n = \Gamma_n p(t)$$

& $$\ddot{D}_n + 2\xi_n\omega_n\dot{D}_n + \omega_n^2 D_n = p(t)$$

$$r_n(t) = r_n^{st}\left[\omega_n^2 D_n(t)\right]$$

$$= \underbrace{r^{st}}_{\text{STEP 3}} \cdot \underbrace{\bar{r}_n}_{\text{STEP 4}} \cdot \underbrace{\omega_n^2 D_n(t)}_{\text{STEP5}}$$

TOTAL RESPONSE INCLUDING ALL MODES at any time, t $$= \underline{\underline{r(t) = \sum_{n=1}^{N} r_n(t) = \sum_{n=1}^{N} r^{st}\,\bar{r}_n\,\omega_n^2 D_n(t)}}$$

PEAK RESPONSE = for n^{th} mode

$$r_{no} = r^{st}\cdot\bar{r}_n\cdot\omega_n^2 D_{no}$$

$$\underline{\underline{r_{no} = p_o\cdot r^{st}\cdot\bar{r}_n\cdot R_{dn}}}$$

$$\left[D_{no} = (D_{n,st})_o\cdot R_{dn} = \frac{p_o}{\omega_n^2\cdot 1}\cdot R_{dn}\right]$$

PEAK RESPONSE OF n^{th} MODE :

$$r_{no} = \underbrace{p_o}_{\text{LOAD EFFECT}} \cdot \underbrace{r^{st} \cdot \bar{r}_n}_{\text{STATIC EFFECT}} \cdot \underbrace{R_{dn}}_{\text{DYNAMIC EFFECT}}$$

STATIC EFFECT: due to spatial distribution of loads in n^{th} mode (i.e., $\underline{s}_n$)

↓

r^{st} = response due to load $\underline{s}$ (static application); $\bar{r}_n$ = dimensionless modal contribution factor

DYNAMIC EFFECT: This is simply the Dynamic Amplification Factor on displacement for a SDOF system with frequency, ω_n; damping ratio, ξ_n and for load, $p(t)$

- Response "r" can be of any type

e.g., $r \equiv$ Roof displacement of N-story building

n^{th} mode response → $r_n(t) = u_{Nn}(t)$

$\underline{u}_n^{st}$ is such that $\underline{s}_n = \underline{k}\,\underline{u}_n^{st}$

also $\underline{s}_n = \Gamma_n \underline{m}\,\underline{\phi}_n = \frac{\Gamma_n}{\omega_n^2} \cdot \underline{k} \cdot \underline{\phi}_n$] $\Rightarrow$ $\underline{u}_n^{st} = \frac{\Gamma_n}{\omega_n^2}\,\underline{\phi}_n$

$$u_{Nn}^{st} = \frac{\Gamma_n}{\omega_n^2}\,\phi_{Nn}$$ → Static Response [Roof Displacement] of n^{th} mode due to load $\underline{s}_n$

$$\boxed{u_{Nn_o} = p_o \cdot \frac{\Gamma_n}{\omega_n^2}\,\underline{\phi}_n \cdot R_{dn}}$$ ← Peak Roof Displacement of n^{th} mode due to given excitation.

e.g., $r \equiv$ Base Shear of N-story building

n^{th} mode response → $r_n(t) = V_{bn}(t)$

$$V_{bn}^{st} = \sum_{j=1}^{N} s_{jn} = \sum_{j=1}^{N} \Gamma_n\, m_j\, \phi_{jn} = \Gamma_n \sum_{j=1}^{N} m_j\, \phi_{jn}$$ [static Base Shear due to load $\underline{s}_n$ in n^{th} mode]

$$\boxed{V_{bn_o} = p_o \cdot \Gamma_n \sum_{j=1}^{N} m_j\, \phi_{jn} \cdot R_{dn}}$$ ← Peak Base Shear of nth mode due to given excitation

26-10

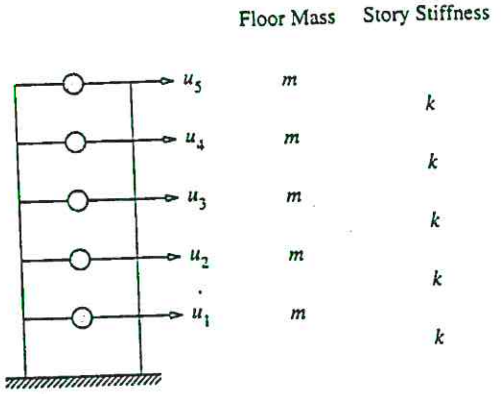


Figure 12.8.1 Uniform five-story shear building.

The mass and stiffness matrices of the structure are

$$\mathbf{m} = m\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \qquad \mathbf{k} = k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

Step 1

Determined by solving the eigenvalue problem, the natural frequencies are

$$\omega_n = \alpha_n \left(\frac{k}{m}\right)^{1/2}$$

where $\alpha_1 = 0.285$, $\alpha_2 = 0.831$, $\alpha_3 = 1.310$, $\alpha_4 = 1.682$, and $\alpha_5 = 1.919$. For a structure with $m = 100$ kips/g, the natural vibration modes, which have been normalized to obtain $M_n = 1$, are (Fig. 12.8.2)

$$\phi_1 = \begin{Bmatrix} 0.334 \\ 0.641 \\ 0.895 \\ 1.078 \\ 1.173 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} -0.895 \\ -1.173 \\ -0.641 \\ 0.334 \\ 1.078 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 1.173 \\ 0.334 \\ -1.078 \\ -0.641 \\ 0.895 \end{Bmatrix} \quad \phi_4 = \begin{Bmatrix} -1.078 \\ 0.895 \\ 0.334 \\ -1.173 \\ 0.641 \end{Bmatrix} \quad \phi_5 = \begin{Bmatrix} 0.641 \\ -1.078 \\ 1.173 \\ -0.895 \\ 0.334 \end{Bmatrix}$$

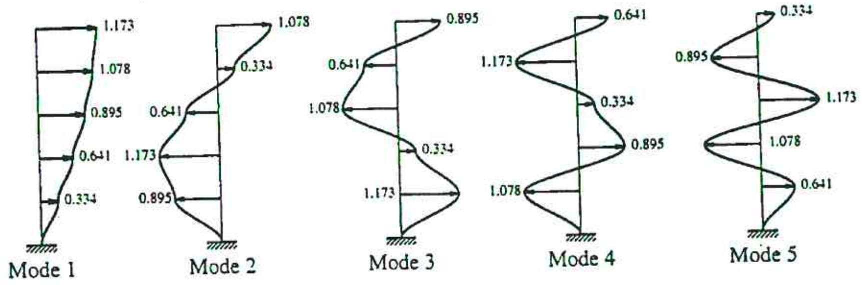


Figure 12.8.2 Natural modes of vibration of uniform five-story shear building.

Natural vibration modes of five-story shear building

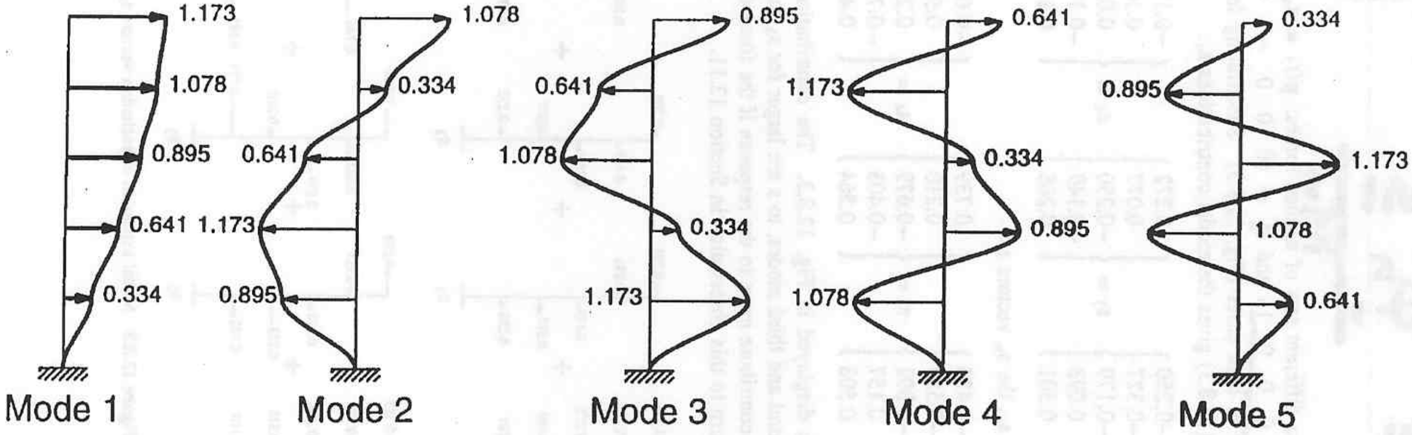


Figure 12.8.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

26-12

Step 2

$$\underline{s}_n = \Gamma_n \, \underline{m} \, \underline{\phi}_n$$

$$\Gamma_n = \frac{\underline{\phi}_n^T \underline{s}}{M_n} \qquad M_n = \underline{\phi}_n^T \underline{m} \, \underline{\phi}_n$$

Consider two different sets of applied forces: $\mathbf{p}(t) = \mathbf{s}_a p(t)$ and $\mathbf{p}(t) = \mathbf{s}_b p(t)$, where $\mathbf{s}_a^T = \langle 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \rangle$ and $\mathbf{s}_b^T = \langle 0 \;\; 0 \;\; 0 \;\; -1 \;\; 2 \rangle$; note that the resultant force is unity in both cases (Fig. 12.8.3). Substituting for $\mathbf{m}$, ϕ_n, and $\mathbf{s} = \mathbf{s}_a$ in Eqs. (12.8.2) and (12.8.3) gives the modal contributions $\mathbf{s}_n$:

$$\mathbf{s}_1 = \left\{\begin{matrix} 0.101 \\ 0.195 \\ 0.272 \\ 0.327 \\ 0.356 \end{matrix}\right\} \quad \mathbf{s}_2 = \left\{\begin{matrix} -0.250 \\ -0.327 \\ -0.179 \\ 0.093 \\ 0.301 \end{matrix}\right\} \quad \mathbf{s}_3 = \left\{\begin{matrix} 0.272 \\ 0.077 \\ -0.250 \\ -0.149 \\ 0.208 \end{matrix}\right\} \quad \mathbf{s}_4 = \left\{\begin{matrix} -0.179 \\ 0.149 \\ 0.055 \\ -0.195 \\ 0.106 \end{matrix}\right\} \quad \mathbf{s}_5 = \left\{\begin{matrix} 0.055 \\ -0.093 \\ 0.101 \\ -0.077 \\ 0.029 \end{matrix}\right\}$$

Similarly, for $\mathbf{s} = \mathbf{s}_b$, the $\mathbf{s}_n$ vectors are

$$\mathbf{s}_1 = \left\{\begin{matrix} 0.110 \\ 0.210 \\ 0.294 \\ 0.354 \\ 0.385 \end{matrix}\right\} \quad \mathbf{s}_2 = \left\{\begin{matrix} -0.423 \\ -0.553 \\ -0.302 \\ 0.157 \\ 0.508 \end{matrix}\right\} \quad \mathbf{s}_3 = \left\{\begin{matrix} 0.739 \\ 0.210 \\ -0.679 \\ -0.403 \\ 0.564 \end{matrix}\right\} \quad \mathbf{s}_4 = \left\{\begin{matrix} -0.685 \\ 0.569 \\ 0.212 \\ -0.746 \\ 0.407 \end{matrix}\right\} \quad \mathbf{s}_5 = \left\{\begin{matrix} 0.259 \\ -0.436 \\ 0.475 \\ -0.363 \\ 0.135 \end{matrix}\right\}$$

These vectors are displayed in Fig. 12.8.3. The contributions of the higher modes, especially the second and third modes, to $\mathbf{s}$ are larger for $\mathbf{s}_b$ than for $\mathbf{s}_a$, suggesting that these modes may contribute more to the response if the force distribution is $\mathbf{s}_b$ than if it is $\mathbf{s}_a$. We will return to this observation in Section 12.11.

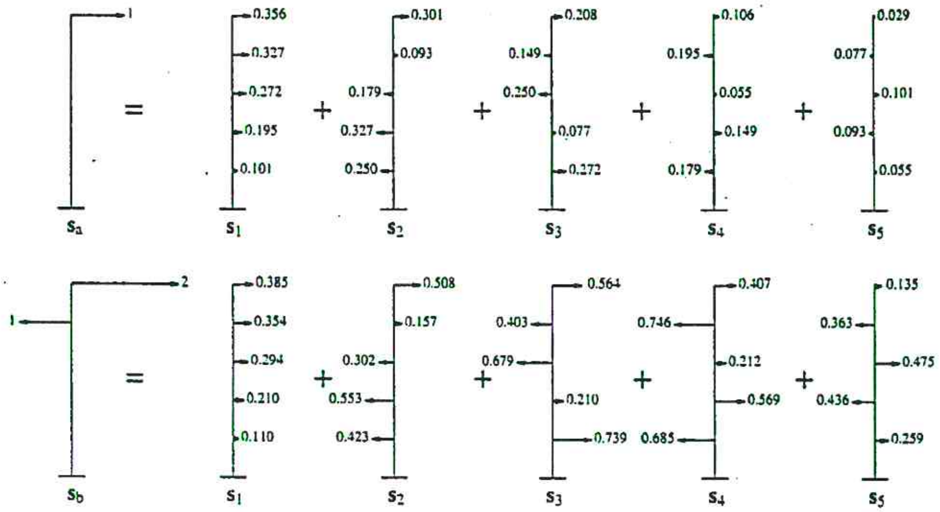


Figure 12.8.3 Modal expansion of excitation vectors $\mathbf{s}_a$ and $\mathbf{s}_b$.

Modal expansion of excitation vectors $\mathbf{s}_a$ and $\mathbf{s}_b$

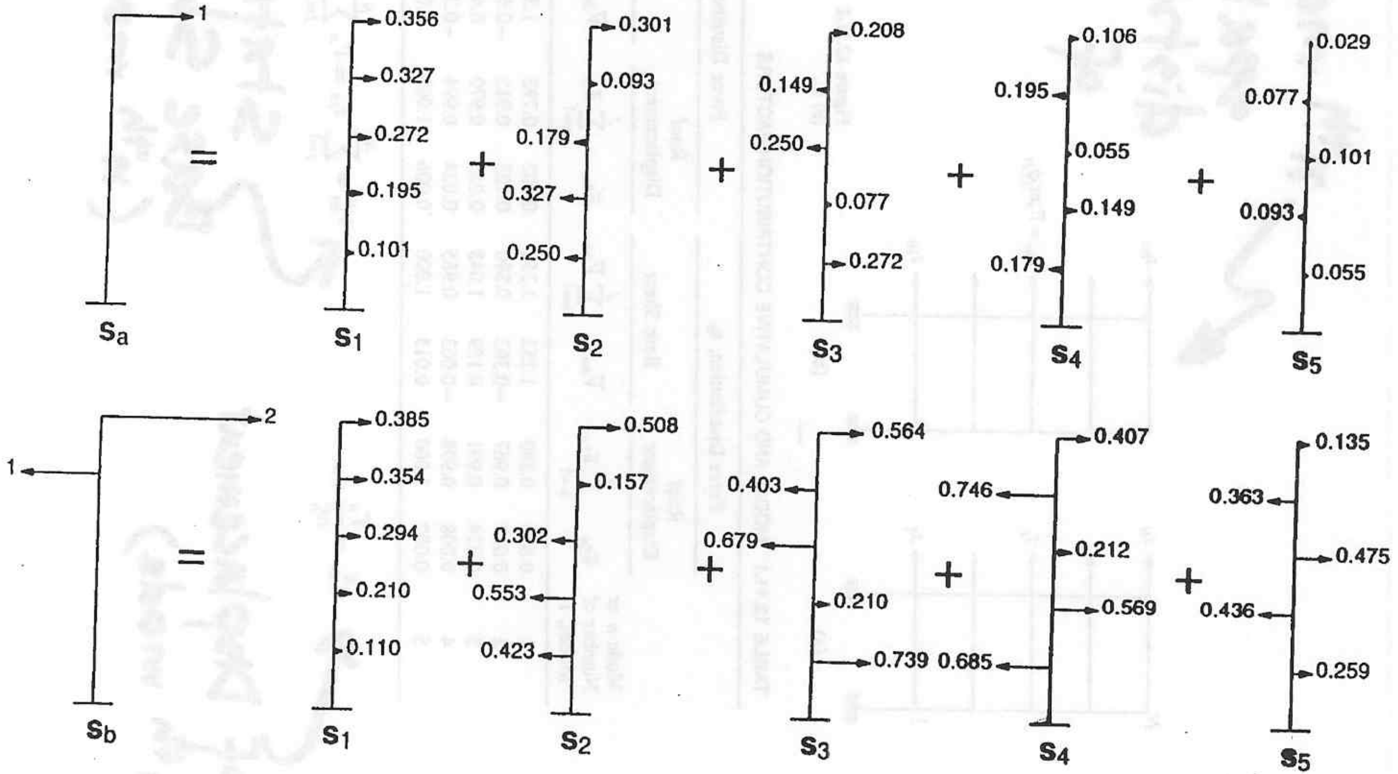


Figure 12.8.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

4

nth mode spatial distribution of forces

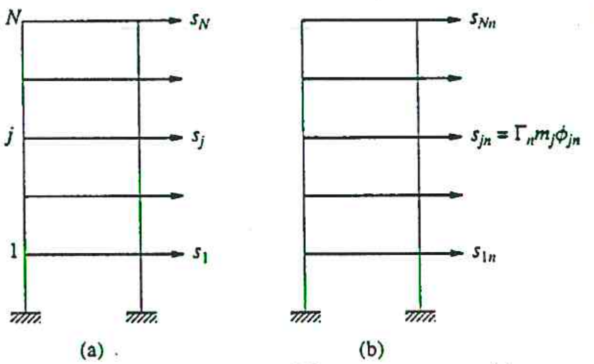


Figure 12.11.1 External forces: (a) **s**: (b) $\mathbf{s}_n$.

TABLE 12.11.1 MODAL AND CUMULATIVE CONTRIBUTION FACTORS

	Force Distribution, $\mathbf{s}_a$				Force Distribution, $\mathbf{s}_b$			
	Roof Displacement		Base Shear		Roof Displacement		Base Shear	
Mode n or Number of Modes, J	$\overline{u}_{5n}$	$\sum_{n=1}^{J} \overline{u}_{5n}$	$\overline{V}_{bn}$	$\sum_{n=1}^{J} \overline{V}_{bn}$	$\overline{u}_{5n}$	$\sum_{n=1}^{J} \overline{u}_{5n}$	$\overline{V}_{bn}$	$\sum_{n=1}^{J} \overline{V}_{bn}$
1	0.880	0.880	1.252	1.252	0.792	0.792	1.353	1.353
2	0.087	0.967	−0.362	0.890	0.123	0.915	−0.612	0.741
3	0.024	0.991	0.159	1.048	0.055	0.970	0.431	1.172
4	0.008	0.998	−0.063	0.985	0.024	0.994	−0.242	0.930
5	0.002	1.000	0.015	1.000	0.006	1.000	0.070	1.000

Step 3

$$u_{Nn}^{st} = \frac{\Gamma_n}{\omega_n^2} \phi_{Nn}$$

Static Roof Displacement (nth mode)

$$V_{bn}^{st} = \sum_{j=1}^{N} s_{jn} = \Gamma_n \sum_{j=1}^{N} m_j \phi_{jn}$$

Static Base Shear (nth mode)

Dynamic response factor for harmonic force

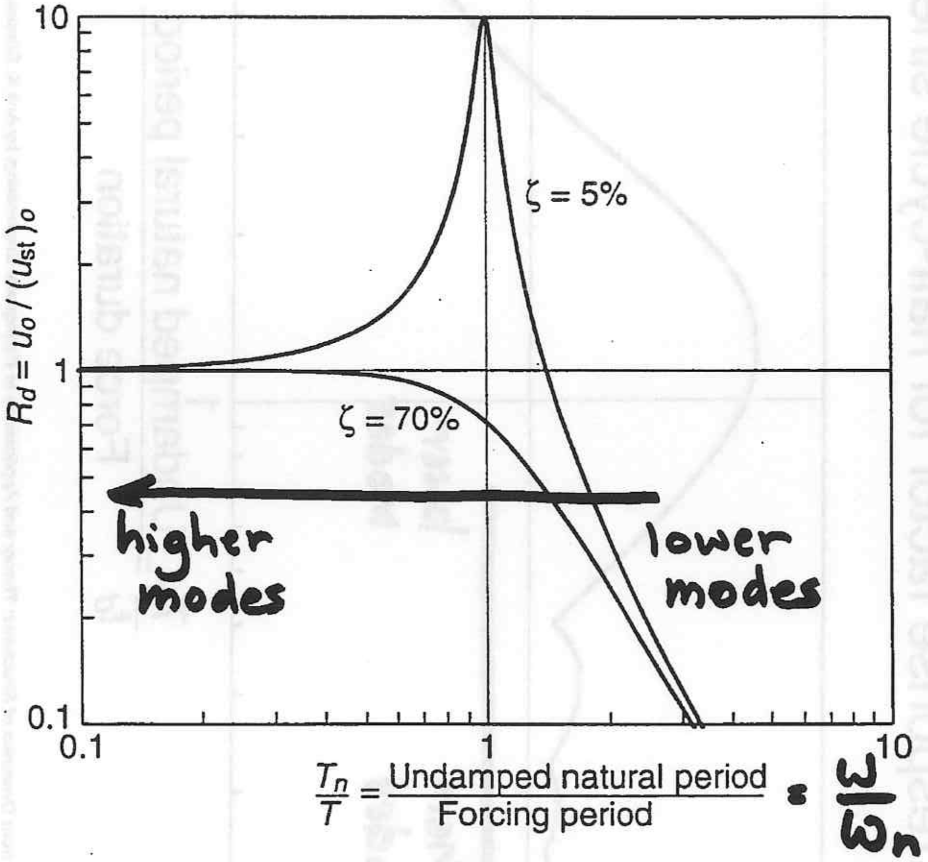


Figure 12.11.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Dynamic response factor for half-cycle sine pulse force

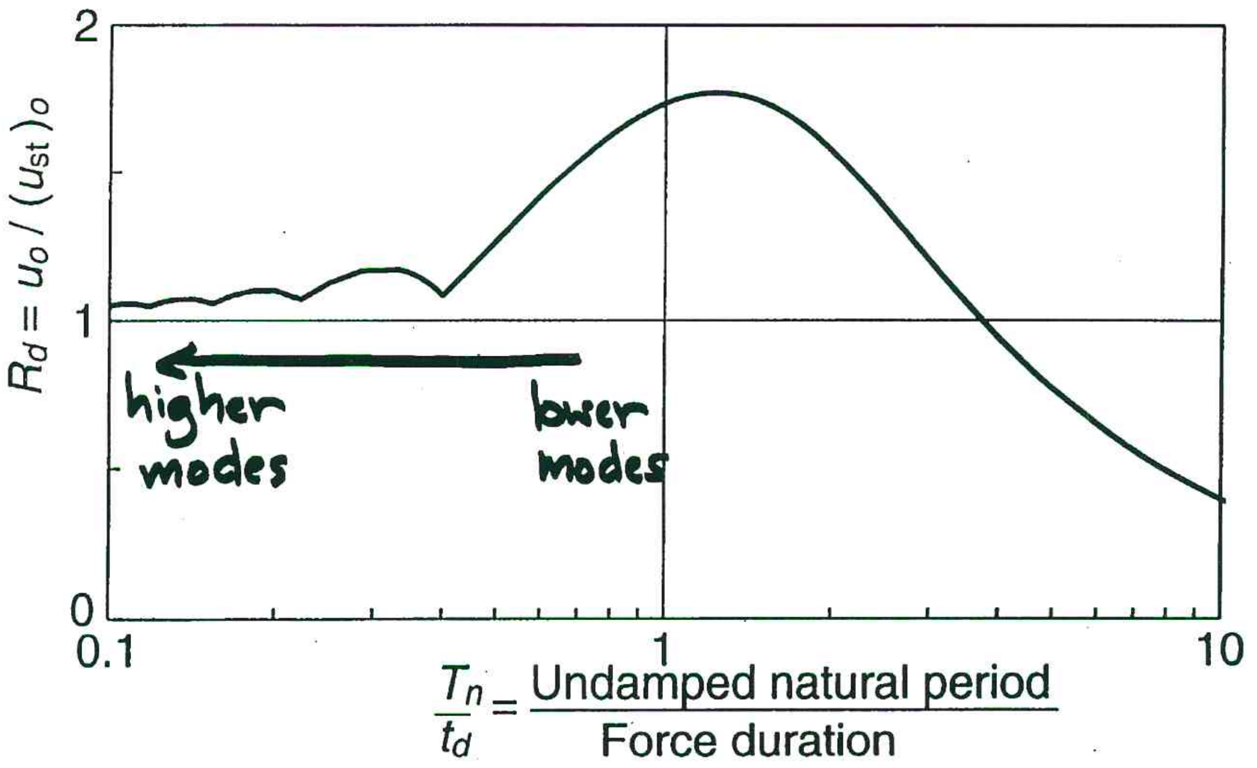


Figure 12.11.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Dynamic response factors R_{dn}

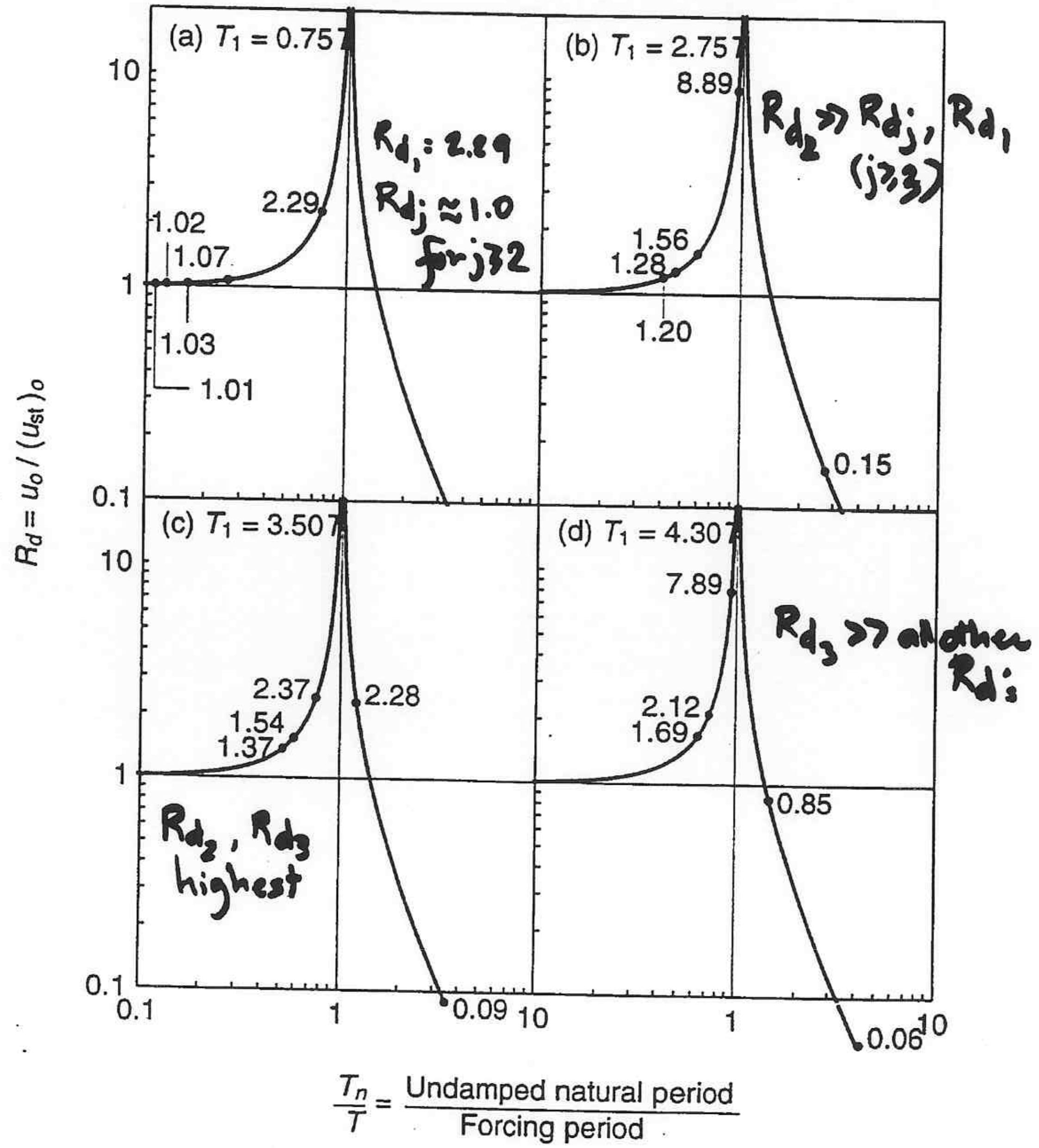


Figure 12.11.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*. by Anil K. Chopra, Prentice-Hall, 1995.

see discussion on pg. 494 in connection with Ξ_b shape and V_b response.

~~28-1~~
27-1

Forced vibration analysis for a multi-degree-of-freedom system when the time-varying excitation has an unchanging spatial distribution, s

Units $kip := 1000 \cdot lbf$ $ksi := \frac{kip}{in^2}$

Define the structural model

$E := 30000 \cdot ksi$ $h_{typ} := 10 \cdot ft$ $I_{typ} := 4320 \cdot in^4$ $m_{typ} := 4 \cdot kip \cdot \frac{s^2}{in}$ $n_{floor} := 2$

$floor := 1 .. n_{floor}$

$m_{1,1} := m_{typ}$

$h_{floor} := h_{typ}$

$m_{2,2} := \frac{1}{2} \cdot m_{typ}$

$h = \begin{pmatrix} 10 \\ 10 \end{pmatrix} ft$

$I_{floor} := I_{typ}$

$kflr_{floor} := 2 \cdot 12 \cdot \frac{E \cdot I_{floor}}{\left(h_{floor}\right)^3}$ $kflr = \begin{pmatrix} 1800.0 \\ 1800.0 \end{pmatrix} \frac{kip}{in}$

$i := 1 .. n_{floor}$ $j := 1 .. n_{floor}$

$k_{i,j} := 0 \cdot \frac{kip}{in}$

$k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \frac{kip}{in}$

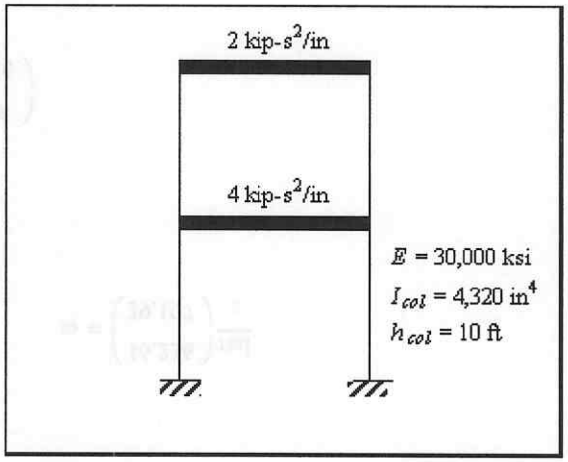
$k_{i,i} := kflr_i$

$i := 1 .. n_{floor} - 1$

$k_{i,i} := k_{i,i} + k_{i+1,i+1}$

$k_{i,i+1} := -kflr_{i+1}$

$k_{i+1,i} := -kflr_{i+1}$



Physical stiffness and mass matrices

$$k = \begin{pmatrix} 3600.0 & -1800.0 \\ -1800.0 & 1800.0 \end{pmatrix} \frac{\text{kip}}{\text{in}} \qquad m = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \text{kip} \cdot \frac{s^2}{\text{in}}$$

Eigenvalues and eigenvectors (modal frequencies and mode shapes)

$$\text{genvals}(k, m) = \begin{pmatrix} 263.604 \\ 1536.396 \end{pmatrix} \frac{\text{rad}}{s^2} \qquad \omega := \sqrt{\text{genvals}(k, m)} \qquad \omega = \begin{pmatrix} 16.236 \\ 39.197 \end{pmatrix} \frac{\text{rad}}{s}$$

$$n_{mode} := n_{floor}$$

$$mode := 1 .. n_{mode}$$

$$floor := 1 .. n_{floor}$$

$$\Phi_{floor,\, mode} := \text{genvecs}(k, m)_{floor,\, mode} \qquad \Phi = \begin{pmatrix} 0.577 & 0.577 \\ 0.816 & -0.816 \end{pmatrix}$$

$$T_n := \frac{2 \cdot \pi}{\omega}$$

$$\omega = \begin{pmatrix} 16.236 \\ 39.197 \end{pmatrix} \frac{\text{rad}}{\text{sec}} \qquad T_n = \begin{pmatrix} 0.387 \\ 0.160 \end{pmatrix} s$$

$$\Phi^{\langle mode \rangle} := \frac{\Phi^{\langle mode \rangle}}{\left(\Phi^{\langle mode \rangle}\right)_{n_{floor}}} \qquad \Phi = \begin{pmatrix} 0.707 & -0.707 \\ 1.000 & 1.000 \end{pmatrix}$$

Mode shapes normalized
so that roof displacement = 1

~~28-3~~
27-3

Sketch the mode shapes

$ii := 1 .. n_{floor} + 1$

$$H_{ii} := \left| \begin{array}{l} 0 \cdot ft \;\; \text{if } ii = 1 \\ \sum_{j=1}^{ii-1} h_j \;\; \text{otherwise} \end{array} \right.$$

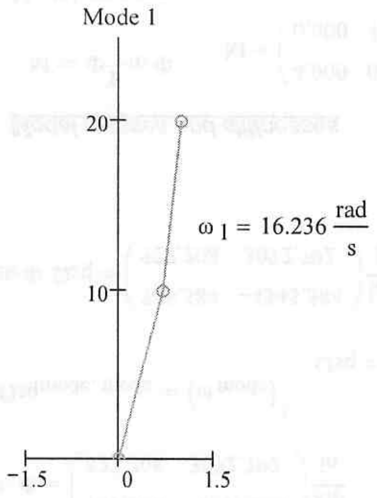
$$H = \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix} ft$$

$mode := 1 .. n_{mode}$

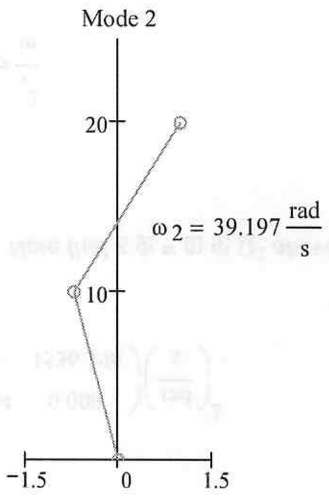
$zeroes_{1, mode} := 0$

$\Phi_{sketch} := stack(zeroes, \Phi)$

$$\Phi_{sketch} = \begin{pmatrix} 0.000 & 0.000 \\ 0.707 & -0.707 \\ 1.000 & 1.000 \end{pmatrix}$$



$\omega_1 = 16.236 \frac{rad}{s}$



$\omega_2 = 39.197 \frac{rad}{s}$

Verification of the eigenvalues and eigenvectors

$$k \cdot \Phi = \begin{pmatrix} 745.584 & -4345.584 \\ 527.208 & 3072.792 \end{pmatrix} \frac{\text{kip}}{\text{in}}$$

$$\Omega sq_{\text{mode, mode}} := \left(\omega_{\text{mode}}\right)^2 \qquad \Omega sq = \begin{pmatrix} 263.604 & 0.000 \\ 0.000 & 1536.396 \end{pmatrix} \left(\frac{\text{rad}}{\text{s}}\right)^2$$

$$m \cdot \Phi \cdot \Omega sq = \begin{pmatrix} 745.584 & -4345.584 \\ 527.208 & 3072.792 \end{pmatrix} \frac{\text{kip}}{\text{in}}$$

Note that $\underline{k}\ \underline{\Phi} = \underline{m}\ \underline{\Phi}\ \underline{\Omega}^2$ *above*

<u>Modal masses and stiffnesses</u>

$$M := \Phi^T \cdot m \cdot \Phi \qquad M = \begin{pmatrix} 4.000 & 0.000 \\ 0.000 & 4.000 \end{pmatrix} \text{kip} \cdot \frac{s^2}{\text{in}}$$

Modal masses

$$M_{\text{mode, mode}} = \begin{pmatrix} 4.000 \\ 4.000 \end{pmatrix} \text{kip} \cdot \frac{s^2}{\text{in}}$$

$$K := \Phi^T \cdot k \cdot \Phi \qquad K = \begin{pmatrix} 1054.416 & -1.227 \times 10^{-13} \\ 2.632 \times 10^{-13} & 6145.584 \end{pmatrix} \frac{\text{kip}}{\text{in}}$$

Modal stiffnesses

$$K_{\text{mode, mode}} = \begin{pmatrix} 1054.416 \\ 6145.584 \end{pmatrix} \frac{\text{kip}}{\text{in}}$$

Note: Off-diagonal elements of $\underline{M}$ *and* $\underline{K}$ *are approximately zero above (as they should be).*

Harmonic excitation with unchanging spatial pattern, s

$p_0 := 5000 \cdot \text{kip}$ $\quad f_0 := 3.0 \cdot \text{Hz}$ $\quad \omega_0 := 2 \cdot \pi \cdot f_0$ $\quad \omega_0 = 18.850 \dfrac{\text{rad}}{\text{s}}$

$$s := \begin{pmatrix} 0.5 \\ 1.0 \end{pmatrix}$$

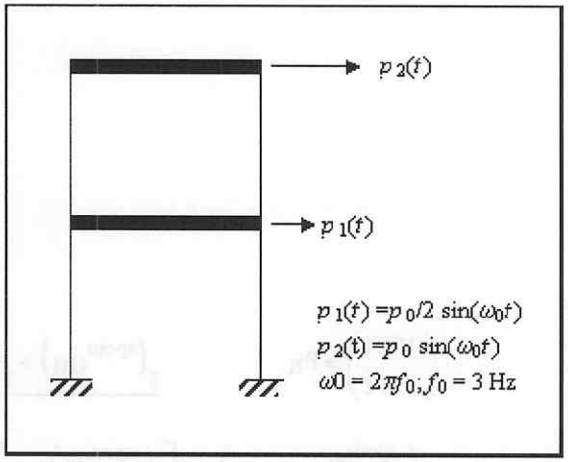
$$\Gamma_{\text{mode}} := \frac{\displaystyle\sum_{\text{floor}=1}^{n_{\text{floor}}} \Phi_{\text{floor, mode}} \cdot s_{\text{floor}}}{M_{\text{mode, mode}}}$$

$$\Gamma = \begin{pmatrix} 0.338 \\ 0.162 \end{pmatrix} \frac{\text{in}}{\text{kip} \cdot \text{sec}^2}$$ Note the units!

$$\text{smode}_{\text{floor, mode}} := \Gamma_{\text{mode}} \cdot \sum_{\text{ifloor}=1}^{n_{\text{floor}}} m_{\text{floor, ifloor}} \cdot \Phi_{\text{ifloor, mode}}$$

$$\text{smode} = \begin{pmatrix} 0.957 & -0.457 \\ 0.677 & 0.323 \end{pmatrix}$$

$$\sum_{j=1}^{2} \text{smode}_{1,j} = 0.500 \qquad \sum_{j=1}^{2} \text{smode}_{2,j} = 1.000$$



Check! ... Correct shape.

Steady-state Response

Consider UNDAMPED case first

$$\text{mode} := 1 .. n_{\text{mode}}$$

$$\zeta_{\text{mode}} := 0.0 \qquad \frac{\omega_0}{\omega_1} = 1.161 \qquad \frac{\omega_0}{\omega_2} = 0.481$$

$$CC_{\text{mode}} := \frac{1 - \left(\frac{\omega_0}{\omega_{\text{mode}}}\right)^2}{\left[1 - \left(\frac{\omega_0}{\omega_{\text{mode}}}\right)^2\right]^2 + \left(2 \cdot \zeta_{\text{mode}} \cdot \frac{\omega_0}{\omega_{\text{mode}}}\right)^2} \qquad DD_{\text{mode}} := \frac{-2 \cdot \zeta_{\text{mode}} \cdot \frac{\omega_0}{\omega_{\text{mode}}}}{\left[1 - \left(\frac{\omega_0}{\omega_{\text{mode}}}\right)^2\right]^2 + \left(2 \cdot \zeta_{\text{mode}} \cdot \frac{\omega_0}{\omega_{\text{mode}}}\right)^2}$$

Chap. 3

$$CC = \begin{pmatrix} -2.875 \\ 1.301 \end{pmatrix} \qquad DD = \begin{pmatrix} 0.000 \\ 0.000 \end{pmatrix} \qquad R_{d_{\text{mode}}} := \sqrt{\left(CC_{\text{mode}}\right)^2 + \left(DD_{\text{mode}}\right)^2} \qquad R_d = \begin{pmatrix} 2.875 \\ 1.301 \end{pmatrix}$$

Note: First mode is closer to resonance.

~~28-7~~
27-7

$i := 1 .. 501$

$\Delta t := \frac{2 \cdot \pi}{\omega_2} \cdot \frac{1}{20}$ $\qquad t_i := (i - 1) \cdot \Delta t$

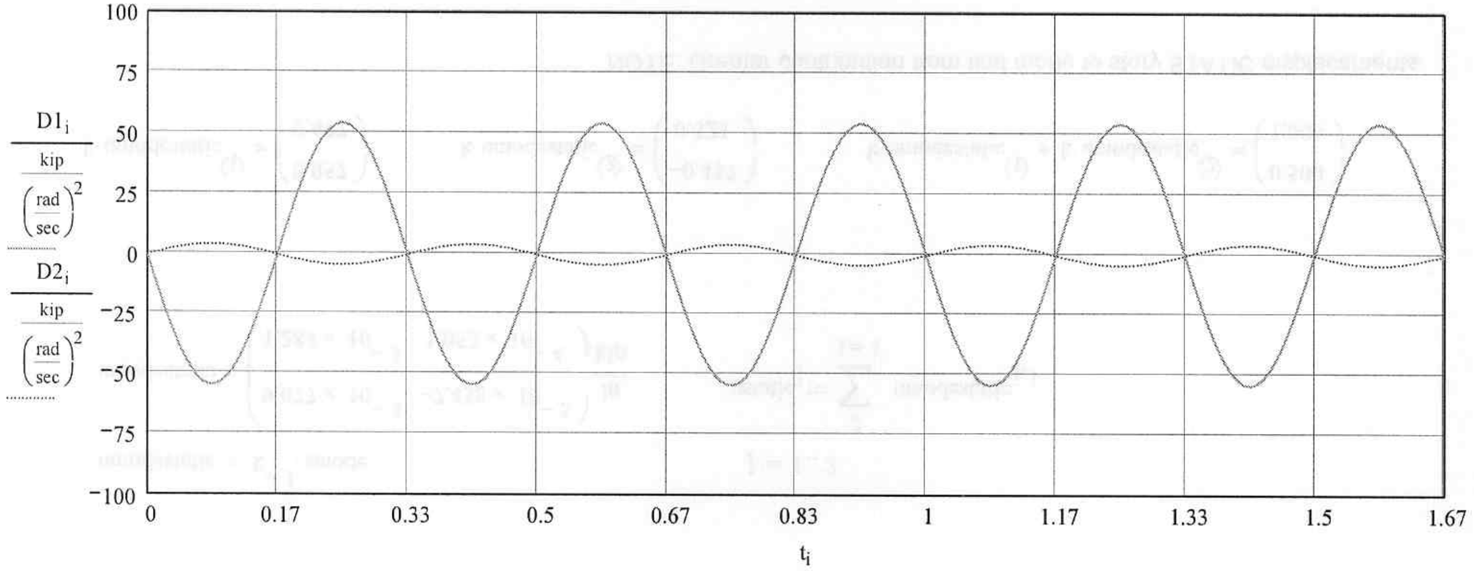
i^{th} SDOF response (mass = 1, dpg ratio = ζ_i, stiffness = ω_i^2).
Force = $p_0 \sin(\omega_0 t)$

Mode 1

Mode 2

$$D1_i := \frac{p0}{(\omega_1)^2} \cdot \left(CC_1 \cdot \sin(\omega_0 \cdot t_i) + DD_1 \cdot \cos(\omega_0 \cdot t_i)\right) \qquad D2_i := \frac{p0}{(\omega_2)^2} \cdot \left(CC_2 \cdot \sin(\omega_0 \cdot t_i) + DD_2 \cdot \cos(\omega_0 \cdot t_i)\right)$$

$$\max(D1) = 54.523 \frac{kip}{\left(\frac{rad}{sec}\right)^2} \qquad \max(D2) = 4.233 \frac{kip}{\left(\frac{rad}{sec}\right)^2} \qquad \frac{p0}{(\omega_1)^2} = 18.968 \frac{kip}{\left(\frac{rad}{sec}\right)^2} \qquad \frac{p0}{(\omega_2)^2} = 3.254 \frac{kip}{\left(\frac{rad}{sec}\right)^2}$$



~~28-8~~
27-8

RESPONSE: Story displacements

$\text{umodestatic} := k^{-1} \cdot \text{smode}$

$\text{umodestatic} = \begin{pmatrix} 9.077 \times 10^{-4} & -7.438 \times 10^{-5} \\ 1.284 \times 10^{-3} & 1.052 \times 10^{-4} \end{pmatrix} \dfrac{\text{in}}{\text{kip}}$

$j := 1 .. 2$

$\text{ustatic}_j := \sum_{i=1}^{2} \text{umodestatic}_{j,i}$

$k \cdot \text{umodestatic}^{\langle 1 \rangle} = \begin{pmatrix} 0.957 \\ 0.677 \end{pmatrix}$

$k \cdot \text{umodestatic}^{\langle 2 \rangle} = \begin{pmatrix} -0.457 \\ 0.323 \end{pmatrix}$

$k \cdot \text{umodestatic}^{\langle 1 \rangle} + k \cdot \text{umodestatic}^{\langle 2 \rangle} = \begin{pmatrix} 0.500 \\ 1.000 \end{pmatrix}$

NOTE: Greater contribution from first mode to story STATIC displacements.

$\text{smode} = \begin{pmatrix} 0.957 & -0.457 \\ 0.677 & 0.323 \end{pmatrix}$

$s = \begin{pmatrix} 0.500 \\ 1.000 \end{pmatrix}$

~~28-9~~
27-9

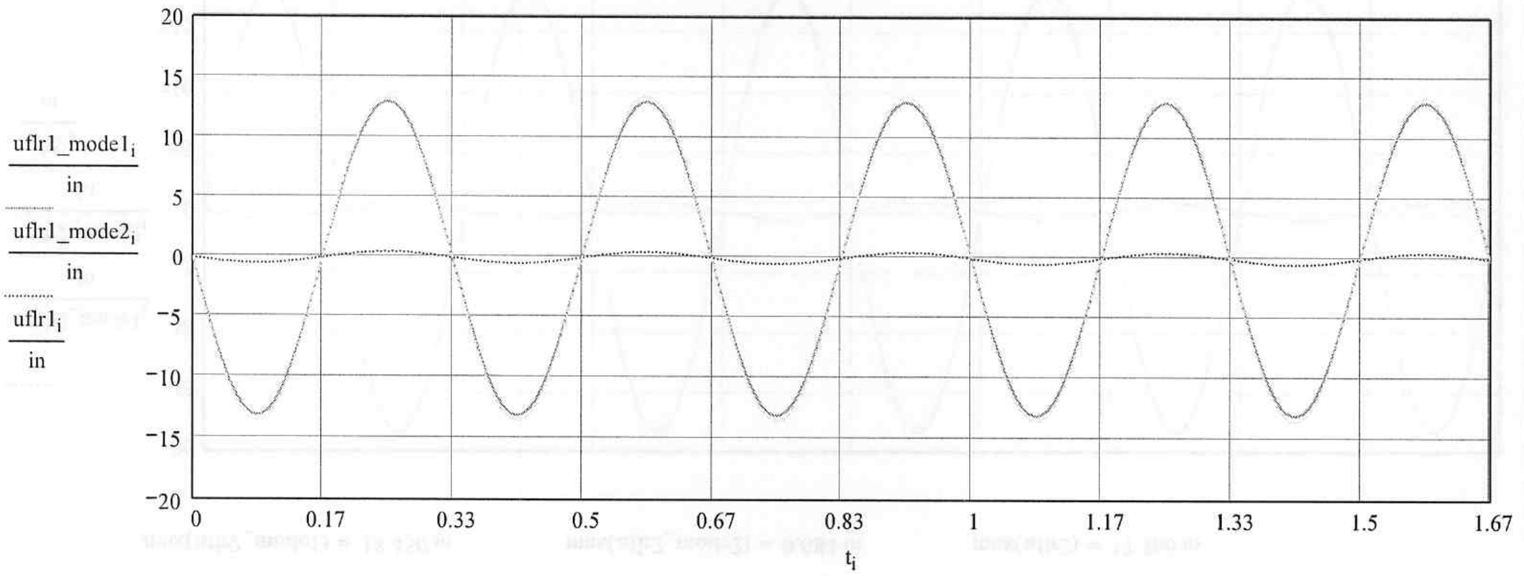
Steady-state Response

FIRST FLOOR DISPLACEMENTS

$$\text{uflr1\_mode1}_i := \text{umodestatic}_{1,1} \cdot \left[(\omega_1)^2 \cdot D1_i \right] \qquad \text{uflr1\_mode2}_i := \text{umodestatic}_{1,2} \cdot \left[(\omega_2)^2 \cdot D2_i \right]$$

$$\text{uflr1}_i := \text{uflr1\_mode1}_i + \text{uflr1\_mode2}_i$$

$\max(\text{uflr1\_mode1}) = 13.046 \text{ in}$ $\max(\text{uflr1\_mode2}) = 0.484 \text{ in}$ $\max(\text{uflr1}) = 13.530 \text{ in}$



Note: Contributions from Modes 1 and 2 are in phase with each other.
Also note that the second mode is very insignificant in the total response.

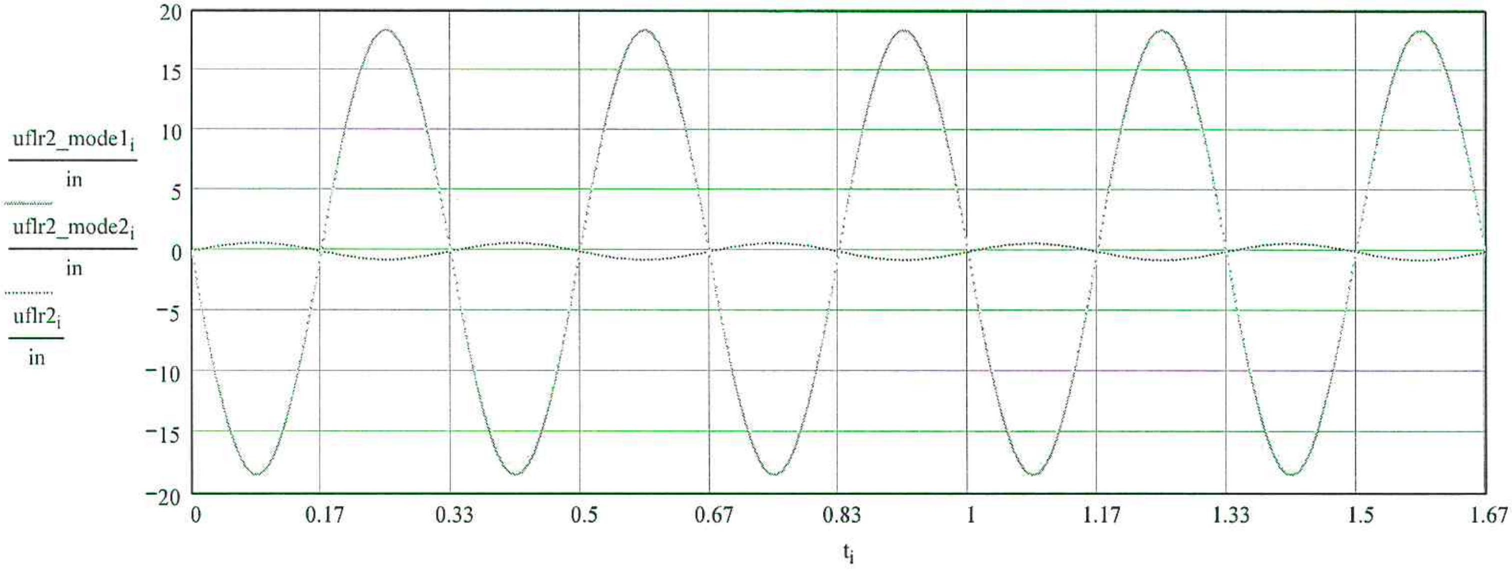
~~28-10~~
27-10

SECOND FLOOR DISPLACEMENTS

$uflr2\_mode1_i := umodestatic_{2,1} \cdot \left[(\omega_1)^2 \cdot D1_i\right]$ $\qquad$ $uflr2\_mode2_i := umodestatic_{2,2} \cdot \left[(\omega_2)^2 \cdot D2_i\right]$

$uflr2_i := uflr2\_mode1_i + uflr2\_mode2_i$

$\max(uflr2\_mode1) = 18.450\ in$ $\qquad$ $\max(uflr2\_mode2) = 0.684\ in$ $\qquad$ $\max(uflr2) = 17.766\ in$



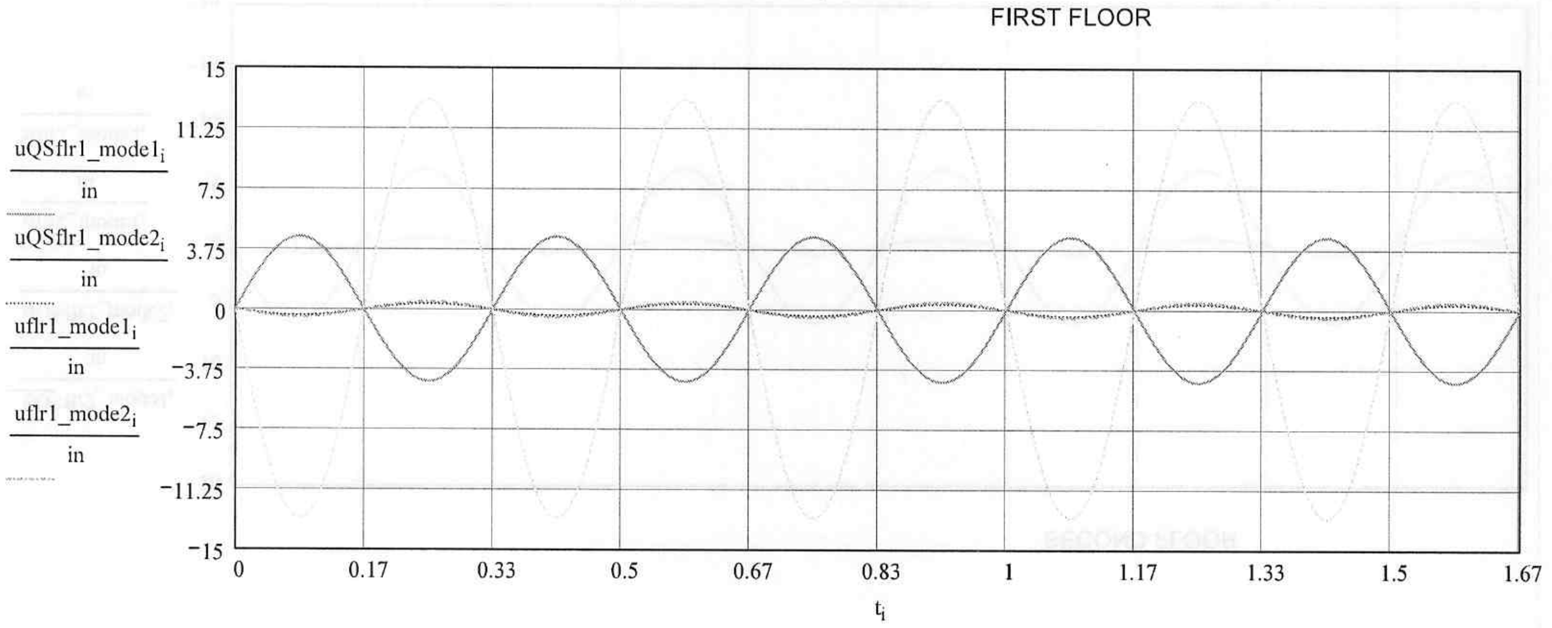
Note: Contributions from Modes 1 and 2 are exactly out of phase with each other.

~~28-11~~
27-11

$$DQS1_i := \frac{p0}{(\omega_1)^2} \cdot \sin(\omega_0 \cdot t_i) \qquad DQS2_i := \frac{p0}{(\omega_2)^2} \cdot \sin(\omega_0 \cdot t_i)$$

$$uQSflr1\_mode1_i := umodestatic_{1,1} \cdot \left[(\omega_1)^2 \cdot DQS1_i\right] \qquad uQSflr1\_mode2_i := umodestatic_{1,2} \cdot \left[(\omega_2)^2 \cdot DQS2_i\right]$$

$$uQSflr1_i := uQSflr1\_mode1_i + uQSflr1\_mode2_i$$

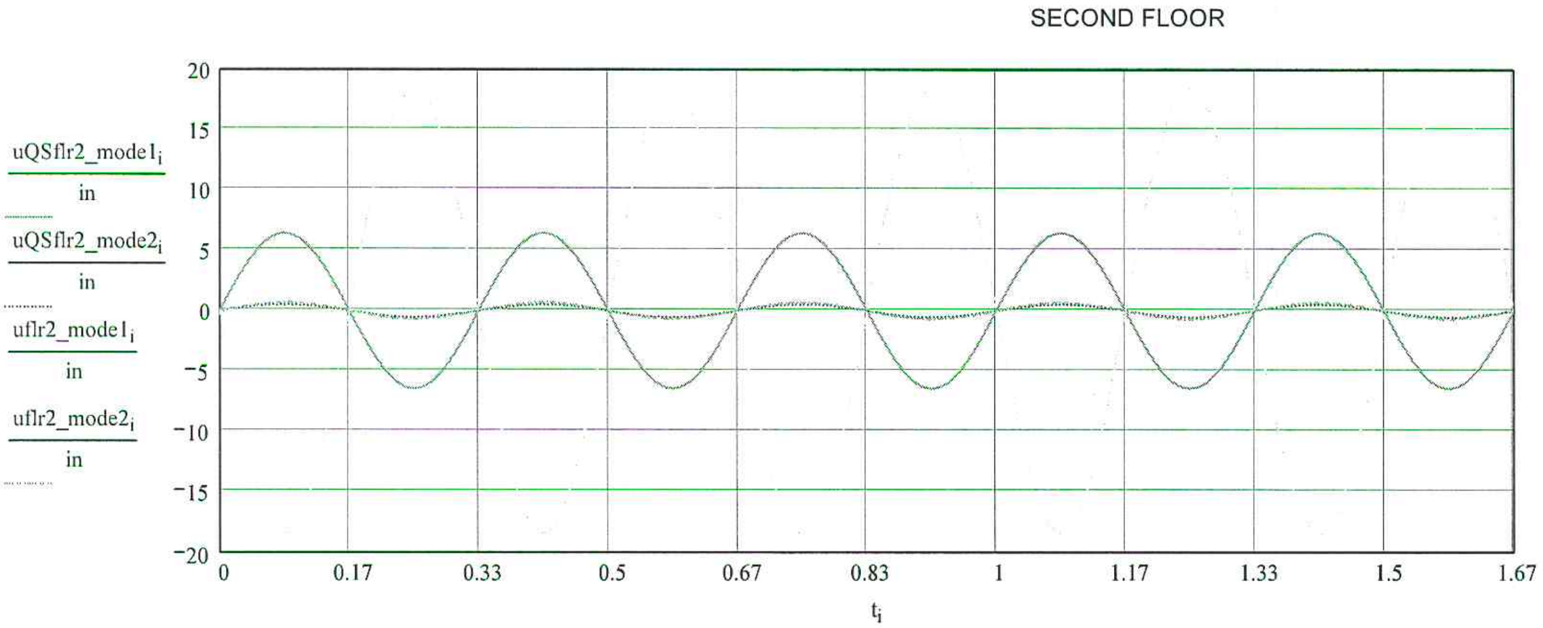


Note: Greater dynamic amplification on mode 1.

~~28-11~~
27-12

$$uQSflr2\_mode1_i := umodestatic_{2,1} \cdot \left[\left(\omega_1 \right)^2 \cdot DQS1_i \right] \qquad uQSflr2\_mode2_i := umodestatic_{2,2} \cdot \left[\left(\omega_2 \right)^2 \cdot DQS2_i \right]$$

$$uQSflr2_i := uQSflr2\_mode1_i + uQSflr2\_mode2_i$$



Note: Again, greater dynamic amplification on mode 1.

~~28-13~~
27-13

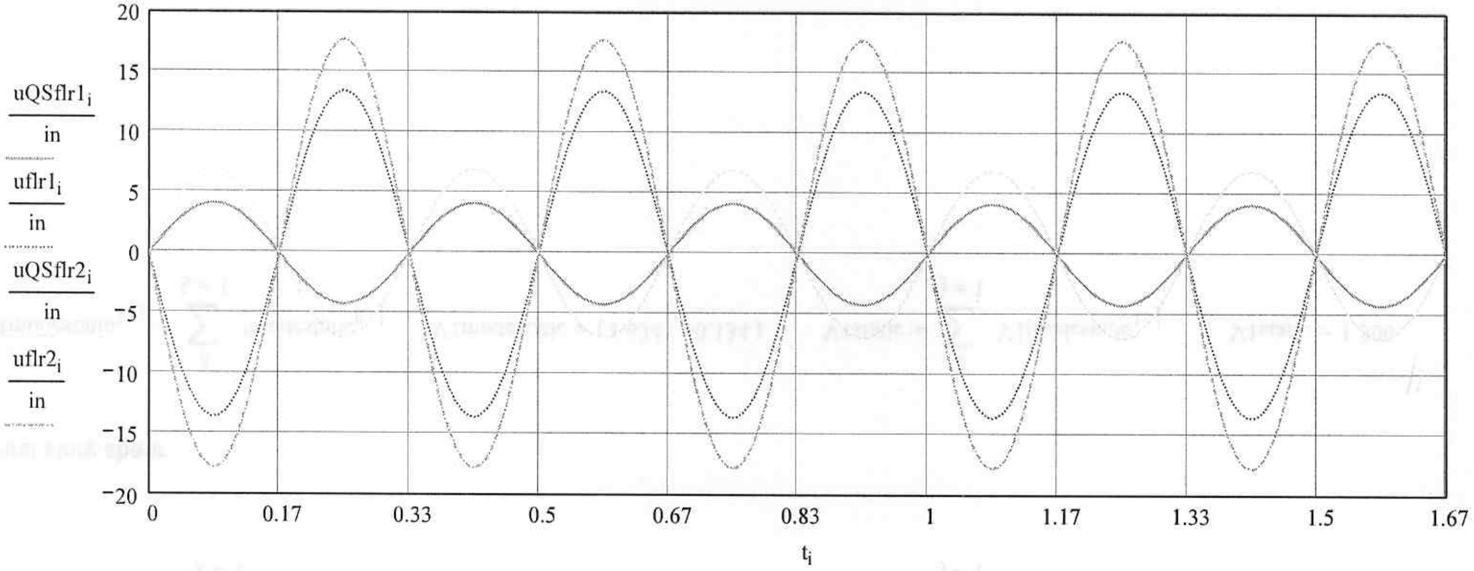


Figure above shows importance of dynamics at each story displacement (after combining both modes)

~~28-14~~
27-14

RESPONSE: Story shears

Get equivalent static forces first

$\text{fmodestatic} := \text{smode}$

$$\text{fmodestatic} = \begin{pmatrix} 0.957 & -0.457 \\ 0.677 & 0.323 \end{pmatrix}$$

$j := 1 .. 2$

Second story shear

$$\text{V2modestatic}_{1,j} := \sum_{k=2}^{2} \text{fmodestatic}_{k,j} \qquad \text{V2modestatic} = (0.677 \quad 0.323) \qquad \text{V2static} := \sum_{j=1}^{2} \text{V2modestatic}_{1,j} \qquad \text{V2static} = 1.000$$

First story shear

$$\text{V1modestatic}_{1,j} := \sum_{k=1}^{2} \text{fmodestatic}_{k,j} \qquad \text{V1modestatic} = (1.634 \quad -0.134) \qquad \text{V1static} := \sum_{j=1}^{2} \text{V1modestatic}_{1,j} \qquad \text{V1static} = 1.500$$

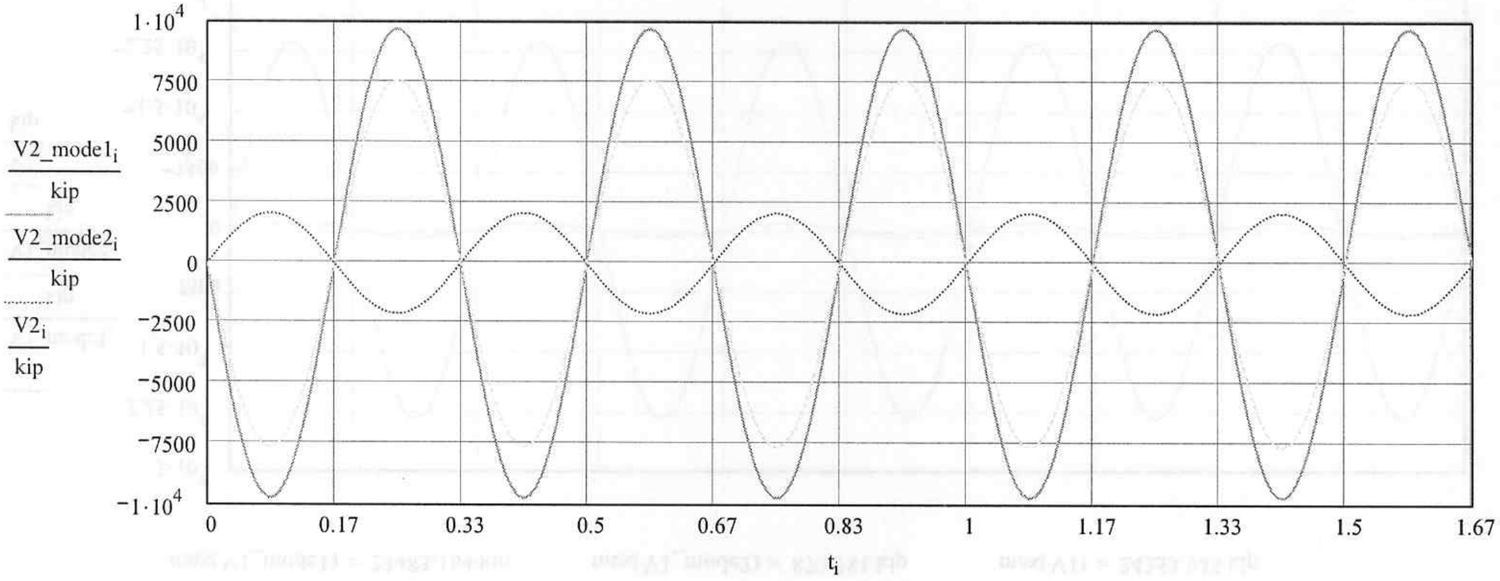
Steady-state Response

SECOND STORY SHEARS

$V2\_mode1_i := V2modestatic_{1,1} \cdot \left[(\omega_1)^2 \cdot D1_i \right]$ $\qquad$ $V2\_mode2_i := V2modestatic_{1,2} \cdot \left[(\omega_2)^2 \cdot D2_i \right]$

$V2_i := V2\_mode1_i + V2\_mode2_i$

$\max(V2\_mode1) = 9727.045 \text{ kip}$ $\qquad$ $\max(V2\_mode2) = 2102.286 \text{ kip}$ $\qquad$ $\max(V2) = 7624.794 \text{ kip}$



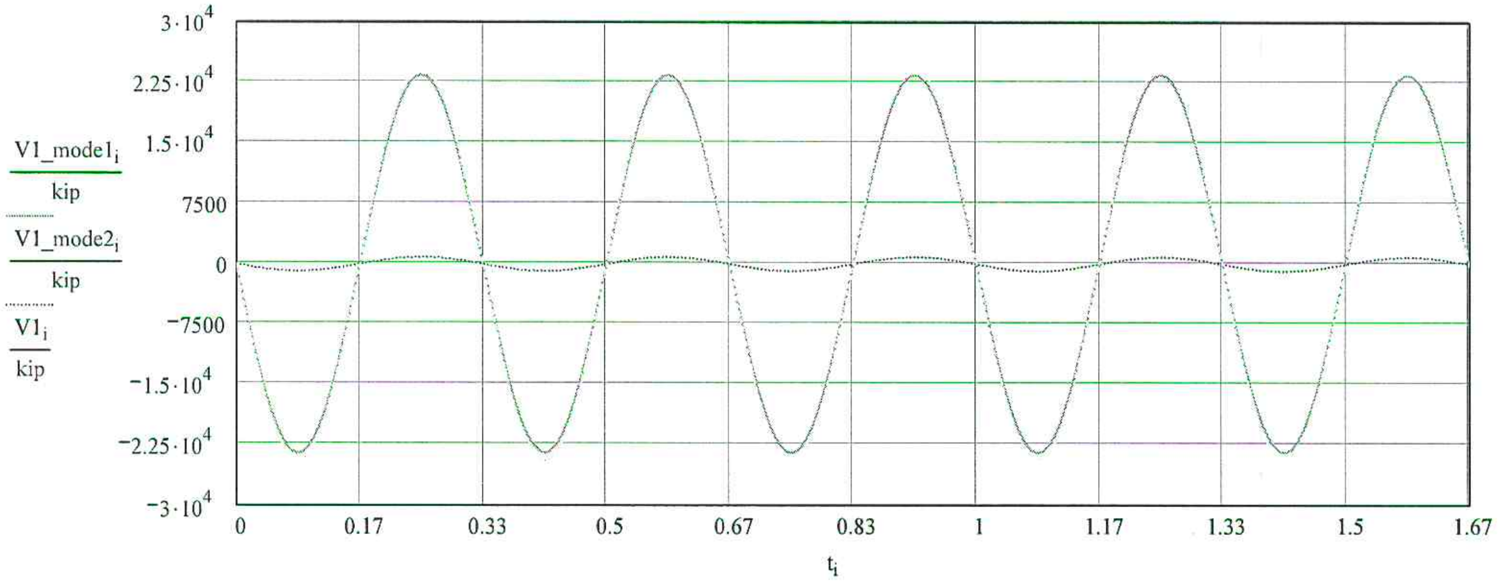
~~28-16~~
27-16

FIRST STORY SHEARS (i.e., Base shear)

$V1\_mode1_i := V1modestatic_{1,1} \cdot \left[(\omega_1)^2 \cdot D1_i\right]$ $V1\_mode2_i := V1modestatic_{1,2} \cdot \left[(\omega_2)^2 \cdot D2_i\right]$

$V1_i := V1\_mode1_i + V1\_mode2_i$

max(V1_mode1) = 23483.164 kip max(V1_mode2) = 870.781 kip max(V1) = 24353.945 kip



Steady-state Response

CHANGE DAMPING TO 10%

$$\text{mode} := 1 .. n_{\text{mode}}$$

$$\zeta_{\text{mode}} := 0.10 \qquad \frac{\omega_0}{\omega_1} = 1.161 \qquad \frac{\omega_0}{\omega_2} = 0.481$$

$$CC_{\text{mode}} := \frac{1 - \left(\frac{\omega_0}{\omega_{\text{mode}}}\right)^2}{\left[1 - \left(\frac{\omega_0}{\omega_{\text{mode}}}\right)^2\right]^2 + \left(2 \cdot \zeta_{\text{mode}} \cdot \frac{\omega_0}{\omega_{\text{mode}}}\right)^2} \qquad DD_{\text{mode}} := \frac{-2 \cdot \zeta_{\text{mode}} \cdot \frac{\omega_0}{\omega_{\text{mode}}}}{\left[1 - \left(\frac{\omega_0}{\omega_{\text{mode}}}\right)^2\right]^2 + \left(2 \cdot \zeta_{\text{mode}} \cdot \frac{\omega_0}{\omega_{\text{mode}}}\right)^2}$$

Chap. 3

$$CC = \begin{pmatrix} -1.989 \\ 1.281 \end{pmatrix} \qquad DD = \begin{pmatrix} -1.327 \\ -0.160 \end{pmatrix} \qquad R_{d_{\text{mode}}} := \sqrt{\left(CC_{\text{mode}}\right)^2 + \left(DD_{\text{mode}}\right)^2} \qquad R_d = \begin{pmatrix} 2.391 \\ 1.291 \end{pmatrix}$$

Note: Smaller amplifciations this time.

~~28-18~~
27-18

Mode 1

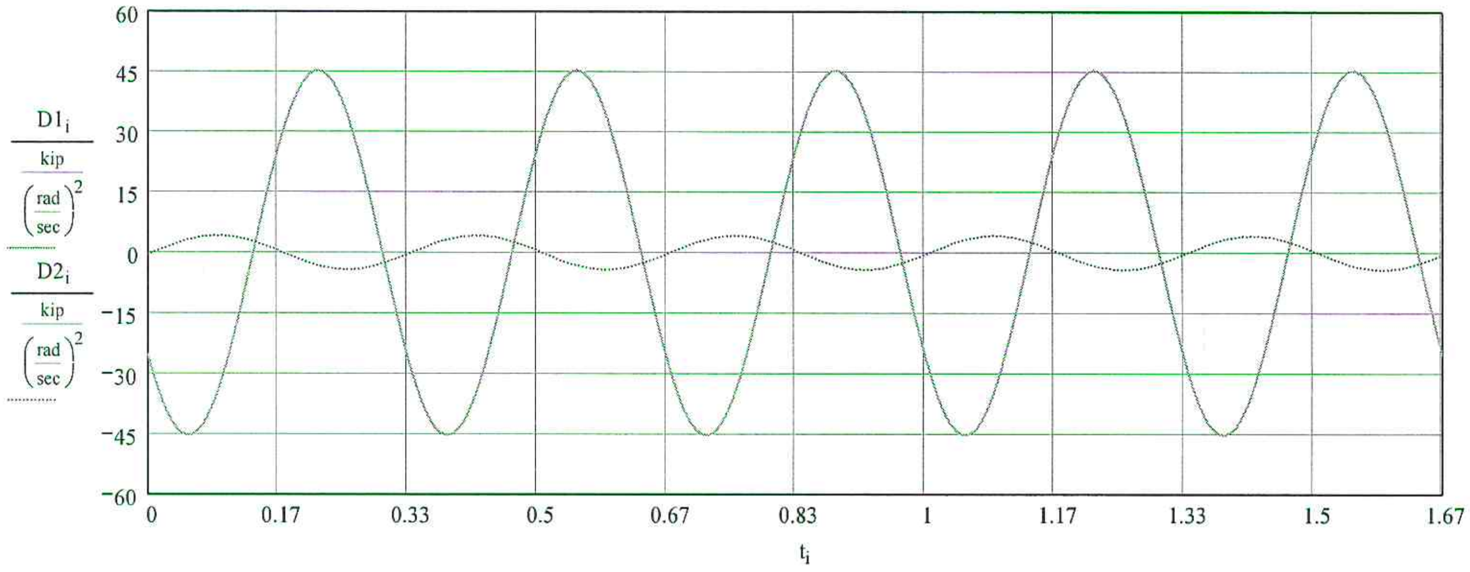
Mode 2

$$D1_i := \frac{p0}{(\omega_1)^2} \cdot \left(CC_1 \cdot \sin(\omega_0 \cdot t_i) + DD_1 \cdot \cos(\omega_0 \cdot t_i)\right)$$

$$D2_i := \frac{p0}{(\omega_2)^2} \cdot \left(CC_2 \cdot \sin(\omega_0 \cdot t_i) + DD_2 \cdot \cos(\omega_0 \cdot t_i)\right)$$

$$\max(D1) = 45.350 \frac{kip}{\left(\frac{rad}{sec}\right)^2} \qquad \max(D2) = 4.201 \frac{kip}{\left(\frac{rad}{sec}\right)^2}$$

$$\frac{p0}{(\omega_1)^2} = 18.968 \frac{kip}{\left(\frac{rad}{sec}\right)^2} \qquad \frac{p0}{(\omega_2)^2} = 3.254 \frac{kip}{\left(\frac{rad}{sec}\right)^2}$$



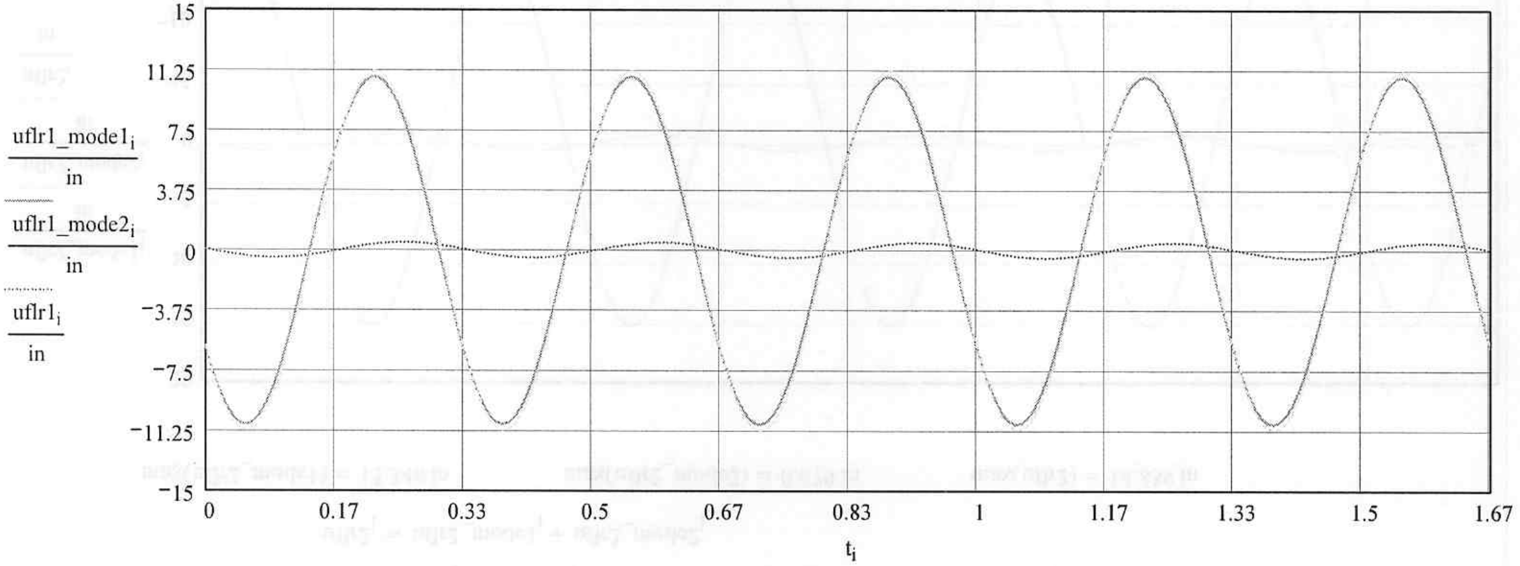
Steady-state Response

FIRST FLOOR DISPLACEMENTS

$\mathrm{uflr1\_mode1}_i := \mathrm{umodestatic}_{1,1} \cdot \left[\left(\omega_1 \right)^2 \cdot \mathrm{D1}_i \right]$ $\mathrm{uflr1\_mode2}_i := \mathrm{umodestatic}_{1,2} \cdot \left[\left(\omega_2 \right)^2 \cdot \mathrm{D2}_i \right]$

$\mathrm{uflr1}_i := \mathrm{uflr1\_mode1}_i + \mathrm{uflr1\_mode2}_i$

$\max(\mathrm{uflr1\_mode1}) = 10.851 \text{ in}$ $\max(\mathrm{uflr1\_mode2}) = 0.480 \text{ in}$ $\max(\mathrm{uflr1}) = 11.219 \text{ in}$



Note: Contributions from Modes 1 and 2 are NOT in phase with each other any more (when damping is present).
The second mode is still insignificant in the total response.

~~28-19~~
27-19

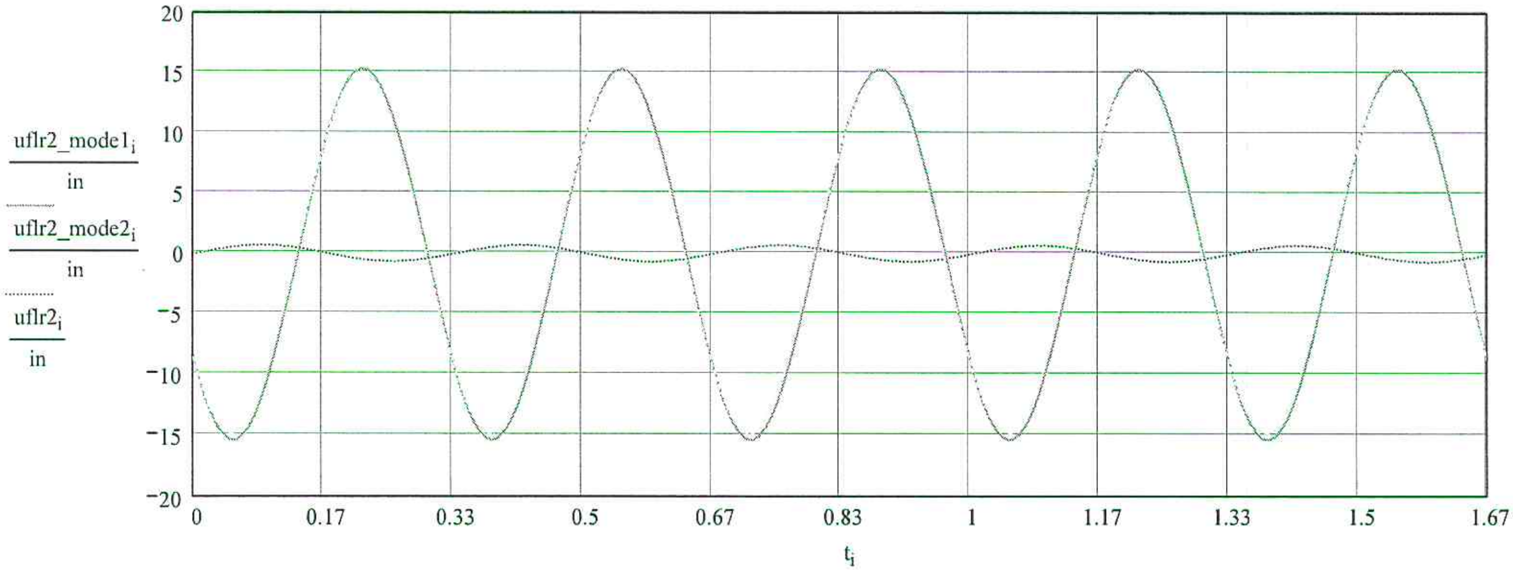
~~20-20~~
27-20

SECOND FLOOR DISPLACEMENTS

$uflr2\_mode1_i := umodestatic_{2,1} \cdot \left[(\omega_1)^2 \cdot D1_i \right]$ $uflr2\_mode2_i := umodestatic_{2,2} \cdot \left[(\omega_2)^2 \cdot D2_i \right]$

$uflr2_i := uflr2\_mode1_i + uflr2\_mode2_i$

max(uflr2_mode1) = 15.346 in max(uflr2_mode2) = 0.679 in max(uflr2) = 14.839 in



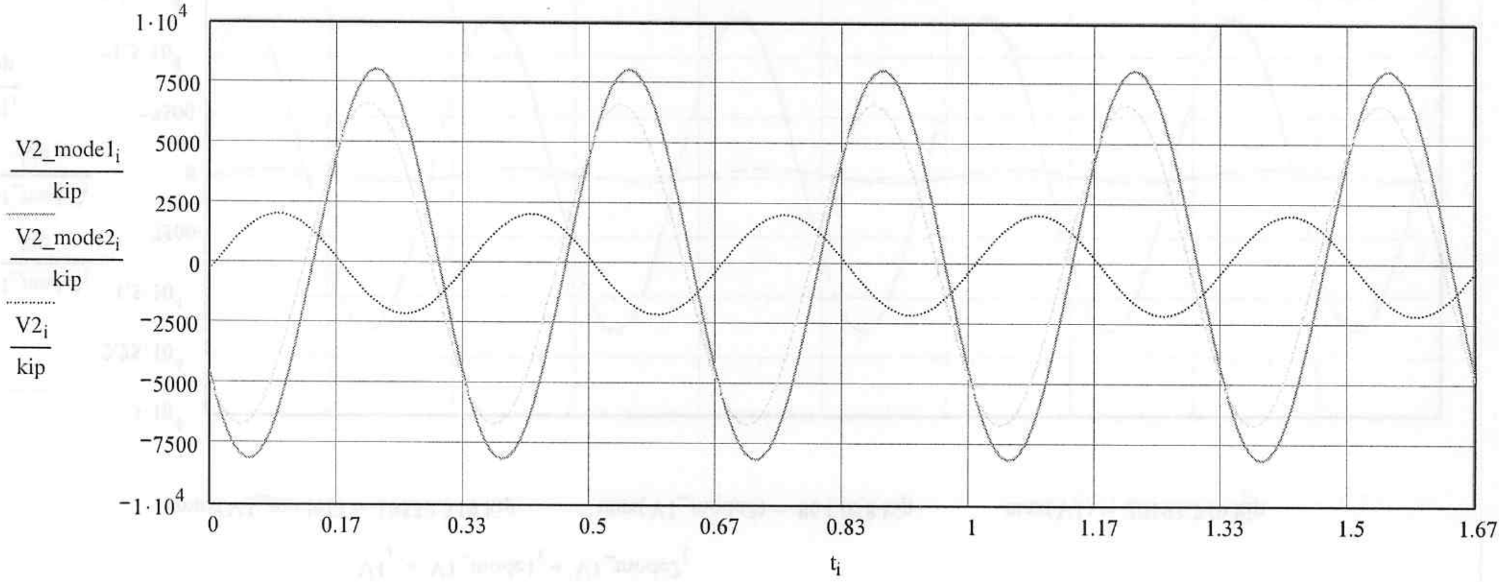
Steady-state Response

SECOND STORY SHEARS

$$V2\_mode1_i := V2modestatic_{1,1} \cdot \left[(\omega_1)^2 \cdot D1_i \right] \qquad V2\_mode2_i := V2modestatic_{1,2} \cdot \left[(\omega_2)^2 \cdot D2_i \right]$$

$$V2_i := V2\_mode1_i + V2\_mode2_i$$

$\max(V2\_mode1) = 8090.549\ \text{kip}$ $\max(V2\_mode2) = 2086.025\ \text{kip}$ $\max(V2) = 6653.901\ \text{kip}$



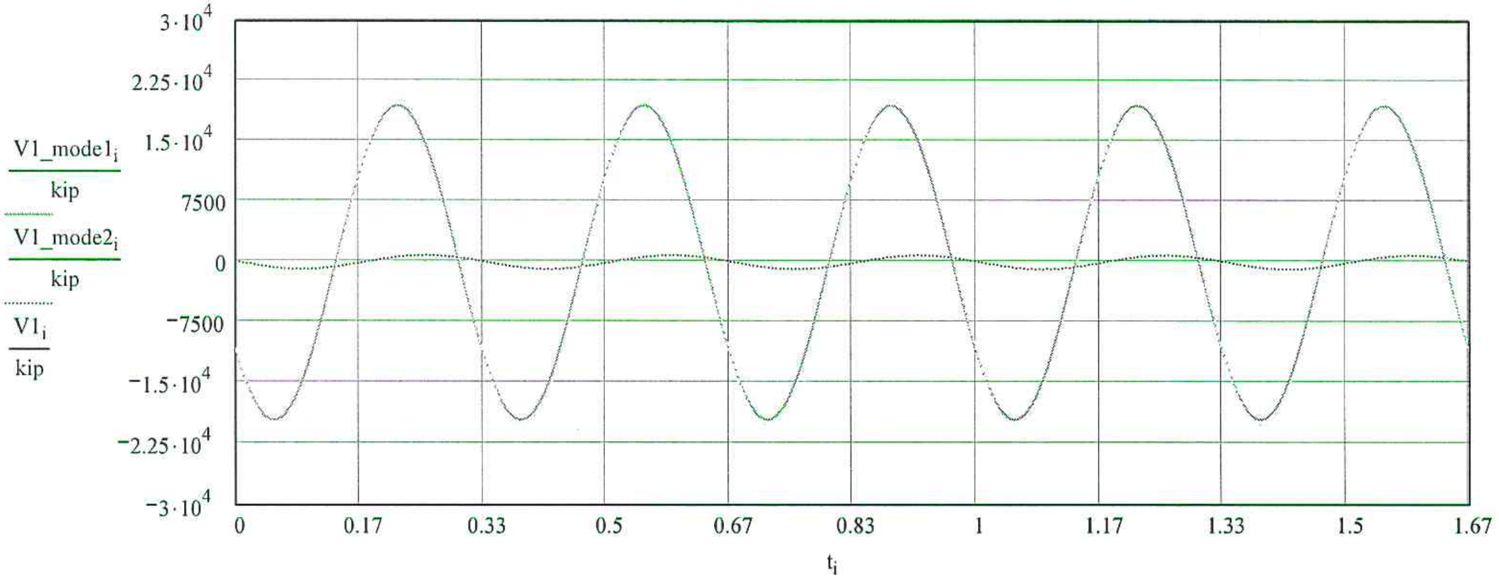
27-21

FIRST STORY SHEARS (i.e., Base shear)

$V1\_mode1_i := V1modestatic_{1,1} \cdot \left[(\omega_1)^2 \cdot D1_i \right]$ $V1\_mode2_i := V1modestatic_{1,2} \cdot \left[(\omega_2)^2 \cdot D2_i \right]$

$V1_i := V1\_mode1_i + V1\_mode2_i$

$\max(V1\_mode1) = 19532.314\ \text{kip}$ $\max(V1\_mode2) = 864.058\ \text{kip}$ $\max(V1) = 20193.749\ \text{kip}$



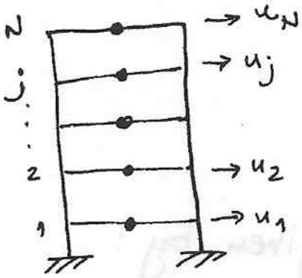
Seismic Response of MDOF Systems

Equations of motion

$$\underline{m}\,\ddot{\underline{u}} + \underline{k}\,\underline{u} = -\underline{m}\,\underline{1}\,\ddot{u}_g(t)$$

(N-story building)

OR



$$\underset{N\times N}{\underline{m}}\,\underset{N\times 1}{\ddot{\underline{u}}} + \underset{N\times N}{\underline{c}}\,\underset{N\times 1}{\dot{\underline{u}}} + \underset{N\times N}{\underline{k}}\,\underset{N\times 1}{\underline{u}} = -\underset{N\times N}{\underline{m}}\,\underset{N\times 1}{\underline{1}}\,\ddot{u}_g(t)$$

$\ddot{u}_g(t)$ ← scalar 1×1

$\underline{1}$ = vector of N one's $\left\{\begin{matrix}1\\1\\\vdots\\1\end{matrix}\right\}$

$$\underline{p}_{eff}(t) = -\underline{m}\,\underline{1}\,\ddot{u}_g(t)$$

Effective earthquake force vector

Replace $\underline{s}$ in general formulation for spatial pattern of loads by $\underline{m}\,\underline{1}$

Then, $\underline{m}\,\underline{1} = \sum_{n=1}^{N} \underline{s}_n = \sum_{n=1}^{N} \Gamma_n\,\underline{m}\,\underline{\phi}_n$

where $\Gamma_n = \dfrac{\underline{\phi}_n^T\,\underline{s}}{M_n} = \dfrac{\underline{\phi}_n^T\,\underline{m}\,\underline{1}}{M_n}$

Note: $\underline{\phi}_n^T\,\underline{m}\,\underline{1} = \sum_{j=1}^{N} m_j\,\phi_{jn}$ (call this L_n^h)

$\Gamma_n = \dfrac{L_n^h}{M_n}$ $\left(M_n = \underline{\phi}_n^T\,\underline{m}\,\underline{\phi}_n = \sum_{j=1}^{N} m_j\,\phi_{jn}^2\right)$

$\underline{s}_n = \Gamma_n\,\underline{m}\,\underline{\phi}_n$ (n^{th} mode contribution to $\underline{m}\,\underline{1}$)

$s_{jn} = \Gamma_n\,m_j\,\phi_{jn}$ (n^{th} mode contribution to $\underline{m}\,\underline{1}$ at story j)

$\underline{s} = \underline{M}\cdot\underline{1}$

$$\ddot{q}_n(t) + 2\xi_n\omega_n\dot{q}_n(t) + \omega_n^2 q_n(t) = -\Gamma_n \ddot{u}_g(t)$$

n^{th} mode SDOF system's equation of motion

Compare with equation of SDOF system (in Chapter 6)

$$\ddot{D}_n(t) + 2\xi_n\omega_n\dot{D}_n(t) + \omega_n^2 D_n(t) = -\ddot{u}_g(t)$$

$$\Rightarrow \underline{q_n(t) = \Gamma_n D_n(t)}$$

Contribution of the n^{th} mode to the displacement vector $\underline{u}(t)$ is given by:

$$\underline{u}_n(t) = \underline{\phi}_n q_n(t) = \Gamma_n \underline{\phi}_n D_n(t)$$

at the j^{th} story: displacement from n^{th} mode $= u_{jn}(t) = \Gamma_n \phi_{jn} D_n(t)$

Inter-story Drift in story $j = u_{jn}(t) - u_{j-1,n}(t)$

$$\hookrightarrow \Delta_{jn}(t) = \Gamma_n(\phi_{jn} - \phi_{j-1,n}) D_n(t)$$

Equivalent static force for the n^{th} mode $= \underline{f}_{sn}(t) = \underline{k}\,\underline{u}_n(t)$

$$= \underline{k}\,\underline{\phi}_n \Gamma_n D_n(t)$$

$$= \omega_n^2 \underline{m}\,\underline{\phi}_n \Gamma_n D_n(t)$$

OR $\underline{f}_{sn}(t) = \underline{s}_n A_n(t)$

since $\underline{s}_n = \Gamma_n \underline{m}\,\underline{\phi}_n$

and $A_n(t) = \omega_n^2 D_n(t)$.

Consider any response quantity $r(t)$

It's n^{th} mode contribution $= r_n(t)$

$r_n(t)$ is determined by static application of $\underline{f}_{sn}(t)$ on the structure.

$$\left.\begin{array}{l}\text{Static part of response}\\ \text{from } n^{th} \text{ mode}\end{array}\right\} = r_n^{st}$$

$$r_n(t) = r_n^{st} A_n(t) \quad \text{——(1)}$$

i.e. obtain r_n^{st} by applying $\underline{s}_n$ (i.e., $\Gamma_n \underline{m} \underline{\Phi}_n$) statically as external forces.

Multiply by $A_n(t)$ (pseudo-acceleration response for system with ω_n and ξ_n)

Note: r_n^{st} may be positive or negative.

Now, since $\underline{k}\,\underline{u}_n^{st} = \underline{s}_n = \Gamma_n \underline{m}\,\underline{\Phi}_n$

we have $\underline{k}\,\underline{u}_n^{st} = \Gamma_n \cancel{m_n}\, \underline{k}\,\underline{\Phi}_n / \omega_n^2$

$$\text{or} \quad \underline{u}_n^{st} = \frac{\Gamma_n}{\omega_n^2}\underline{\phi}_n$$

$$\underline{u}_n(t) = \underline{u}_n^{st} A_n(t) \quad \text{— from (1)}$$

or $\underline{u}_n(t) = \dfrac{\Gamma_n}{\omega_n^2}\phi_n A_n(t)$ in terms of $A_n(t)$

Total displacement $\underline{u}(t) = \sum_{n=1}^{N} \underline{u}_n(t)$

$$\text{OR} \quad \underline{u}(t) = \sum_{n=1}^{N} \Gamma_n \, \phi_n \, D_n(t)$$

Response $r(t) = \sum_{n=1}^{N} r_n(t)$

$$= \sum_{n=1}^{N} r_n^{st} A_n(t)$$

For very short periods $T_n \leq \frac{1}{33}$, we know that $A_n(t) \rightarrow -\ddot{u}_g(t)$

$$\Rightarrow r(t) = \sum_{n=1}^{N_d} r_n^{st} A_n(t) - \sum_{n=N_d+1}^{N} r_n^{st} \cdot \ddot{u}_g(t)$$

where $T_{N_d+1}, T_{N_d+2} \ldots, T_N \leq \frac{1}{33}$

$$\text{OR} \quad r(t) = \sum_{n=1}^{N_d} \underbrace{r_n^{st}} A_n(t) - \ddot{u}_g(t) \left[r^{st} - \sum_{n=1}^{N_d} \underbrace{r_n^{st}} \right]$$

$$r^{st} = \sum_{n=1}^{N} r_n^{st}$$

1 static analysis

N_d static analyses

$(N_d + 1)$ static analyses instead of N static analyses.

Seismic Response - Summary (Time History of Response is of Interest)

1. Define $\ddot{u}_g(t)$ at time steps Δt

2. a) Derive $\underline{k}$, $\underline{m}$ (Chapter 9)

   b) Estimate ξ_n (Chapter 11 table)

3. Solve eigenvalue problem:

   Obtain ω_n, $\underline{\phi}_n$ $(n = 1, 2 \ldots N)$

4. Find $\underline{s}_n$ (Remember $\underline{s}_n$ is the n^{th} mode contribution of the earthquake force distribution)

5. For each mode n

   (a) Apply $\underline{s}_n$ statically to determine any response r_n^{st} (if interest is in response paramcter, r)

   (b) Obtain $A_n(t)$ using $\ddot{u}_g(t)$ from step 1 assuming a SDOF system (with ω_n, ξ_n) and using numerical methods of Chapter 5

   (c) Obtain $r_n(t) = r_n^{st} \cdot A_n(t)$

6. Combine all modes

$$r(t) = \sum_{n=1}^{N} r_n(t)$$

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Conceptual explanation of modal analysis

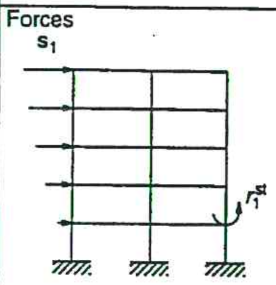
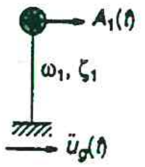
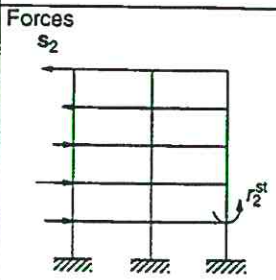
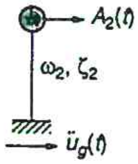
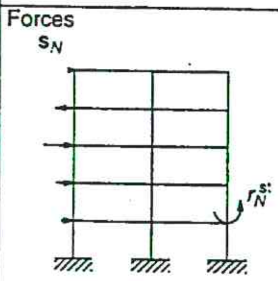
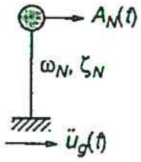
Mode	Static Analysis of Structure	Dynamic Analysis of SDF System	Modal Contribution to Dynamic Response
1			$r_1(t) = r_1^{st} A_1(t)$
2			$r_2(t) = r_2^{st} A_2(t)$
• • •	• • • • • •	• • • • • •	• • • • • •
N			$r_N(t) = r_N^{st} A_N(t)$
Total response **(combining all modes)**			$r(t) = \sum_{n=1}^{N} r_n(t)$

Figure 13.1.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Compare with fig. 12.9.1 ; Replace $A_n(t)$ by $\omega_n^2 D_n(t)$

and $\ddot{u}_g(t)$ by $p(t)$ on unit mass

Example 13.2

A two-story shear frame has the mass and story stiffnesses properties shown in Fig. E13.2a. Determine the modal expansion of the effective earthquake force distribution associated with horizontal ground acceleration $\ddot{u}_g(t)$.

Solution The stiffness and mass matrices (from Example 9.1) are

$$\mathbf{k} = k\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{m} = m\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

where $k = 24EI_c/h^3$, and the natural frequencies and modes (from Example 10.4) are

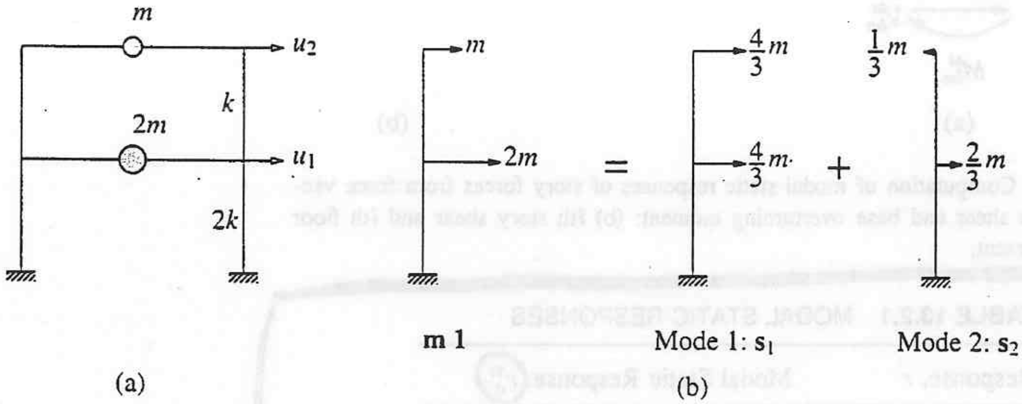
$$\omega_1 = \sqrt{\frac{k}{2m}} \qquad \omega_2 = \sqrt{\frac{2k}{m}}$$

$$\phi_1 = \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$\longrightarrow M_n = \sum_{j=1}^{N} m_j \phi_{jn}^2 \; ; \; L_n^h = \sum_{j=1}^{N} m_j \phi_{jn}$

$\Gamma_n = L_n^h / M_n$

The modal properties M_n, L_n^h, and Γ_n are computed from Eq. (13.2.3). For the first mode: $M_1 = 2m\left(\frac{1}{2}\right)^2 + m(1)^2 = 3m/2$; $L_1^h = 2m\left(\frac{1}{2}\right) + m(1) = 2m$; $\Gamma_1 = L_1^h/M_1 = \frac{4}{3}$. Similarly,



$\underline{s}_n = \Gamma_n \underline{m} \phi_n$

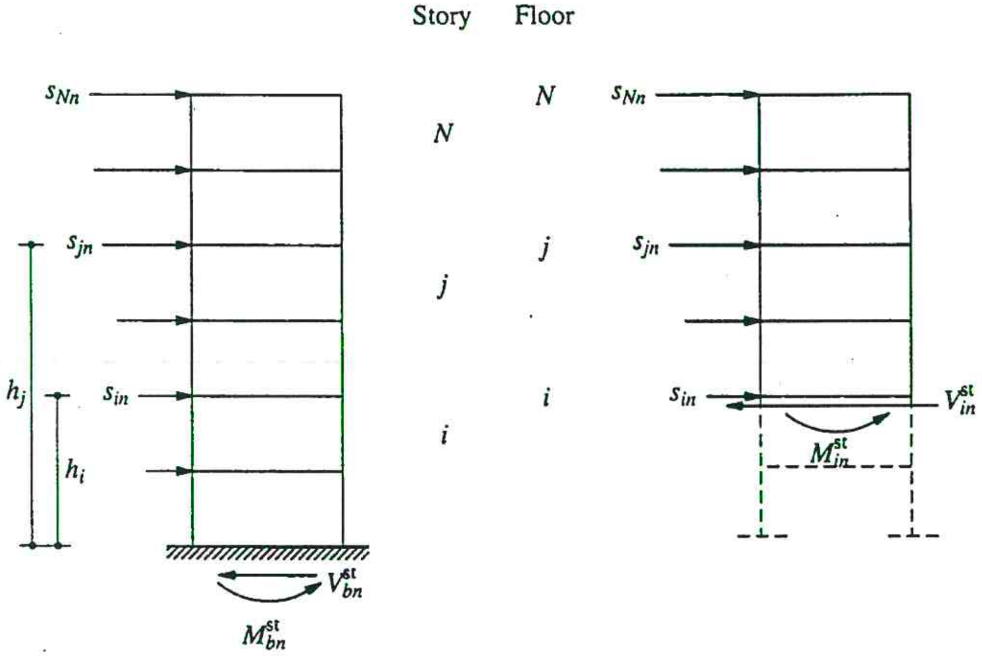
Figure E13.2 (a) Two-story shear frame; (b) modal expansion of **m1**.

for the second mode: $M_2 = 3m$, $L_2^h = -m$, and $\Gamma_2 = -\frac{1}{3}$. Substituting for Γ_n, **m**, and ϕ_n in Eq. (13.2.4) gives

$$\mathbf{s}_1 = \frac{4}{3}m\begin{bmatrix} 2 & \\ & 1 \end{bmatrix}\begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} = \frac{4}{3}m\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\mathbf{s}_2 = -\frac{1}{3}m\begin{bmatrix} 2 & \\ & 1 \end{bmatrix}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = -\frac{1}{3}m\begin{Bmatrix} -2 \\ 1 \end{Bmatrix}$$

The modal expansion of **m1** is displayed in Fig. E13.2b.



(a) (b)

Figure 13.2.2 Computation of modal static responses of story forces from force vector $\mathbf{s}_n$: (a) base shear and base overturning moment; (b) ith story shear and ith floor overturning moment.

TABLE 13.2.1 MODAL STATIC RESPONSES

Response, r	Modal Static Response, r_n^{st}
V_i	$V_{in}^{st} = \sum_{j=i}^{N} s_{jn}$
M_i	$M_{in}^{st} = \sum_{j=i}^{N} (h_j - h_i) s_{jn}$
V_b	$V_{bn}^{st} = \sum_{j=1}^{N} s_{jn} = \Gamma_n L_n^h \equiv M_n^*$
M_b	$M_{bn}^{st} = \sum_{j=1}^{N} h_j s_{jn} = \Gamma_n L_n^\theta \equiv h_n^* M_n^*$
u_j	$u_{jn}^{st} = (\Gamma_n / \omega_n^2) \phi_{jn}$
Δ_j	$\Delta_{jn}^{st} = (\Gamma_n / \omega_n^2)(\phi_{jn} - \phi_{j-1,n})$

IMPORTANT TABLE

MULTIPLY THESE BY $A_n(t)$

$\boxed{r_n(t)}$

OR Multiply by A_n

$\boxed{r_{no}}$

PEAK

28-9

Example 13.3

Derive equations for (a) the floor displacements and (b) the story shears for the shear frame of Example 13.2 subjected to ground motion $\ddot{u}_g(t)$.

Solution Steps 1 to 4 of the procedure summary have already been implemented in Example 13.2.

(a) *Floor displacements.* Substituting Γ_n and ϕ_{jn} from Example 13.2 in Eq. (13.2.5) gives the floor displacements due to the each mode:

$$\underline{u}_n = \Gamma_n \underline{\phi}_n D_n(t) \quad \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}_1 = \frac{4}{3}\begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} D_1(t) \qquad \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}_2 = -\frac{1}{3}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} D_2(t) \tag{a}$$

Combining the contributions of the two modes gives the floor displacements:

$$u_1(t) = u_{11}(t) + u_{12}(t) = \tfrac{2}{3}D_1(t) + \tfrac{1}{3}D_2(t) \tag{b}$$

$$u_2(t) = u_{21}(t) + u_{22}(t) = \tfrac{4}{3}D_1(t) - \tfrac{1}{3}D_2(t) \tag{c}$$

(b) *Story shears.* Static analysis of the frame for external floor forces $\mathbf{s}_n$ gives V_{in}^{st}, $i = 1$ and 2, shown in Fig. E13.3. Substituting these results in Eq. (13.2.8) gives

Mode 1 $$V_{11}(t) = \tfrac{8}{3}mA_1(t) \qquad V_{21}(t) = \tfrac{4}{3}mA_1(t) \tag{d}$$

Mode 2 $$V_{12}(t) = \tfrac{1}{3}mA_2(t) \qquad V_{22}(t) = -\tfrac{1}{3}mA_2(t) \tag{e}$$

Combining the contributions of two modes gives the story shears

$$V_1(t) = V_{11}(t) + V_{12}(t) = \tfrac{8}{3}mA_1(t) + \tfrac{1}{3}mA_2(t) \tag{f}$$

$$V_2(t) = V_{21}(t) + V_{22}(t) = \tfrac{4}{3}mA_1(t) - \tfrac{1}{3}mA_2(t) \tag{g}$$

The floor displacements and story shears have been expressed in terms of $D_n(t)$ and $A_n(t)$. These responses of the nth-mode SDF system to prescribed $\ddot{u}_g(t)$ can be determined by numerical time-stepping methods (Chapter 5).

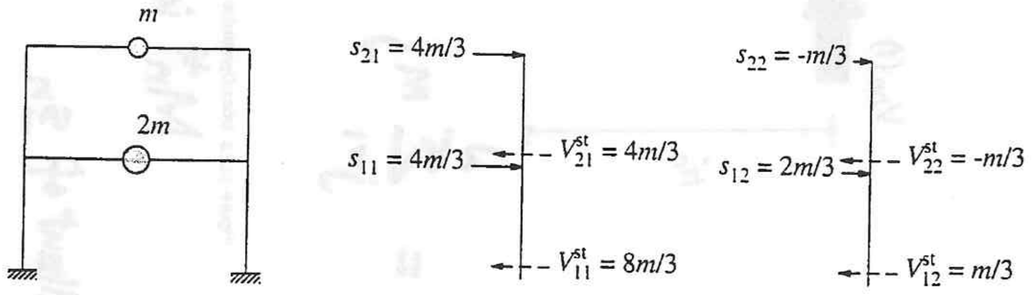


Figure E13.3

Effective modal mass and modal height

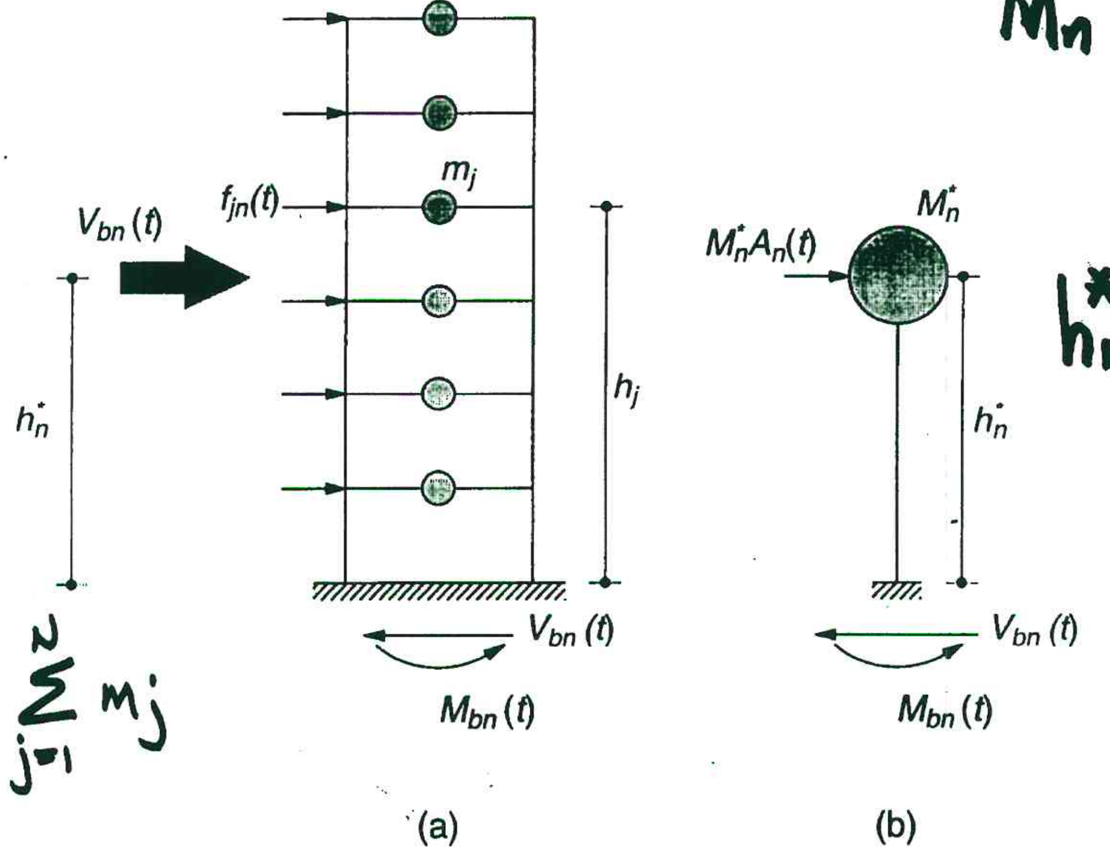
$$M_n^* = \Gamma_n L_n^h$$

$$= \Gamma_n \sum_{j=1}^{N} m_j \phi_{jn}$$

$$h_n^* = L_n^\theta \Big/ L_n^h$$

where

$$L_n^\theta = \sum_{j=1}^{N} h_j m_j \phi_{jn}$$



$$\sum_{n=1}^{N} M_n^* = \sum_{j=1}^{N} m_j$$

Figure 13.2.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

M_n^* is the "Effective" mass for mode n in computing V_{bn}

since $V_{bn} = M_n^* A_n(t)$

h_n^*: Ht. of resultant of $\underline{s}_n$

5-story Shear Frame

$\underline{m}$, $\underline{k}$ } → see pg. 445, 446 (1st Ed.)
ω's, $\underline{\phi}$'s } or pg. 526, 527 (2nd Ed.)

Modal expansion of **m1**

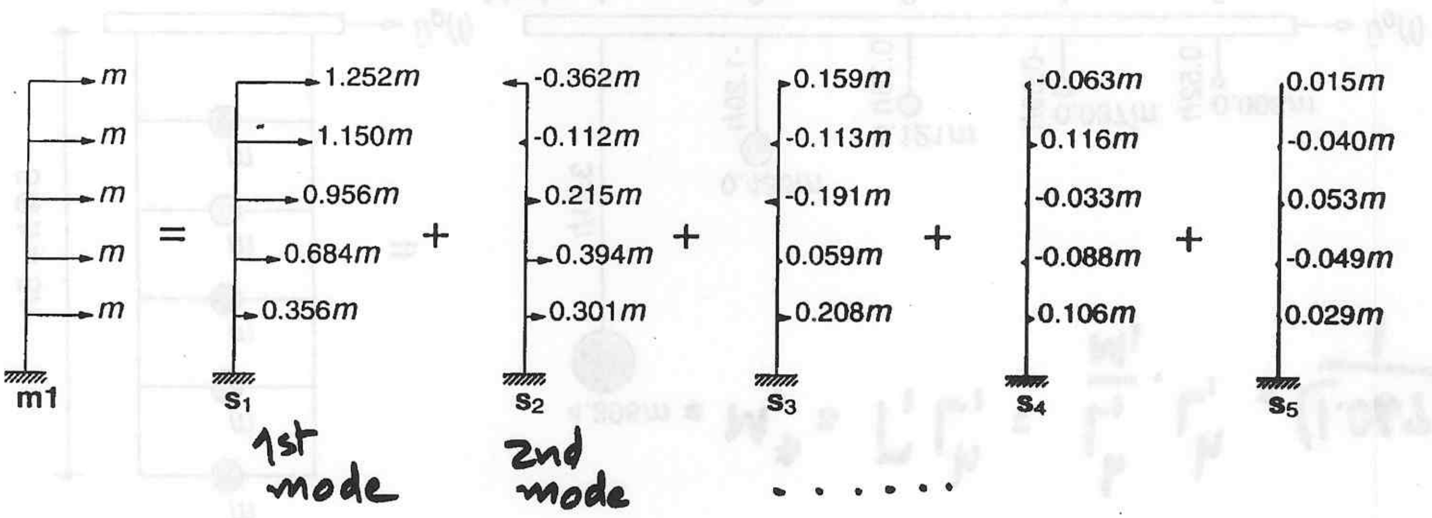


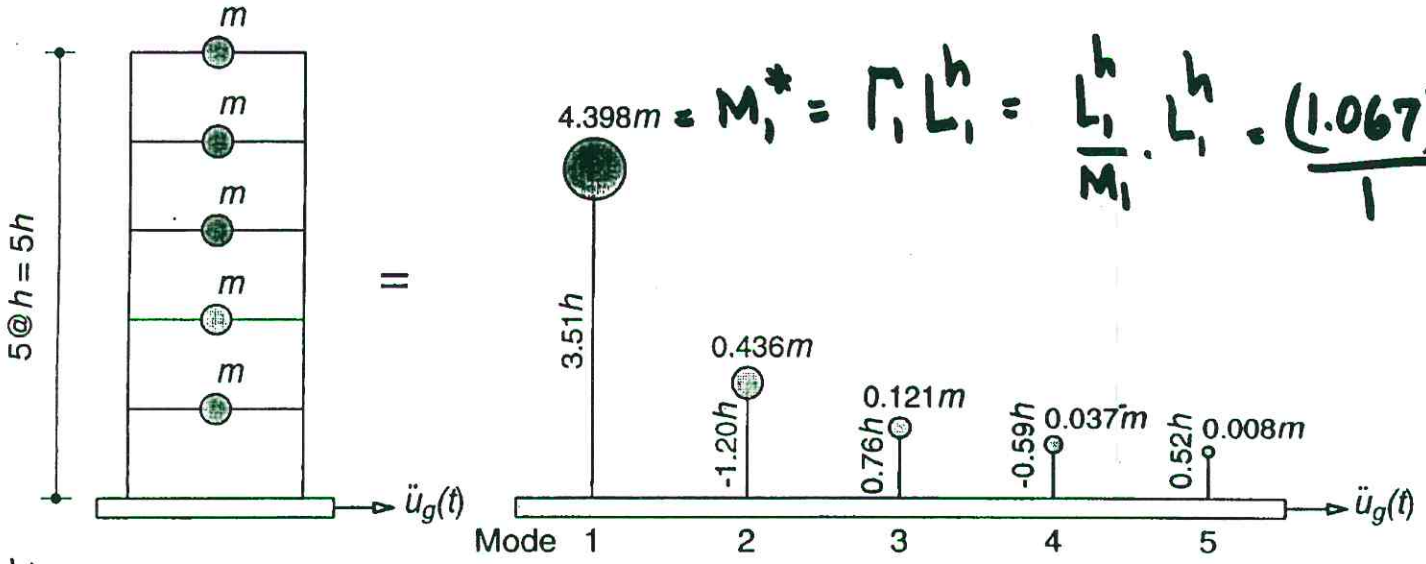
Figure 13.2.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

$\underline{s}_n = \Gamma_n \underline{m} \underline{\phi}_n$ where $\Gamma_n = \dfrac{L_n^h}{M_n}$

& $L_n^h = \sum_{j=1}^{N} m_j \phi_{jn}$

2-8-11

Effective modal masses and effective modal heights



$$4.398m = M_1^* = \Gamma_1 L_1^h = \frac{L_1^h}{M_1} \cdot L_1^h = \frac{(1.067)^2}{1} (kip\text{-}s^2/in)$$

$m = 100 \frac{kips}{g}$

Figure 13.2.5 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Displacement and pseudo-acceleration of modal SDF systems

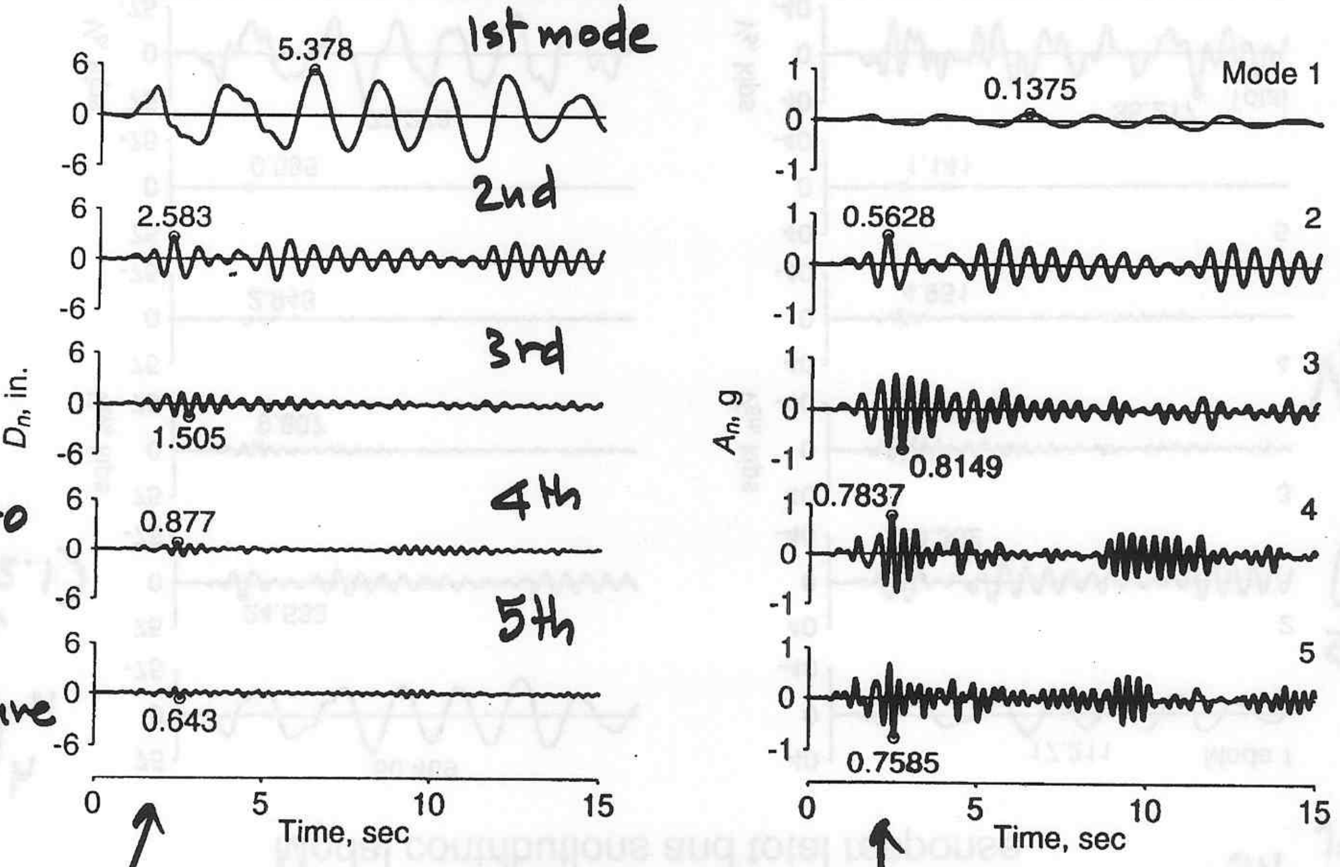


Figure 13.2.6 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

look at
Ch. 6
for El Centro
motion
→ input to
this structure

Using methods
of Chap. 5

$A_n(t) = \omega_n^2 D_n(t)$

$V_{bn} = V^{st}_{bn} A_n(t)$

where

$V^{st}_{bn} = \Gamma_n L^h_n$

(see Table 13.2.1)

Base shear and fifth-story shear

Modal contributions and total response

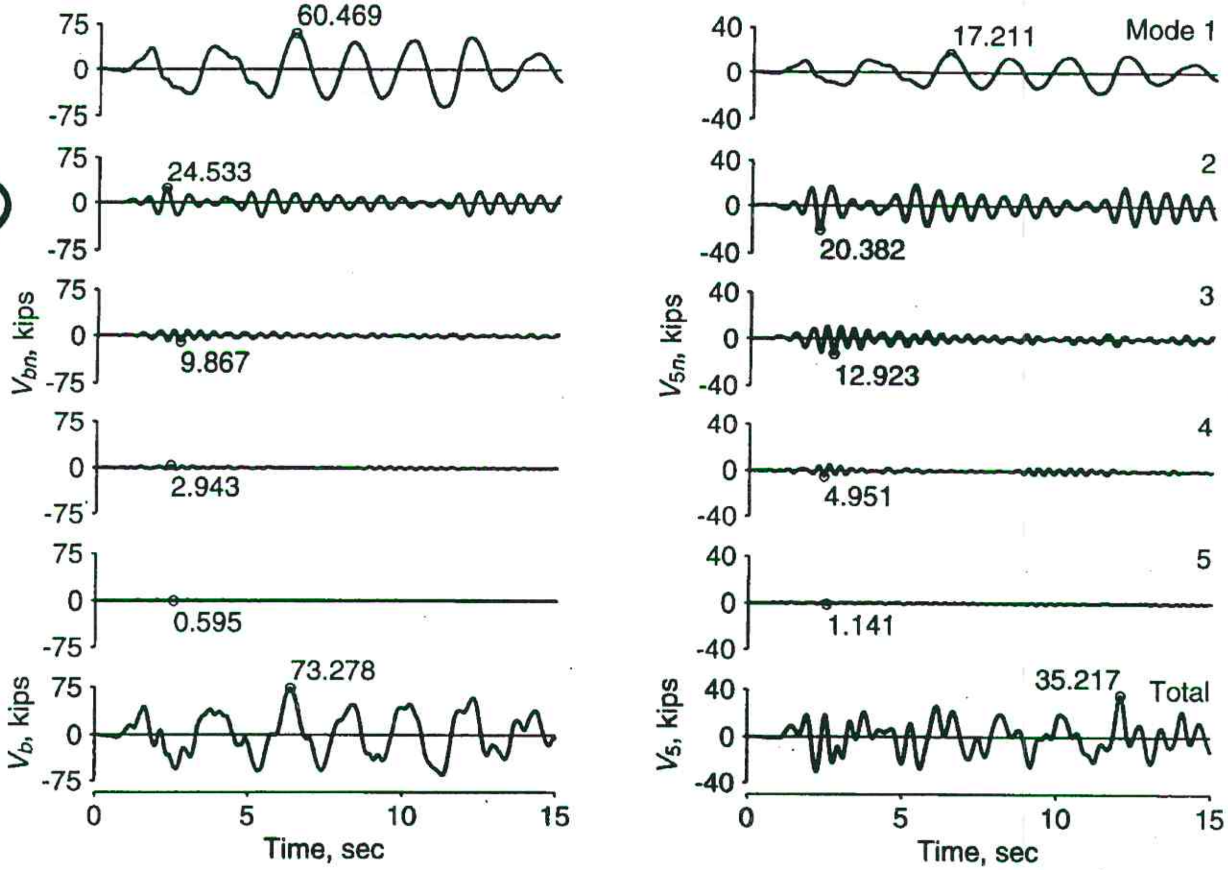


Figure 13.2.7 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Table 13.2.1 $\rightarrow$ $V^{st}_{5n} = \sum_{\underset{i}{j=5}}^{\overset{N}{5}} s_{jn} = s_{5n}$

and

$s_{5n} = \Gamma_n m_5 \phi_{5n}$

(Eq. 13.2.4

$V_{5n} = V^{st}_{5n} A_n(t)$

$u_{5n} = u^{st}_{5n} A_n(t)$ ← From Table 13.2.1

OR

$u_{5n} = \Gamma_n \phi_{5n} D_n(t)$

↑ eq. 13.2.5

Roof displacement and base overturning moment

Modal contributions and total response

$M_{bn} = M^{st}_{bn} A_n(t)$

where

$M^{st}_{bn} = \Gamma_n L^{\theta}_n$

where

$L^{\theta}_n = \sum_{j=1}^{N} h_j m_j \phi_{jn}$

[see Table 13.2.1]

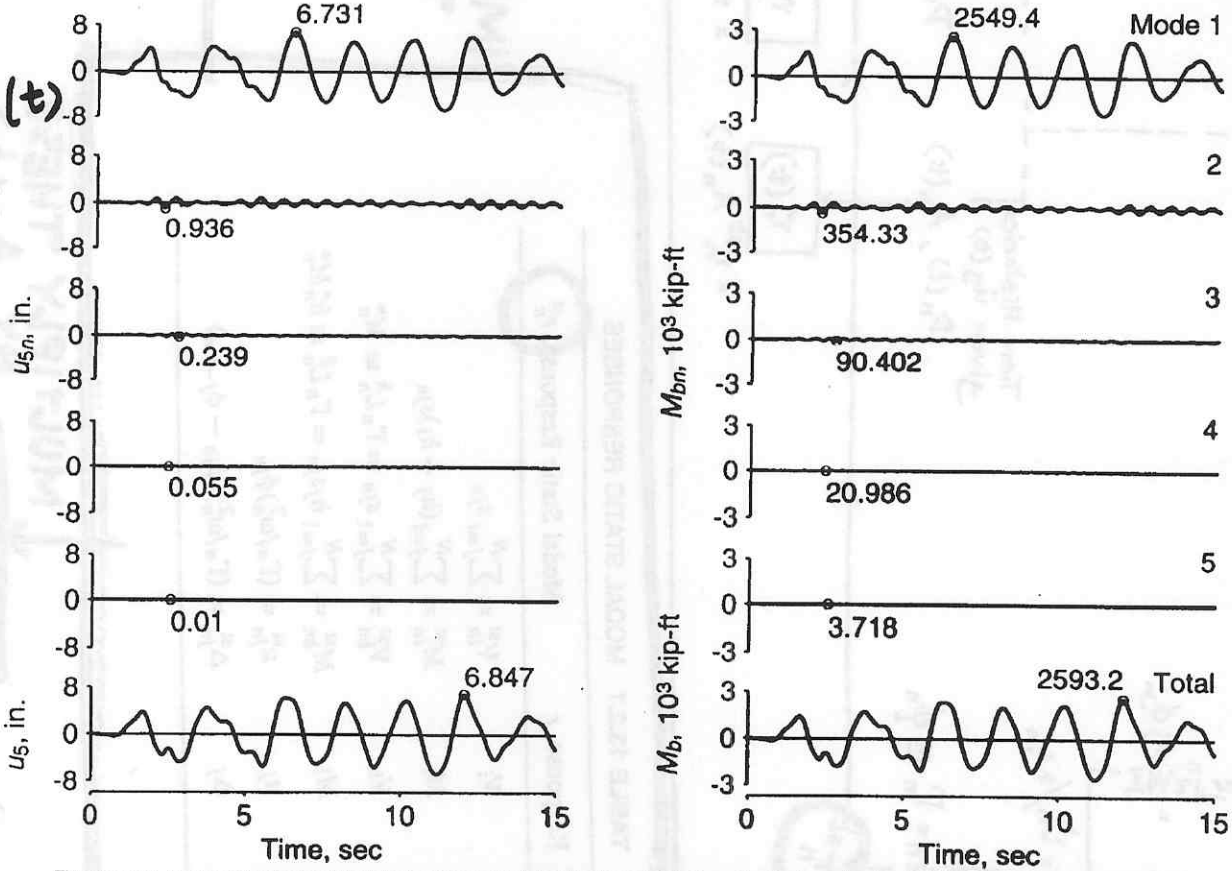


Figure 13.2.8 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

28-16

MDOF SYSTEMS - SEISMIC RESPONSE

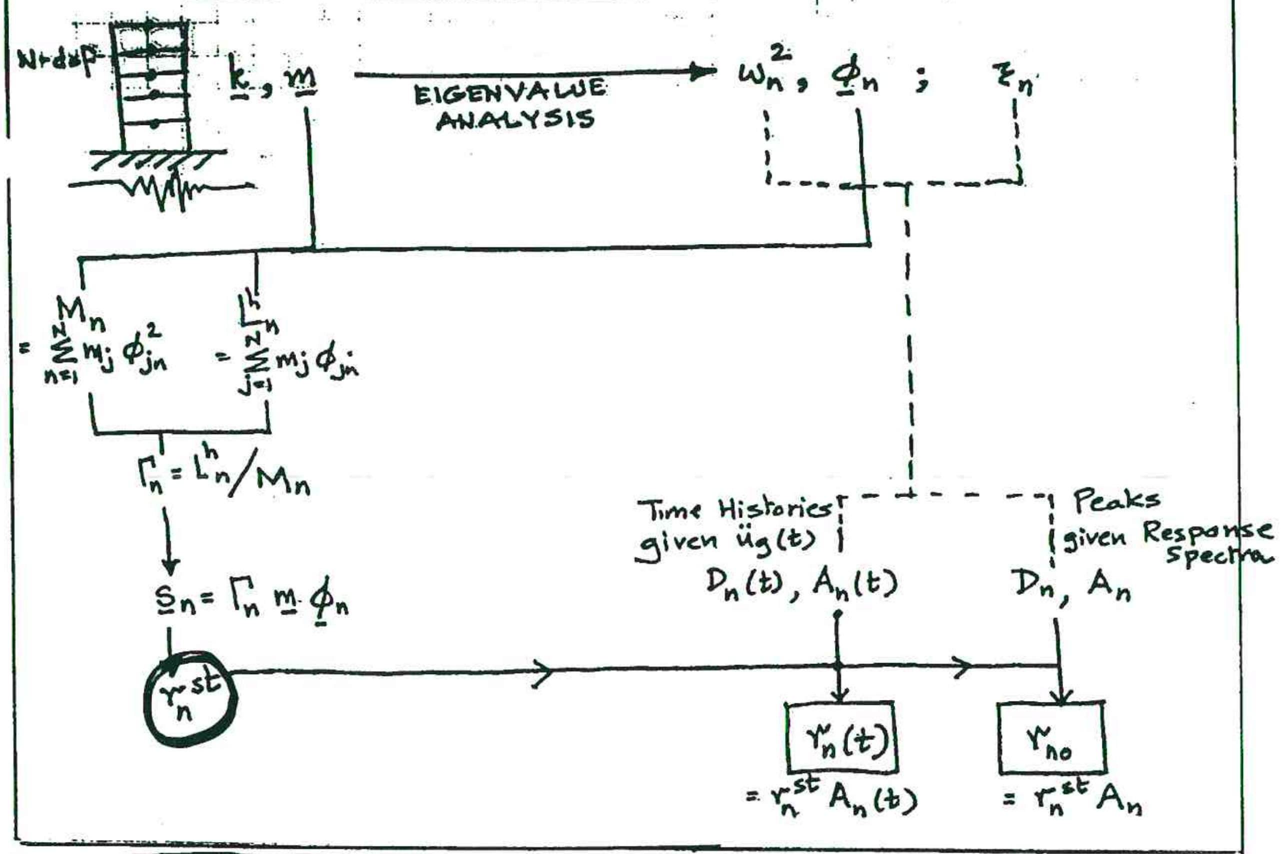


TABLE 13.2.1 MODAL STATIC RESPONSES

Response, r	Modal Static Response, r_n^{st}
V_i	$V_{in}^{st} = \sum_{j=i}^{N} s_{jn}$
M_i	$M_{in}^{st} = \sum_{j=i}^{N} (h_j - h_i) s_{jn}$
V_b	$V_{bn}^{st} = \sum_{j=1}^{N} s_{jn} = \Gamma_n L_n^h \equiv M_n^*$
M_b	$M_{bn}^{st} = \sum_{j=1}^{N} h_j s_{jn} = \Gamma_n L_n^\theta \equiv h_n^* M_n^*$
u_j	$u_{jn}^{st} = (\Gamma_n / \omega_n^2) \phi_{jn}$
Δ_j	$\Delta_{jn}^{st} = (\Gamma_n / \omega_n^2)(\phi_{jn} - \phi_{j-1,n})$

IMPORTANT TABLE

MULTIPLY THESE BY $A_n(t)$

$\boxed{r_n(t)}$

OR Multiply by A_n

$\boxed{r_{no}}$

PEAK

1-PG SUMMARY

[13.2.1]

Response Spectrum Analysis (for Seismic Loads)

If interest is in PEAK response for a MDOF system.

Modal Combination Rules

$$r_0 \cong \left[\sum_{n=1}^{N} r_{no}^2 \right]^{1/2}$$

Peak of $r(t)$

if natural frequencies are well separated

This is the SRSS [Square-root-of-sum of squares] rule.

Steps in Analysis

1. a) obtain $\underline{m}$, $\underline{k}$
   b) Estimate ζ_n
2. Eigenvalue Analysis :- Get ζ_n, ω_n (T_n)
3. For each mode n :
   (a) Using ζ_n, T_n, get D_n & A_n
   (b) Compute response peaks in terms of D_n & A_n if displacements, drifts are of interest.
   (c) Apply $\underline{f}_{sn_0}$ (peak static equivalent forces) & obtain Bending moments, shears, etc.
4. Use SRSS rule to combine modal peak responses & get Peak Response (TOTAL)

$$r_0 \cong \left[\sum_{n=1}^{N} r_{no}^2 \right]^{1/2}$$

29-2

Spectral ordinates for example structure

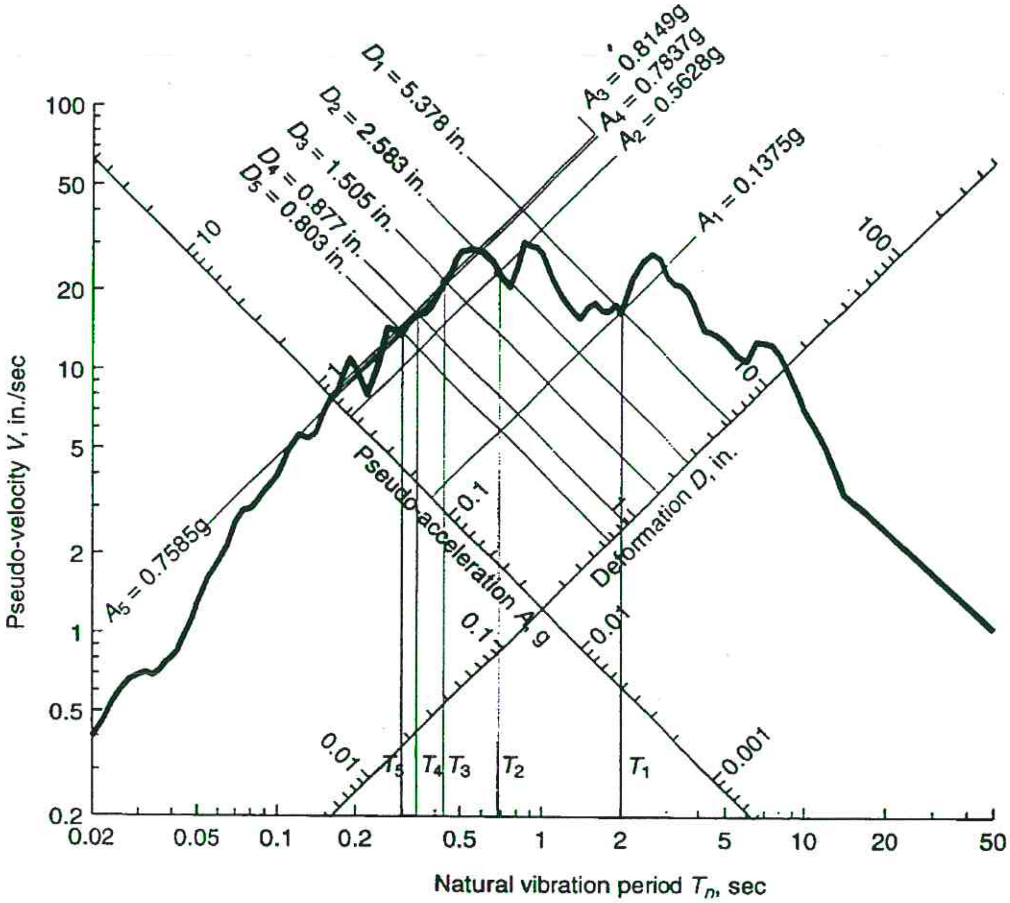


Figure 13.8.2 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

eigenvalue analysis

from spectrum

$\underline{u}_1 = \Gamma_1 \underline{\phi}_1 D_1$

get Γ_1 using Eq. 13.2.3

Peak modal displacements and forces

Eq. 13.8.2

$\underline{f}_1 = \underline{s}_1 A_1$

$\underline{s}_1 = \Gamma_1 \underline{m} \, \underline{\phi}_1$

$V_{bn} = \sum_{j=1}^{N} f_{jn}$

$M_{bn} = \sum_{j=1}^{N} h_j f_{jn}$

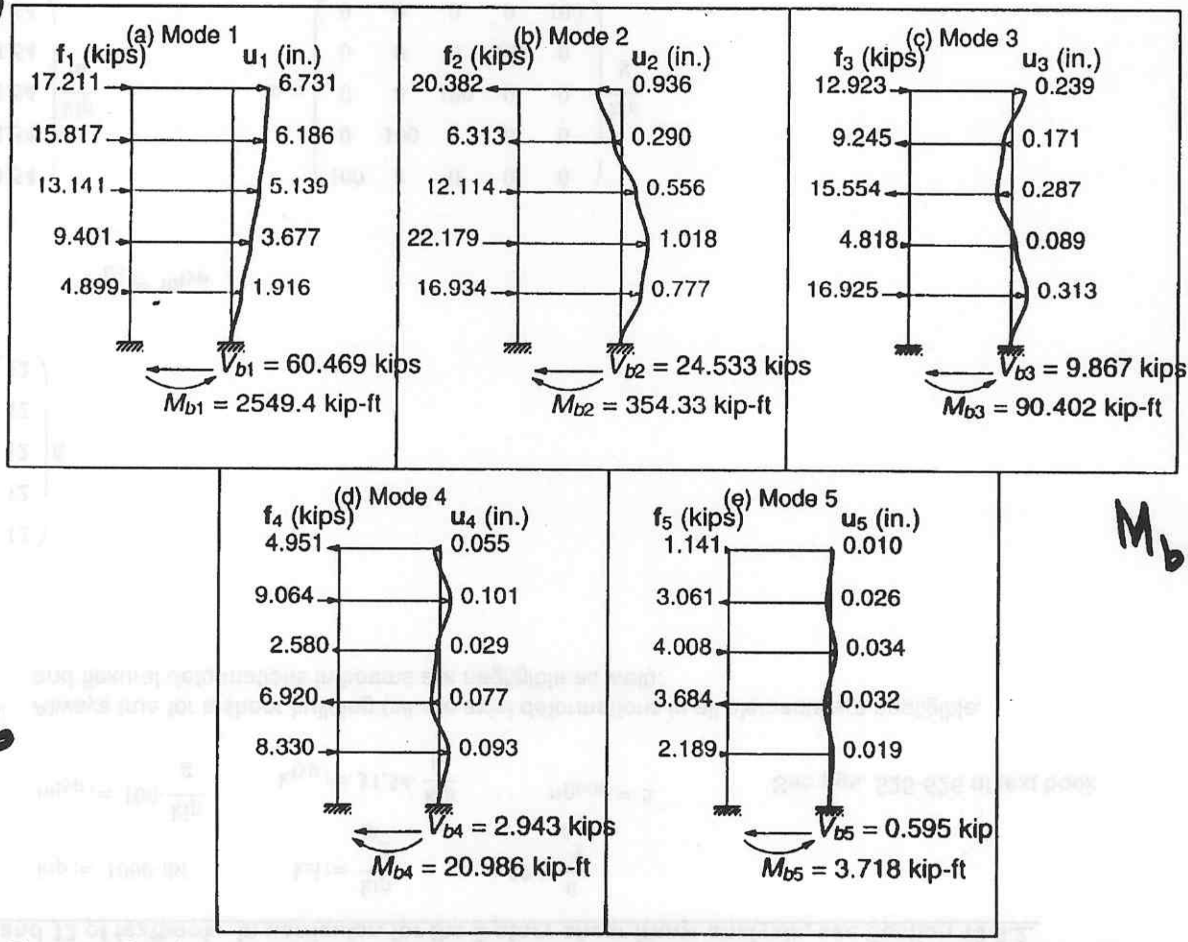


Figure 13.8.3 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 1995.

Use methods of Chapter 6

Use SRSS rule to get peak response (displacement, shear, bending moment)

29-4

Peak Response of MDOF Systems subject to Earthquake Ground Motion
Use of a Response Spectrum and Modal Combination Rule
Refer to Chs. 12 and 13 of textbook. In particular, for the 5-story shear frame analysis, see Section 13.8.2.

Units $\quad \mathrm{kip} := 1000 \cdot \mathrm{lbf} \quad \mathrm{ksi} := \frac{\mathrm{kip}}{\mathrm{in}^2} \quad g := 32.2 \cdot \frac{\mathrm{ft}}{\mathrm{s}^2}$

$h_{typ} := 12 \cdot \mathrm{ft} \quad m_{typ} := 100 \cdot \frac{\mathrm{kip}}{g} \quad k_{typ} := 31.54 \cdot \frac{\mathrm{kip}}{\mathrm{in}} \quad n_{floor} := 5$ See pgs. 525-526 of text book

$n_{mode} := n_{floor}$ Always true for a shear building (where axial deformations in all elements are negligible, and flexural deformations in beams are negligible as well).

$i := 1 .. n_{floor}$

$h_i := h_{typ}$

$$h = \begin{pmatrix} 12 \\ 12 \\ 12 \\ 12 \\ 12 \end{pmatrix} \mathrm{ft}$$

$k_{flr_i} := k_{typ} \qquad m_{i,i} := m_{typ}$

$$k_{flr} = \begin{pmatrix} 31.54 \\ 31.54 \\ 31.54 \\ 31.54 \\ 31.54 \end{pmatrix} \frac{\mathrm{kip}}{\mathrm{in}} \qquad m = \begin{pmatrix} 100 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 100 \end{pmatrix} \frac{\mathrm{kip}}{g}$$

$i := 1 .. n_{floor}$ $\qquad$ $j := 1 .. n_{floor}$

$k_{i,j} := 0 \cdot \frac{kip}{in}$

$$k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \frac{kip}{in}$$

$k_{i,i} := k_{flr_i}$

$i := 1 .. n_{floor} - 1$

$k_{i,i} := k_{i,i} + k_{i+1,i+1}$

$k_{i,i+1} := -k_{flr_{i+1}}$

$k_{i+1,i} := -k_{flr_{i+1}}$

$$k = \begin{pmatrix} 63.08 & -31.54 & 0.00 & 0.00 & 0.00 \\ -31.54 & 63.08 & -31.54 & 0.00 & 0.00 \\ 0.00 & -31.54 & 63.08 & -31.54 & 0.00 \\ 0.00 & 0.00 & -31.54 & 63.08 & -31.54 \\ 0.00 & 0.00 & 0.00 & -31.54 & 31.54 \end{pmatrix} \frac{kip}{in}$$

Stiffness matrix for 5-story shear building

29-5

29-6

Eigenvalues and eigenvectors (modal frequencies and mode shapes)

$$\text{genvals}(k, m) = \begin{pmatrix} 448.789 \\ 344.995 \\ 209.053 \\ 84.125 \\ 9.873 \end{pmatrix} \frac{\text{rad}}{\text{s}^2}$$

$$\lambda := \sqrt{\text{genvals}(k, m)}$$

$$\lambda = \begin{pmatrix} 21.185 \\ 18.574 \\ 14.459 \\ 9.172 \\ 3.142 \end{pmatrix} \frac{\text{rad}}{\text{s}}$$

$$\text{mode} := 1 .. n_{\text{mode}}$$

$$\text{index}_{\text{mode}} := \text{mode}$$

$$\lambda I := \text{augment}\left(\frac{\lambda}{\frac{\text{rad}}{\text{s}}}, \text{index}\right)$$

$$\lambda I = \begin{pmatrix} 21.185 & 1.000 \\ 18.574 & 2.000 \\ 14.459 & 3.000 \\ 9.172 & 4.000 \\ 3.142 & 5.000 \end{pmatrix}$$

$$\omega := \text{csort}(\lambda I, 1)^{\langle 1 \rangle} \cdot \frac{\text{rad}}{\text{s}}$$

$$\omega = \begin{pmatrix} 3.142 \\ 9.172 \\ 14.459 \\ 18.574 \\ 21.185 \end{pmatrix} \frac{\text{rad}}{\text{s}} \qquad \text{csort}(\lambda I, 1)^{\langle 2 \rangle} = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{index}_{\text{mode}} := \text{csort}(\lambda I, 1)_{\text{mode}, 2}$$

$$\text{floor} := 1 .. n_{\text{floor}}$$

$$\Phi_{\text{floor, mode}} := \text{genvecs}(k, m)_{\text{floor, index}_{\text{mode}}}$$

$$\Phi = \begin{pmatrix} -0.170 & -0.456 & 0.597 & 0.549 & 0.326 \\ -0.326 & -0.597 & 0.170 & -0.456 & -0.549 \\ -0.456 & -0.326 & -0.549 & -0.170 & 0.597 \\ -0.549 & 0.170 & -0.326 & 0.597 & -0.456 \\ -0.597 & 0.549 & 0.456 & -0.326 & 0.170 \end{pmatrix} \qquad \omega = \begin{pmatrix} 3.142 \\ 9.172 \\ 14.459 \\ 18.574 \\ 21.185 \end{pmatrix} \frac{\text{rad}}{\text{sec}}$$

$$T_n := \frac{2 \cdot \pi}{\omega} \qquad T_n = \begin{pmatrix} 2.000 \\ 0.685 \\ 0.435 \\ 0.338 \\ 0.297 \end{pmatrix} \text{s}$$

$$\frac{\omega}{\sqrt{\frac{k_{\text{typ}}}{m_{\text{typ}}}}} = \begin{pmatrix} 0.285 \\ 0.831 \\ 1.310 \\ 1.683 \\ 1.919 \end{pmatrix}$$

Compare with pg. 484 of text book

29-8

$$\Phi^{\langle mode \rangle} := \frac{\Phi^{\langle mode \rangle}}{\Phi_{n_{floor},\, mode}}$$

Normalize mode shapes so that roof displacement equals unity

$$\Phi = \begin{pmatrix} 0.285 & -0.831 & 1.310 & -1.683 & 1.919 \\ 0.546 & -1.088 & 0.373 & 1.398 & -3.229 \\ 0.764 & -0.594 & -1.204 & 0.521 & 3.513 \\ 0.919 & 0.310 & -0.715 & -1.831 & -2.683 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{pmatrix}$$

$$M_{mode} := \frac{\left(\sqrt{\Phi^{\langle mode \rangle^{T}} \cdot m \cdot \Phi^{\langle mode \rangle}}\right)_{1}}{\left(kip \cdot \frac{s^{2}}{in}\right)^{\frac{1}{2}}}$$

$$M = \begin{pmatrix} 0.852 \\ 0.927 \\ 1.116 \\ 1.560 \\ 2.994 \end{pmatrix}$$

Note the units of each modal mass is (kip-s^2/in)$^{1/2}$

$$\Phi^{\langle mode \rangle} := \frac{\Phi^{\langle mode \rangle}}{M_{mode}}$$

$$\Phi := \Phi \cdot \left(kip \cdot \frac{s^{2}}{in}\right)^{-\frac{1}{2}}$$

Mass-normalize all the mode shapes

$$\Phi = \begin{pmatrix} 0.334 & -0.896 & 1.173 & -1.078 & 0.641 \\ 0.641 & -1.173 & 0.334 & 0.896 & -1.078 \\ 0.896 & -0.641 & -1.078 & 0.334 & 1.173 \\ 1.078 & 0.334 & -0.641 & -1.173 & -0.896 \\ 1.173 & 1.078 & 0.896 & 0.641 & 0.334 \end{pmatrix} \left(\text{kip}\cdot\frac{s^2}{in}\right)^{-\frac{1}{2}}$$

Mass-normalized modes
(compare with pg. 484 of text book)

Sketch mode shapes

$$ii := 1 .. n_{floor} + 1$$

$$H_{ii} := \left| \begin{array}{l} 0 \cdot ft \ \text{ if } \ ii = 1 \\ \displaystyle\sum_{j=1}^{ii-1} h_j \ \text{ otherwise} \end{array} \right.$$

$$zeroes := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

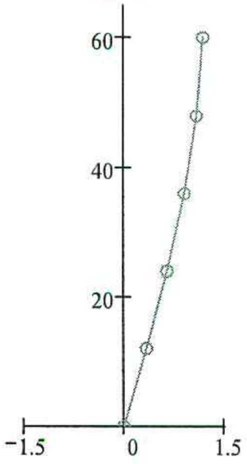
$$\Phi_{sketch} := \text{stack}\left[zeroes, \frac{\Phi}{\left(\text{kip}\cdot\frac{s^2}{in}\right)^{-\frac{1}{2}}} \right]$$

$$H = \begin{pmatrix} 0 \\ 12 \\ 24 \\ 36 \\ 48 \\ 60 \end{pmatrix} ft$$

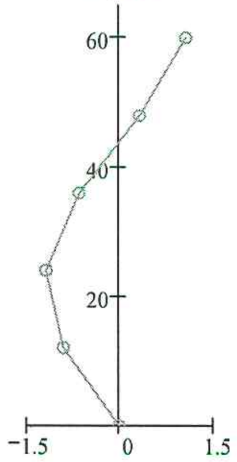
$$\Phi_{sketch} = \begin{pmatrix} 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.334 & -0.896 & 1.173 & -1.078 & 0.641 \\ 0.641 & -1.173 & 0.334 & 0.896 & -1.078 \\ 0.896 & -0.641 & -1.078 & 0.334 & 1.173 \\ 1.078 & 0.334 & -0.641 & -1.173 & -0.896 \\ 1.173 & 1.078 & 0.896 & 0.641 & 0.334 \end{pmatrix}$$

29-10

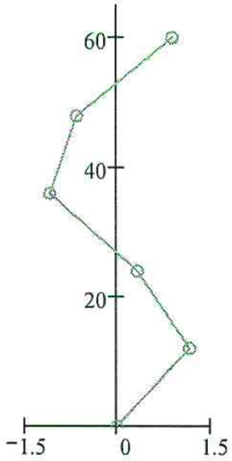
Mode 1



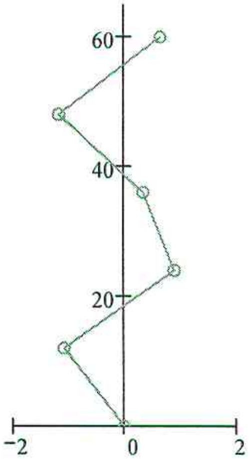
Mode 2



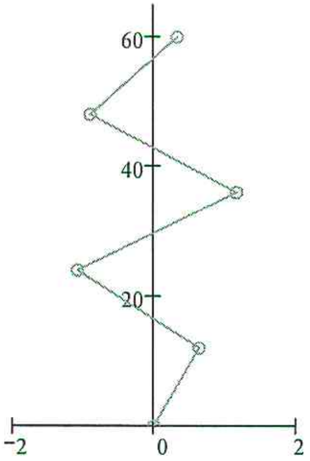
Mode 3



Mode 4



Mode 5



$$M_{mode} := \left(\left(\Phi^{\langle mode \rangle}\right)^{T} \cdot m \cdot \Phi^{\langle mode \rangle}\right)_{1}$$

$$M = \begin{pmatrix} 1.000 \\ 1.000 \\ 1.000 \\ 1.000 \\ 1.000 \end{pmatrix} \qquad \Phi^{T} \cdot m \cdot \Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Check!

$$Lh_{mode} := \sum_{j=1}^{n_{floor}} m_{j,j} \cdot \Phi_{j,mode}$$

$$\Gamma_{mode} := \frac{Lh_{mode}}{M_{mode}}$$

$$\Gamma = \begin{pmatrix} 1.067 \\ -0.336 \\ 0.177 \\ -0.099 \\ 0.045 \end{pmatrix} \left(kip \cdot \frac{s^{2}}{in}\right)^{\frac{1}{2}}$$

Compare with Table 13.2.2 of text book

29-11

29-12

Modal Expansion of **m*1**

$\text{mode} := 1 .. n_{mode}$ $\text{floor} := 1 .. n_{floor}$

$$s_{floor,\, mode} := \Gamma_{mode} \cdot \sum_{k=1}^{n_{floor}} m_{floor,\, k} \cdot \Phi_{k,\, mode}$$

This is the same as: $\underline{s}_n = \Gamma_n * \underline{m} * \underline{\phi}_n$

$$\frac{s}{m_{typ}} = \begin{pmatrix} 0.356 & 0.301 & 0.208 & 0.106 & 0.029 \\ 0.684 & 0.394 & 0.059 & -0.088 & -0.049 \\ 0.956 & 0.215 & -0.191 & -0.033 & 0.053 \\ 1.150 & -0.112 & -0.113 & 0.116 & -0.040 \\ 1.252 & -0.362 & 0.159 & -0.063 & 0.015 \end{pmatrix}$$

Compare with Figure 13.2.4 in text book

$$\text{sum\_cols\_of\_s}_{floor} := \sum_{i=1}^{n_{mode}} s_{floor,\, i}$$

$$\text{sum\_cols\_of\_s} = \begin{pmatrix} 100 \\ 100 \\ 100 \\ 100 \\ 100 \end{pmatrix} \frac{kip}{g}$$

$$m \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \\ 100 \\ 100 \\ 100 \end{pmatrix} \frac{kip}{g}$$

Check!

Given response spectrum values at the five modal periods (see Figure 13.8.2)

5 modal periods: $T_n = \begin{pmatrix} 2.000 \\ 0.685 \\ 0.435 \\ 0.338 \\ 0.297 \end{pmatrix} s$ Peak Pseudo-acceleration response: $A := \begin{pmatrix} 0.1375 \\ 0.5628 \\ 0.8149 \\ 0.7837 \\ 0.7585 \end{pmatrix} \cdot g$

Peak deformation response: $D_{mode} := \dfrac{A_{mode}}{(\omega_{mode})^2}$

$$D = \begin{pmatrix} 5.381 \\ 2.585 \\ 1.506 \\ 0.878 \\ 0.653 \end{pmatrix} in$$

29-14

DISPLACEMENTS

$mode := 1 .. n_{mode}$

$floor := 1 .. n_{floor}$

$u0_{floor,mode} := \Gamma_{mode} \cdot \Phi_{floor,mode} \cdot D_{mode}$

$$u0 = \begin{pmatrix} 1.917 & 0.778 & 0.313 & 0.093 & 0.019 \\ 3.679 & 1.019 & 0.089 & -0.078 & -0.032 \\ 5.143 & 0.556 & -0.287 & -0.029 & 0.035 \\ 6.190 & -0.290 & -0.171 & 0.102 & -0.026 \\ 6.736 & -0.936 & 0.239 & -0.055 & 0.010 \end{pmatrix} in$$

Compare with Figure 13.8.3 (Pg. 566)

$$u_{sketch} := \text{stack}\left(\text{zeroes}, \frac{u0}{in}\right)$$

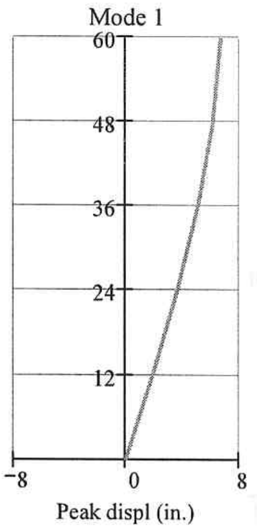
Peak story displacements using the SRSS rule for combination of modes

$$utotal_{floor} := \sqrt{\sum_{mode=1}^{n_{mode}} \left(u0_{floor,mode}\right)^2}$$

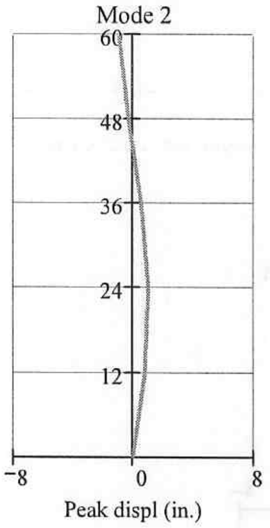
$$utotal = \begin{pmatrix} 2.095 \\ 3.819 \\ 5.181 \\ 6.200 \\ 6.805 \end{pmatrix} in$$

Compare with Table 13.8.5

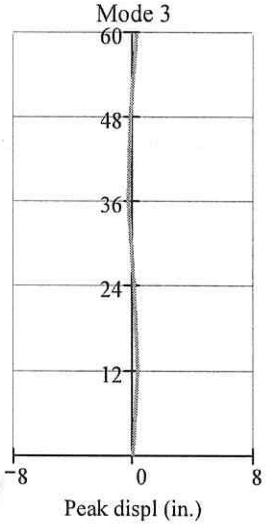
Sketch of peak displacements in each mode



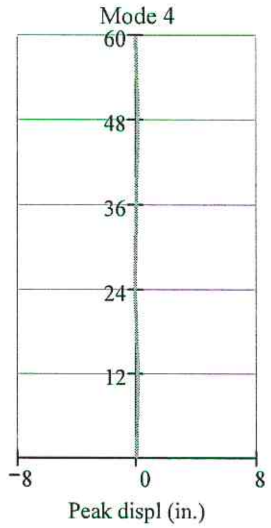
$$u_0^{\langle 1\rangle} = \begin{pmatrix} 1.917 \\ 3.679 \\ 5.143 \\ 6.190 \\ 6.736 \end{pmatrix} \text{in}$$

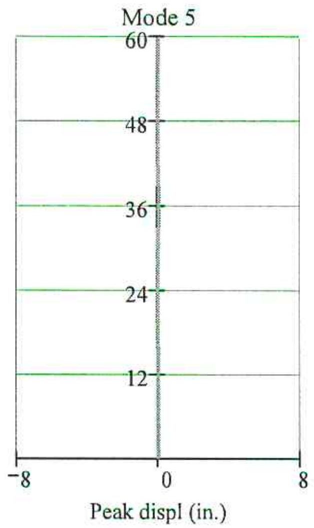


$$u_0^{\langle 2\rangle} = \begin{pmatrix} 0.778 \\ 1.019 \\ 0.556 \\ -0.290 \\ -0.936 \end{pmatrix} \text{in}$$



$$u_0^{\langle 3\rangle} = \begin{pmatrix} 0.313 \\ 0.089 \\ -0.287 \\ -0.171 \\ 0.239 \end{pmatrix} \text{in}$$





$$u_0^{\langle 4\rangle} = \begin{pmatrix} 0.093 \\ -0.078 \\ -0.029 \\ 0.102 \\ -0.055 \end{pmatrix} \text{in}$$

$$u_0^{\langle 5\rangle} = \begin{pmatrix} 0.019 \\ -0.032 \\ 0.035 \\ -0.026 \\ 0.010 \end{pmatrix} \text{in}$$

EQUIVALENT STATIC FORCES

$$f0_{floor,\,mode} := s_{floor,\,mode} \cdot A_{mode}$$

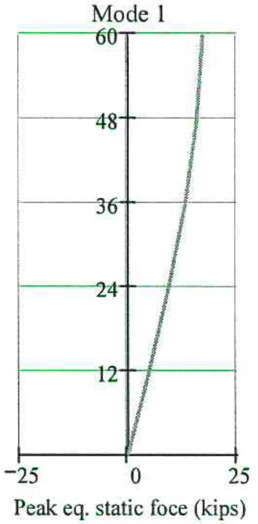
$$f0 = \begin{pmatrix} 4.899 & 16.934 & 16.925 & 8.330 & 2.189 \\ 9.401 & 22.178 & 4.817 & -6.921 & -3.683 \\ 13.141 & 12.114 & -15.554 & -2.580 & 4.008 \\ 15.817 & -6.313 & -9.244 & 9.064 & -3.060 \\ 17.211 & -20.382 & 12.923 & -4.951 & 1.141 \end{pmatrix} \text{kip}$$

Compare with Figure 13.8.3 (pg. 566)

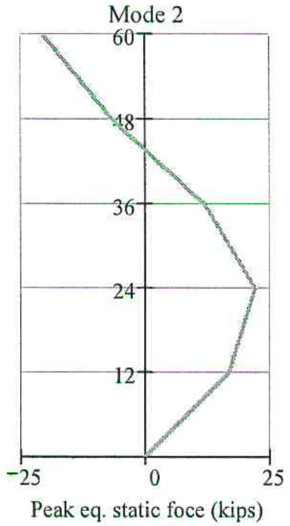
$$f_{sketch} := \text{stack}\left(\text{zeroes}, \frac{f0}{\text{kip}}\right)$$

29-18

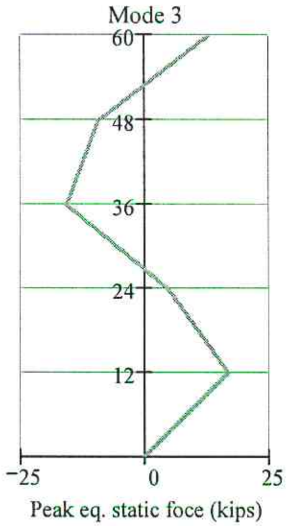
Sketch of peak equivalent static forces in each mode



$$f_0^{\langle 1\rangle} = \begin{pmatrix} 4.899 \\ 9.401 \\ 13.141 \\ 15.817 \\ 17.211 \end{pmatrix} \text{kip}$$

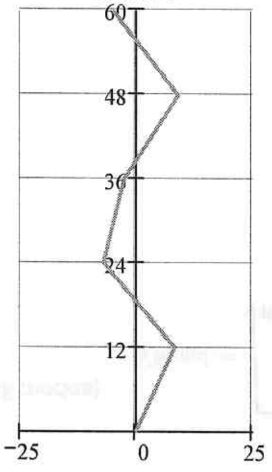


$$f_0^{\langle 2\rangle} = \begin{pmatrix} 16.934 \\ 22.178 \\ 12.114 \\ -6.313 \\ -20.382 \end{pmatrix} \text{kip}$$



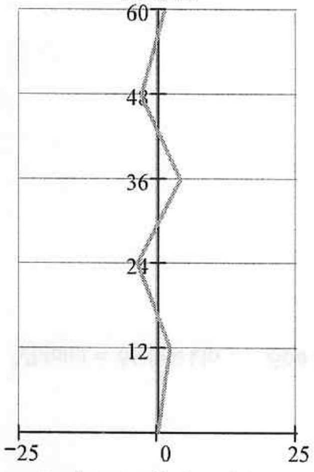
$$f_0^{\langle 3\rangle} = \begin{pmatrix} 16.925 \\ 4.817 \\ -15.554 \\ -9.244 \\ 12.923 \end{pmatrix} \text{kip}$$

Mode 4



Peak eq. static foce (kips)

Mode 5



Peak eq. static foce (kips)

$$f_0^{\langle 4\rangle} = \begin{pmatrix} 8.330 \\ -6.921 \\ -2.580 \\ 9.064 \\ -4.951 \end{pmatrix} \text{kip}$$

$$f_0^{\langle 5\rangle} = \begin{pmatrix} 2.189 \\ -3.683 \\ 4.008 \\ -3.060 \\ 1.141 \end{pmatrix} \text{kip}$$

29-19

SHEARS

$$V0_{floor,\,mode} := \sum_{flr\,=\,floor}^{n_{floor}} f0_{flr,\,mode}$$

$$V_0 = \begin{pmatrix} 60.468 & 24.532 & 9.867 & 2.943 & 0.595 \\ 55.569 & 7.598 & -7.058 & -5.387 & -1.595 \\ 46.168 & -14.580 & -11.876 & 1.533 & 2.089 \\ 33.027 & -26.694 & 3.678 & 4.113 & -1.919 \\ 17.211 & -20.382 & 12.923 & -4.951 & 1.141 \end{pmatrix} \text{kip}$$

Peak Base Shear: $Vb0_{mode} := V0_{1,\,mode}$

$$Vb0^T = (60.468\;\; 24.532\;\; 9.867\;\; 2.943\;\; 0.595)\,\text{kip}$$ Compare with Figure 13.8.3 (pg. 566)

Peak Base Shear (including all modes)
using the SRSS rule

$$\text{Vbtotal} := \sqrt{\sum_{mode\,=\,1}^{n_{mode}} \left(Vb0_{mode}\right)^2} \qquad \text{Vbtotal} = 66.064\,\text{kip}$$ See pg. 567 and Table 13.8.5.

MOMENTS

$$ht_{floor} := floor \cdot h_{typ}$$

$$ht = \begin{pmatrix} 12 \\ 24 \\ 36 \\ 48 \\ 60 \end{pmatrix} ft$$

$$M0_{floor,\,mode} := \sum_{flr = floor}^{n_{floor}} ht_{flr} \cdot f0_{flr,\,mode}$$

$$M_0 = \begin{pmatrix} 2549.320 & -354.322 & 90.401 & -20.987 & 3.718 \\ 2490.536 & -557.526 & -112.699 & -120.944 & -22.553 \\ 2264.921 & -1089.810 & -228.315 & 45.151 & 65.849 \\ 1791.849 & -1525.910 & 331.622 & 138.028 & -78.445 \\ 1032.654 & -1222.903 & 775.353 & -297.050 & 68.450 \end{pmatrix} kip \cdot ft$$

Peak Base Overturning Moment: $M_{b0_{mode}} := M0_{1,\,mode}$

$$M_{b0}^{T} = (2549.320 \;\; -354.322 \;\; 90.401 \;\; -20.987 \;\; 3.718)\, kip \cdot ft$$

Compare with Figure 13.8.3 (pg. 566)

Peak Base Overurning Moment
(including all modes)
using the SRSS rule

$$Mbtotal := \sqrt{\sum_{mode = 1}^{n_{mode}} \left(M_{b0_{mode}}\right)^2}$$

$$Mbtotal = 2575.501\, kip \cdot ft$$

See Table 13.8.5.

MDOF SUMMARY

Load = $\underline{s}\,p(t)$	Load = Earthquake
$\underline{m}\ddot{\underline{u}} + \underline{c}\dot{\underline{u}} + \underline{k}\underline{u} = \underline{s}\,p(t)$	$\underline{m}\ddot{\underline{u}} + \underline{c}\dot{\underline{u}} + \underline{k}\underline{u} = -\underline{m}\cdot\underline{1}\cdot\ddot{u}_g(t)$

$$\underline{s} \longrightarrow \underline{m}\cdot\underline{1}$$

$$p(t) \longrightarrow -\ddot{u}_g(t)$$

$$\boxed{\underline{s}_n = \Gamma_n\,\underline{m}\,\underline{\phi}_n}$$

$\Gamma_n = \dfrac{\underline{\phi}_n^T\,\underline{s}}{M_n}$	$\Gamma_n = \dfrac{\underline{\phi}_n^T\cdot\underline{m}\cdot\underline{1}}{M_n} = \dfrac{L_n^h}{M_n}$ $\left(L_n^h = \sum_{j=1}^{N} m_j\,\phi_{jn}\right)$

$$\boxed{M_n\ddot{q}_n + C_n\dot{q}_n + K_n q_n = P_n(t)}$$

$P_n(t) = \Gamma_n M_n\,p(t)$	$P_n(t) = -\Gamma_n M_n\,\ddot{u}_g(t)$
$\ddot{q}_n + 2\xi_n\omega_n\dot{q}_n + \omega_n^2 q_n = \Gamma_n\,p(t)$ modal EOM	$\ddot{q}_n + 2\xi_n\omega_n\dot{q}_n + \omega_n^2 q_n = -\Gamma_n\,\ddot{u}_g(t)$ modal EOM
$\ddot{D}_n + 2\xi_n\omega_n\dot{D}_n + \omega_n^2 D_n = p(t)$ (mass = 1; ξ_n, ω_n) SDOF EOM load = $p(t)$	$\ddot{D}_n + 2\xi_n\omega_n\dot{D}_n + \omega_n^2 D_n = -\ddot{u}_g(t)$ [ξ_n, ω_n; $\ddot{u}_g(t)$] SDOF EOM
$q_n(t) = \Gamma_n D_n(t)$ ↳ Chs 2,3,4,5	$q_n(t) = \Gamma_n D_n(t)$ ↳ Chs 6,5
$\underline{f}_{s,n}(t) = \underline{s}_n\left[\omega_n^2 D_n(t)\right]$	$\underline{f}_{s,n}(t) = \underline{s}_n\left[A_n(t)\right]$
$r_n(t) = r_n^{st}\left[\omega_n^2 D_n(t)\right]$	$r_n(t) = r_n^{st} A_n(t)$

$$\boxed{r(t) = \sum_{n=1}^{N} r_n(t)}$$

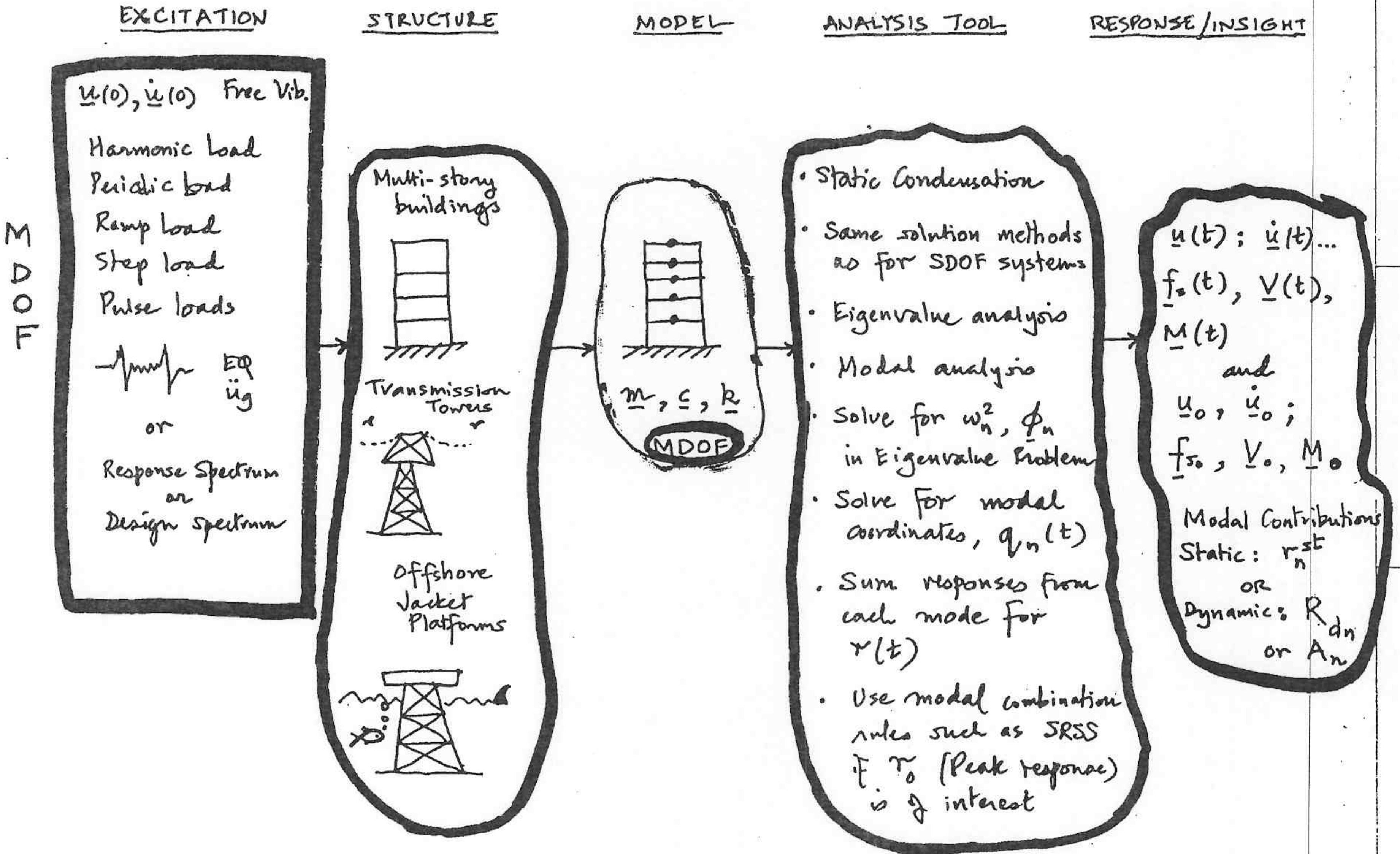
Example: $p(t) = p_0 \sin \omega t$

$$r_n(t) = r_n^{st}\cdot p_0\cdot \underbrace{R_{d_n}\cdot\sin[\omega t - \theta_n]}_{f(\omega_n,\,\omega,\,\xi_n)}$$

$r_{n_0} = r_n^{st}\cdot p_0\cdot R_{d_n}$ ↑ use Ch.3 or shock spectra, for example	$r_{n_0} = r_n^{st}\cdot A_n$ ↑ use response spectrum

$$\boxed{r_0 \simeq \sqrt{\sum_{n=1}^{N} r_{n_0}^2}}$$

(OVERVIEW - Continued)



EXCITATION

$\underline{u}(0), \dot{\underline{u}}(0)$ Free Vib.

Harmonic Load
Periodic load
Ramp load
Step load
Pulse loads
EQ $\ddot{u}_g$
or
Response Spectrum
or
Design spectrum

ANALYSIS TOOL

- Static Condensation
- Same solution methods as for SDOF systems
- Eigenvalue analysis
- Modal analysis
- Solve for ω_n^2, $\underline{\phi}_n$ in Eigenvalue Problem
- Solve for modal coordinates, $q_n(t)$
- Sum responses from each mode for $r(t)$
- Use modal combination rules such as SRSS If r_o (Peak response) is of interest

RESPONSE/INSIGHT

$\underline{u}(t)$; $\dot{\underline{u}}(t)$...
$\underline{f}_s(t)$, $\underline{V}(t)$, $\underline{M}(t)$
and
$\underline{u}_o$, $\dot{\underline{u}}_o$;
$\underline{f}_{so}$, $\underline{V}_o$, $\underline{M}_o$

Modal Contributions
Static: r_n^{st}
OR
Dynamic: R_{dn}
or A_n

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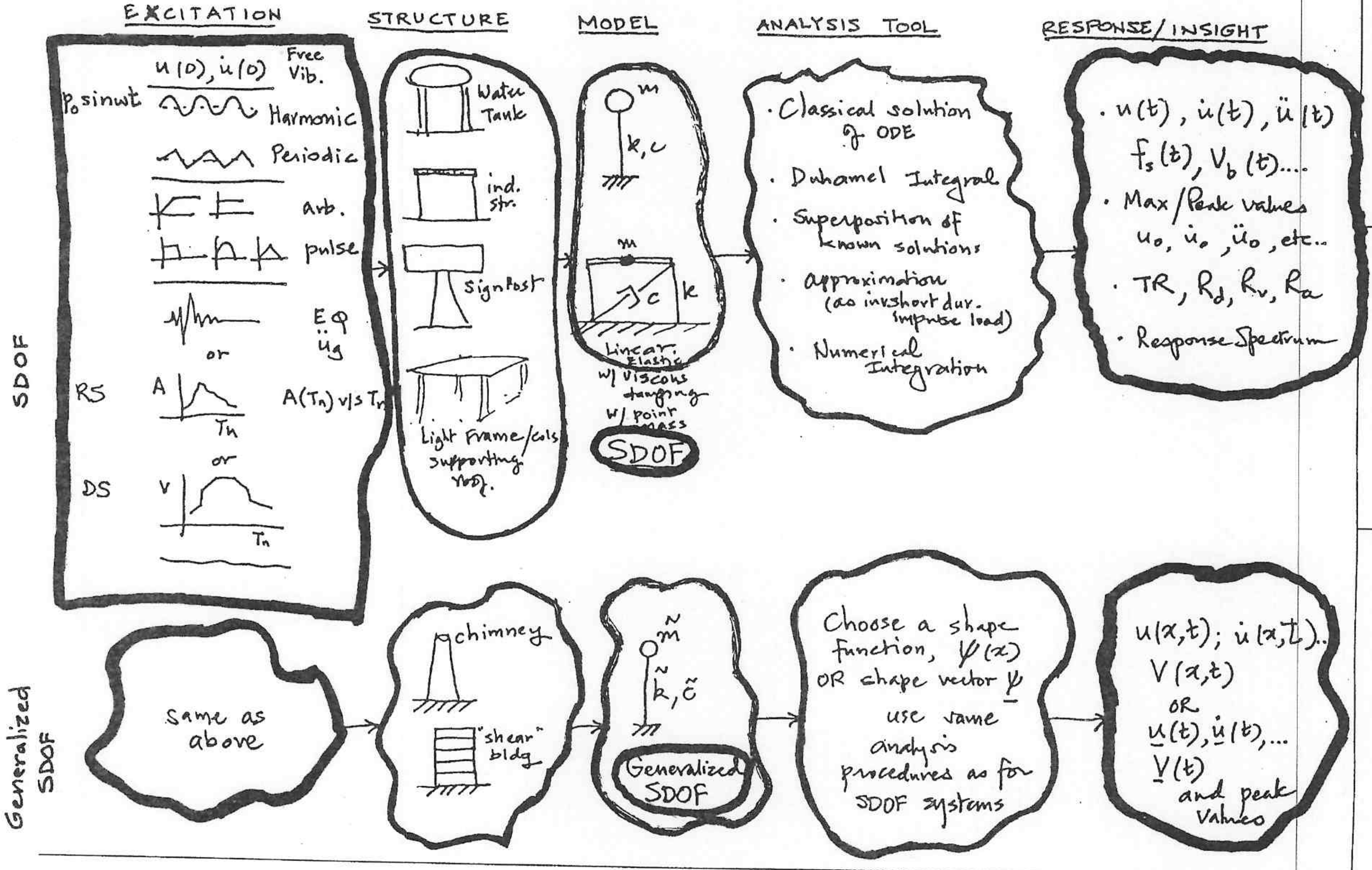
STRUCTURAL DYNAMICS - AN OVERVIEW



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Summary for CE 384P (Dynamic Response of Structures) – Spring 2006 Semester

The following topics were covered in the course:

- Introduction to the subject of structural dynamics
- Single-degree-of-freedom (SDOF) systems
  - Equations of Motion
  - Free vibration
  - Response to harmonic and periodic loading
  - Unit impulse response; response to arbitrary loading
  - Response to step and pulse excitations; shock spectra
  - Numerical methods
  - Earthquake response of linear elastic systems; response and design spectra
- Generalized SDOF systems
  - Systems with distributed mass and elasticity
  - Lumped-mass systems
  - Rayleigh's method for estimating natural vibration frequency
  - Selection of shape functions
- Multi-degree-of-freedom (MDOF) systems
  - Equations of motion
  - Free vibration; natural vibration frequencies and mode shapes
  - Modal analysis of linear systems
  - Damping in structures
  - Response of linear systems to seismic excitation
  - Response spectrum analysis
- Other topics (of important current relevance) that were introduced/discussed
  - Designing office buildings to minimize floor vibrations
  - Earthquake design spectrum as specified in the IBC 2000 code
  - Retrieval of ground motions from a public database from past earthquakes and use in analysis (example used was 1994 Northridge earthquake)
- If you liked this course, consider learning next about…
  - Dynamics of inelastic/nonlinear structures
  - Dynamics of offshore structures
  - Earthquake engineering
  - Random vibration, dynamics in the frequency domain
  - Abnormal loading (e.g., blast) and progressive collapse analysis
  - Vibration control in structures
  - Wind engineering
  - Health monitoring and sensors.

That's all, folks! Thank you.
Lance Manuel, (512) 232-5691, lmanuel@mail.utexas.edu
05/04/06

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Handout No. 11

Design

by Thomas M. Murray, Ph.D., P.E.

10 Floor Vibrations:

TIPS FOR DESIGNERS OF OFFICE BUILDINGS

What's happening? *ENR* has a cover story entitled, "Bad Vibes" (May 19, 1997). Vibration problems with London's Millennium Bridge are the subject of a recent *Time* magazine article. For over 30 years, I have been involved with the problem of floor vibrations due to human activity, and I can honestly say that there have been more problem floors reported in the last 18 months than in the previous 28+ years. What's the reason? There are a number: higher strength steels and concretes, computer-optimized designs, longer spans, less inherent damping and much lighter live loads due to the ubiquitous electronic office, to name but a few.

Fortunately, all of these can be accounted for if a little care is taken in the design process. Following are 10 tips to help produce steel-framed floors that are not annoying to office-building occupants. If you are not familiar with floor vibration analysis, I recommend a study of the AISC/CISC publication *Design Guide 11: Floor Vibrations Due to Human Activity.*

1 Don't blame vibration problems on LRFD–it's not the cause of serviceability problems.

Sure, LRFD results in lighter floor systems, especially if composite construction is used. Sure, the profession has a hang-up about LRFD. Yes, composite systems rarely satisfy floor vibration criteria, but that's not the fault of LRFD. A stretched-to-the-limit ASD design will result in the same serviceability problems as LRFD. The designer needs to accept that 50 ksi steels, higher strength concrete, optimized computer-based designs, longer spans and much lighter actual live loads result in lively floors (as the British say), and therefore require a little more design time. Better yet, think of it as the need for a little art in your floor designs.

2 Use the AISC/CISC design guide criteria.

Study the American Institute of Steel Construction and the Canadian Institute of Steel Construction's *Design Guide 11: Floor Vibrations Due to Human Activity.* Unlike older publications such as *Modified Reiher-Meister Scale, Murray Criterion* and the *Canadian Standards Association Rule* that use a heel-drop impact, the new criteria are based on resonance with walking.

Resonance can occur when the exciting frequency (rate of walking) or a multiple of that frequency (harmonic) equals the natural frequency of the floor system. Resonance results in very large amplitudes of displacement, velocity or acceleration, as seen in Figure 1. The criteria ensure that resonance does not occur for the first three harmonics associated with walking. That is, if a person is walking at 2 steps per second (2 Hz), the floor system is checked for resonance at 2, 4 and 6 Hz.

The design guide criteria state that a floor is satisfactory if the following inequality is satisfied:

$$\frac{a_p}{g} = \frac{P_o \exp(-0.35 f_n)}{\beta W} < \frac{a_o}{g}$$

where a_p/g is the predicted peak acceleration of the floor due to walking as a

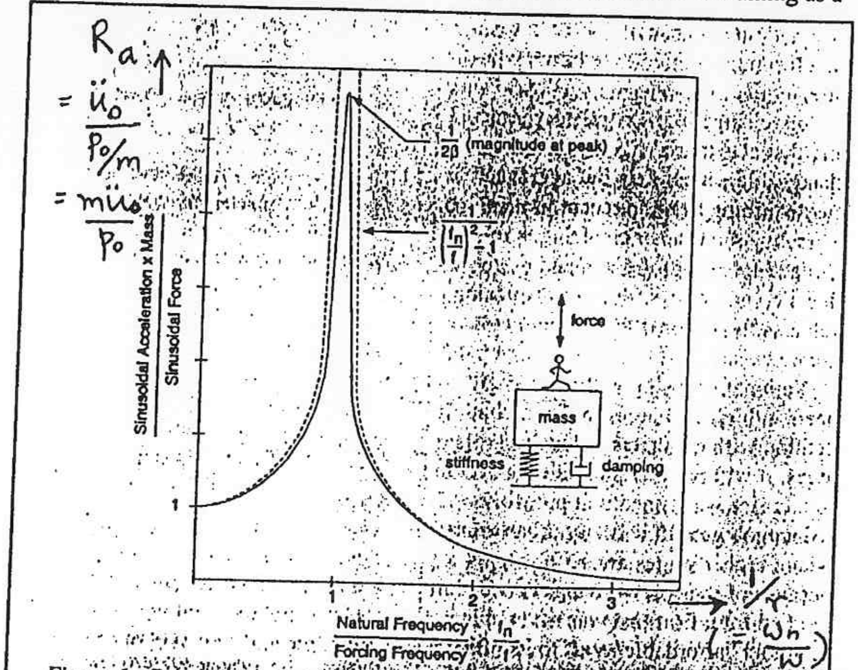

Figure 1. Resonance response (Figure 1.3 of the AISC/CISC design guide).

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function of gravity, a_o/g is the tolerance acceleration for the environment, P_o is a constant force representing the excitation, f_n is the natural frequency of the floor system, β is the modal damping in the floor system and W is the effective weight that moves because of the excitation. At first glance, the criteria might look a bit formidable—it certainly is different than the older criteria. In reality, only f_n and W require calculations. P_o is 65 lb. for office floors, a_o/g is 0.005g (0.5%g) for office environments and β is a number between 0.01 and 0.05 (see Tip 3).

But why learn the new criteria? All floor vibration criteria have two parts: a prediction of the floor response and a human tolerance level. Furthermore, all criteria must be calibrated and thus are empirical in nature (the necessary fundamental studies of human response to low frequency/very low amplitude vertical vibration have not been done). The *Modified Reiher-Meister*, *Murray Criterion* and *Canadian Rule* were all calibrated using floors built at least 25 years ago. However, construction and the office environment have changed. Today, we use lighter structural members, thinner concrete decks and longer spans. Actual office live loads are probably less than one-half of what they were 25 years ago, and permanent partitions are more scarce resulting in less damping. The older methods simply do not account for these changes. For instance, the *Modified Reiher-Meister Scale* assumes 5 to 8% log decrement damping, a level very unlikely for today's floors.

Consider the floor framing shown in Figure 2. The structural system is 3¼" normal-weight concrete on 0.6" C deck, supported by 24K8 joists at 24" on center and spanning 38'. The joists are supported by W24×76 girders spanning 30'. Nothing about this system is really unusual except that the live load deflection for the joists is less than *L*/480. The *Modified Reiher-Meister Criterion* predicts a "slightly perceptible" floor. The *Murray Criterion* requires 4.1% damping, which is easily justified. The *Design Guide* predicts a peak acceleration of 0.66%g, which is greater than the office environment tolerance acceleration of 0.50%g and is an unacceptable floor. The framing shown is nearly identical to a recently investigated floor where the building occupants had complained quite vigorously and where damping posts were installed to reduce vibration.

3 Consider the consequences of an electronic office.

The electronic office is virtually paperless; I have been in one where the only papers were a few newspapers (mostly the financial section) scattered around the computer terminals. The result is much less live load and much less damping. Desks, filing cabinets and bookcases are live load and great sources of damping. In their absence, the potential for annoying floor vibrations mounts. Adding to the problem are modern floor layouts–open, with few fixed partitions, widely spaced demountable partitions or no partitions at all. Atrium-type areas are more common and curtain walls are less stiff. What's the solution? Use the AISC/CISC design guide methods, assume actual live loads in the 6 to 9 psf range, and modal damping of 2 to 2.5% of critical.

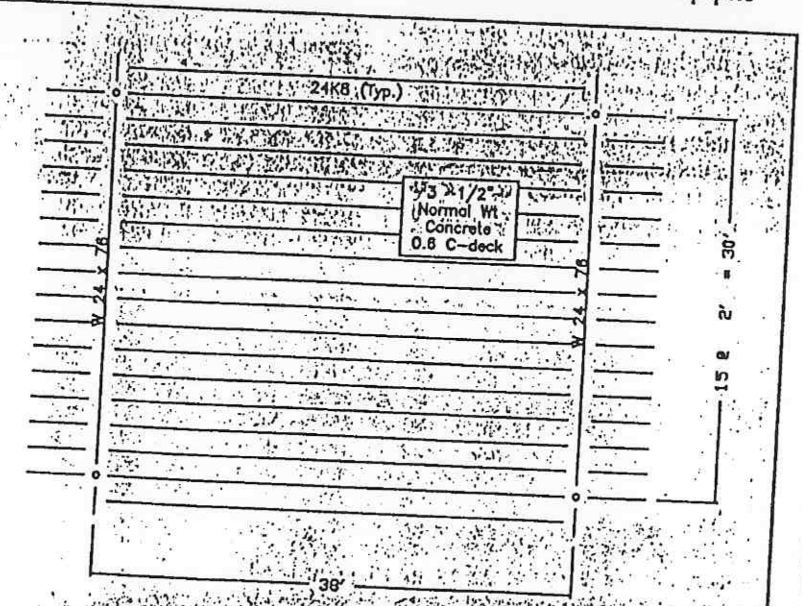

Figure 2. Floor framing.

Recently, because of an annoying floor, the office contents in one building were actually weighed–the result was an equivalent weight of 8 psf! Throw in the humans, and you may get 9 psf! The floor design live load was 125 psf. Do we need to change our code live loads? Probably, but that's a question for the ASCE-7 Committee.

What about damping? Read on.

4 Don't mix-up Log Decrement and Modal Damping.

Now for some jargon: log decrement damping was used to develop the older heel-drop-based floor vibration tolerance criteria. Unfortunately, log decrement damping overestimates the damping as it measures not only energy dissipation (true damping) but also the transmission of vibrational energy to other structural components. The design guide criteria use modal damping or "true" damping (it's interesting that we call modal damping "true damping" when we cannot measure it very accurately, at least in floors). What's the difference? Only about 50% to 100%, so be careful! Modal damping is one-half to two-thirds of log decrement damping, so if you are accustomed to estimating damping for heel-drop based criteria, you will need to adjust your design office practices.

What are good modal damping estimates? Damping is usually expressed as a ratio of critical damping. Critical damping is the damping required to bring a system to rest in one-half of a cycle. That is, if you hit something and it has 1.00 or 100% critical damping, it

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will come to rest without oscillating. For offices with fixed partitions, a good estimate is 0.05 or 5%; for conventional or paper offices, i.e. good old structural engineering offices, with demountable partitions, use 3%; and for the paperless or electronic office, I recommend 2 to 2.5%. Note again that these numbers are much less than those recommended for heel-drop based criteria.

5 Do not design floors with a natural frequency below 3 Hz

Walking speed in an office can be 1.25 to 1.5 steps per second (or Hz). Resonance at the second harmonic, 2.5 to 3 Hz, is then a real possibility if the floor's natural frequency is below 3 Hz. I have caused a floor to vibrate at its natural frequency by running a shaker (an electrically-powered oscillating mass) at one-half of the floor frequency. The result is quite unsettling; if this happened in an office building, complaints would be loud and clear.

However, a 3 Hz or less floor can be made to work if it is made very heavy, say 100+ psf.

6 Remember that joists and joist-girders require special consideration.

The stiffness of trusses is affected by shear deformations in the webs. An age-old rule-of-thumb is that the effective moment of inertia of a parallel chord truss is 0.85 times the moment of inertia of the chords. This rule is used to compute the $L/360$ deflection limit live load in the Steel Joist Institute load tables. This rule works well if the span-to-depth ratio of the truss is greater than about 18; if the ratio is less, the deflection will be greater than predicted.

Joists and joist-girders have another problem–they are not fabricated with work points. Panel point eccentricities of up to 2", as shown in Figure 3, are common. Surprisingly, this has no effect on strength although member stiffness is reduced, especially if the span-to-depth ratio is less than about 18. The design guide offers the following expressions that are used to predict the effective moment of inertia of joist and joists girders:

- for angle web members with $6 < L/D < 24$:

$$C_r = 0.90 \, (-e^{-0.28(L/D)})^{2.8}$$

- for round rod web members with $10 < L/D < 24$:

$$C_r = 0.721 + 0.00725 \, (L/D)$$

where L is the member span and D is the nominal depth; and

$$I_{mod} = C_r \, I_{chords}$$

This moment of inertia is then used to calculate the effective transformed moment of inertia of the composite section. The above expressions were developed using static analysis and tests and apply equally well to static live load deflections.

For many years, I maintained that joist seats provided enough stiffness so that the supporting girder or joist-girder could be considered fully composite for floor vibration analysis. I was very wrong. Using floors constructed in the Virginia Tech Structures and Materials Laboratory, we found that joist seats are not, in fact, very good shear connectors. The design guide recommends that the composite moment of inertia of a girder or joist girder be approximated using:

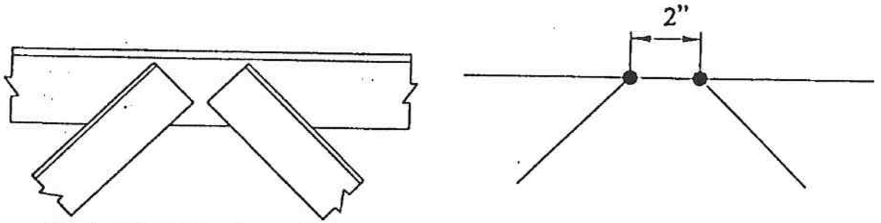
$$I_g = I_{nc} + (I_c - I_{nc}) / 4$$

where I_{nc} and I_c are the non-composite and fully composite moments of inertia, respectively. Recent field tests have shown this expression is a bit conservative if the joists are closely spaced, say not more than 30", and unconservative if there are only two or three joists being supported by the girder or joist-girder. Testing is currently being conducted to develop better approximations.

7 Improve a design that does not satisfy the criterion.

The criteria for heel-drop based methods indicates that increasing the stiffness has very little effect on the floor performance. With these methods, the only way to effectively improve a proposed floor design is to increase the mass. A different result is found when the design guide methods for office floors are used. With this method, the tolerance criterion can be satisfied by either increasing the mass or increasing the stiffness. A stiffer floor is always a better floor so the latter result is logical–no one has ever had a vibration problem with a 10' span.

If the design guide method is being used and a proposed framing scheme does not satisfy the tolerance criterion, e.g. 0.5% of gravity, there are two approaches to improving the design. First, you can increase the mass by adding concrete or changing from lightweight to normal weight concrete. This approach will result in a slightly lower fundamental frequency but a larger effective weight, W in the criteria. The lower frequency will increase the pre-



Typical Joist Configuration

Finite Element Model

Figure 3. Joist panel point eccentricity.

9-4

dicted acceleration and the larger effective weight will decrease it, usually more than the frequency-caused increase, resulting in a better floor. Second, you can first stiffen the member (beam or girder) with the lower frequency until both frequencies are approximately the same. If the system is still not satisfactory, member types can be stiffened until a satisfactory design is achieved. My experience has shown that the latter method is more cost effective for most designs.

8 Don't believe the myth that certain beam spans should be avoided.

In the late 60s or early 70s a paper was written describing a number of joist-supported problem floors where the joist spans were in the 24' to 28' range. Somehow this was interpreted to mean that bays with beam or joist spans in this range should be not be designed, and this belief has become part of the folklore (if I may use that term) of the structural engineering community. Even some joist manufacturer engineers will tell you to avoid these spans. The problem floors described in the original paper were typical of the time, meaning that the spans and the problems were connected. But, in fact, there is no correlation between span and occupant complaints. Span alone is not the reason a particular floor is annoying to occupants.

Likewise, long span floors, say spans greater than 40', are not inherently problem floors. I have made measurements on composite joist supported floors with spans between 40' and 118' (that's not a typo, there truly is an office building with a 118' span). The design guide criteria predicted the floors would not be annoying and they were not.

The bottom line is that floors of any span can be designed such that occupants will not feel annoying vibrations. Just be sure the design satisfies the design guide criteria and the frequency is above 3 Hz.

9 Be careful when designing crossovers (elevated walks).

Atrium crossovers can be a design challenge. Crossovers typically have long spans; therefore, the frequency is quite low. Further, there is very little damping, generally about 1% modal damping. The result is that deep, stiff supporting members are required.

Also, the location of the slab needs to be considered. I know of two problem crossovers where the structural engineers relied on previous experience with floors of similar framing and did not check the crossover design. In both cases, complaints were received even before the buildings were opened. The major cause of the problems was that the crossover slab was located between the supporting beams at about mid-depth as shown in Figure 4. The result was that the moment of inertia of the crossover was twice the moment of inertia of the supporting beams, which, of course, is much less than the composite moment of inertia would had been if the slab was on top of the beams. The result was a much lower frequency than expected and an expensive fix in both cases.

10 Be even more careful when designing health clubs in office buildings.

Aerobics classes are part of any health club's activities, and an aerobics class is probably the most severe building floor loading for vibration concerns. The energy from aerobics can travel much farther than you might expect. I know of an instance where aerobics on the 60th floor of a building were felt on the 40th floor but not on the floors in between or below the 40th floor! Aerobics in one corner on the second level of a two-story strip mall has been felt several hundred feet away. Solutions are costly: a 400% increase in steel weight over the strength design would have been required in a strip mall to solve the problem (the owner decided to move the health club to the lower level instead).

The design guide has criteria for designing floors supporting rhythmic activities. Basically, the floor frequency must be above a limiting value that depends on an acceleration limit, which is determined considering the activity and what is called the "affected occupancy" and the weight of the floor. The acceleration limits for aerobics alone, aerobics in conjunction with weightlifting and aerobics near offices are 5 to 10%, 2% and 0.5%, respectively. It turns out that weight-lifters are sensitive folks, thus the lower limit. Also, some of them are big, so you have to be extra careful! The required floor frequencies for the three conditions and a 100 psf floor are 8.8 Hz, 9.2 Hz and 16 Hz. For a 50 psf floor, the corresponding frequencies are 9.2 Hz, 10.6 Hz and 22.1 Hz.

If the spans are less than, say 30', use of lightweight concrete and closely spaced, deep joists will result in a floor frequency in the range of 10 to 12 Hz without too much expense. The floor system would be satisfactory for aerobics alone or in conjunction with

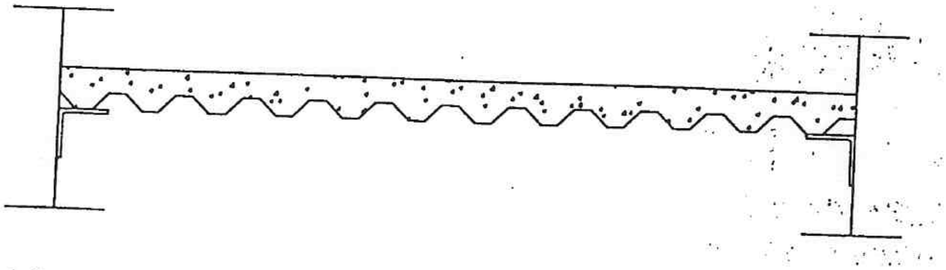


Figure 4. Crossover cross-section. Note: transverse members allow the deck to run parallel to the girders as shown.

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weight-lifting but not near offices. Generally, it is cost prohibitive to design a floor system that supports both aerobics and offices.

If the aerobics activity cannot be moved to a slab on grade, then I suggest either a separate framing system for the aerobics floor or the use of a floating floor. Separate framing is an easy solution for two story buildings.

When using this approach, the aerobics floor slab must be completely separated from the surrounding slabs, and the ceiling below cannot be supported from the aerobics floor framing. Separate cold-formed framing connected only to the columns has been used to support the ceiling below.

Floating floors may be the only solution in a tall building. The concept of a floating floor is similar to that used for isolating machinery vibration. A floating floor is simply a separate floor supported by very soft springs attached to the structural floor. The natural frequency of the floating floor should be quite low, less than 2 to 3 Hz, which generally requires a heavy slab, 50 to 100 psf. Also, the space between the two floors must be vented or the change in air pressure due to the movement of the floating floor will cause the structural floor to move.

A Final Thought

A number of structural engineers have told me that they now design for serviceability and then check strength. As Hardy Cross once wrote: Strength is essential but otherwise not important.

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